THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2008
COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

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SECTION 1 – MULTIPLE CHOICE QUESTIONS

1	2	3	4	5	6	7	8	9	10	11
B	C	E	A	D	A	B	D	C	E	B

12	13	14	15	16	17	18	19	20	21	22
D	D	A	B	E	E	E	B	A	B	A

QUESTION 1

$$\frac{\Delta f}{\Delta t} = \frac{f(-2) - f(0)}{(-2) - 0} = \frac{f(-2) - f(0)}{-2}$$

$$f(-2) = -8 + 4 = -4$$

$$f(0) = 0$$

Therefore $\frac{\Delta f}{\Delta t} = \frac{-4 - 0}{-2} = 2$

The answer is B.

QUESTION 2

Decreasing for values of x such that $f'(x) < 0$.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2).$$

Therefore $3(x-2)(x+2) < 0$ is required.

Therefore $-2 < x < 2$.

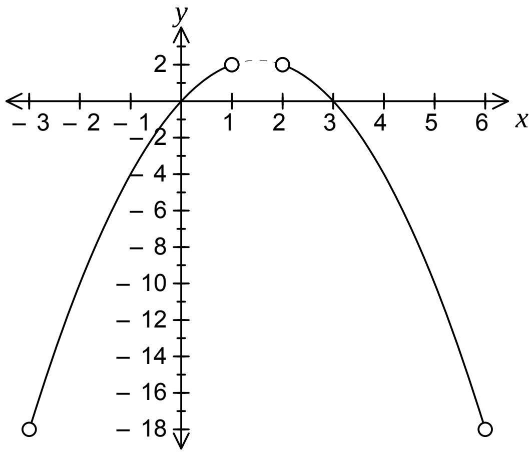
An option that is a subset of $-2 < x < 2$ is therefore required.

The answer is C.

QUESTION 3

$f(x) = -18 \Rightarrow x = -3, 6$

$f(x) = 2 \Rightarrow x = 1, 2$



Maximum turning point at $x = \frac{3}{2}$. $f\left(\frac{3}{2}\right) = \frac{9}{4}$.

From the graph it is therefore clear that D is either $-3 < x < 1$ or $2 < x < 6$.

The answer is E.

QUESTION 4

Two lines are perpendicular if $m_1 m_2 = -1$.

$3x - y + 4 = 0 \Rightarrow y = 3x + 4 \Rightarrow m_1 = 3$.

$ax + 2y - 3 = 0 \Rightarrow y = -\frac{a}{2}x + \frac{3}{2} \Rightarrow m_2 = -\frac{a}{2}$.

Therefore $m_1 m_2 = (3)\left(-\frac{a}{2}\right) = -\frac{3a}{2}$.

Therefore $-\frac{3a}{2} = -1 \Rightarrow a = \frac{2}{3}$.

The answer is A.

QUESTION 5

It is required that $\frac{x^2 - x - 6}{4 + 3x - x^2} \geq 0$, where $4 + 3x - x^2 = -(x-4)(x+1) \neq 0 \Rightarrow x \neq 4, -1$.

Therefore:

Case 1: $x^2 - x - 6 = (x-3)(x+2) \geq 0$ AND $4 + 3x - x^2 = -(x-4)(x+1) > 0$

$\Rightarrow x \leq -2$ or $x \geq 3$ AND $-1 < x < 4$

$\Rightarrow 3 \leq x < 4$

Case 2: $x^2 - x - 6 = (x-3)(x+2) \leq 0$ AND $4 + 3x - x^2 = -(x-4)(x+1) < 0$.

$\Rightarrow -2 \leq x \leq 3$ AND $x < -1$ or $x > 4$

$\Rightarrow -2 \leq x < -1$

Therefore $3 \leq x < 4$ or $-2 \leq x < -1$.

The answer is D.

QUESTION 6

$$\int_b^a 3g(x) - x - h(x)\ dx \quad = 3\int_b^a g(x)\ dx - \int_b^a x\ dx - \int_b^a h(x)\ dx$$

$$= 3(2) - \int_b^a x\ dx - (-3)$$

$$= 9 - \int_b^a x\ dx$$

$$= 9 - \left[\frac{1}{2}x^2\right]_b^a$$

$$= 9 - \left(\frac{a^2}{2} - \frac{b^2}{2}\right)$$

$$= 9 - \frac{a^2}{2} + \frac{b^2}{2}$$

$$= 9 + \frac{b^2 - a^2}{2}$$

The answer is A.

QUESTION 7

$$\frac{dy}{dx} = 2ax + b$$

Substitute $\frac{dy}{dx} = 4$ when $x = 1$: $\quad 4 = 2a + b \quad$ …. (1)

Substitute (1, 2) into $y = ax^2 + bx$: $\quad 2 = a + b \quad$ …. (2)

Solve equations (1) and 92) simultaneously: $a = 2$ and $b = 0$.

The answer is B.

QUESTION 8

Substitute $\left(\log_e 3,\ \frac{82\sqrt{3}}{9}\right)$ into $y = 3e^{x/a} + e^{-x/a}$:

$$\frac{82\sqrt{3}}{9} = 3e^{\frac{1}{a}\log_e 3} + e^{-\frac{1}{a}\log_e 3} = 3e^{\log_e 3^{1/a}} + e^{\log_e 3^{-1/a}} = 3\left(3^{1/a}\right) + 3^{-1/a}$$

Test each option by comparing decimal approximations on each side of the equation.

The answer is D.

QUESTION 9

$f(x)$ requires a 'join' at $x = 0$: $(0-a)^3 + 2 = b(0) + \cos(0)$

$\Rightarrow -a^3 + 2 = 1$

$\Rightarrow a^3 = 1$

$\Rightarrow a = 1$

$$f'(x) = \begin{cases} 3(x-a)^2, & x \le 0 \\ b - \sin x, & x > 0 \end{cases}$$

$f'(x)$ requires a 'join' at $x = 0$: $3(0-a)^2 = b - \sin(0)$

$\Rightarrow 3a^2 = b$

Substitute $a = 1$: $b = 3$.

The answer is C.

QUESTION 10

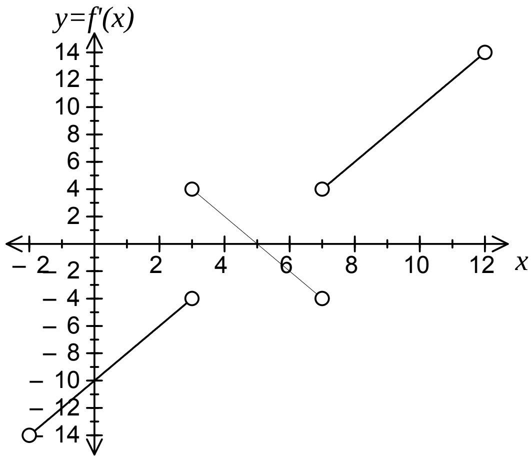
$$f(x) = \begin{cases} (x-5)^2 - 4, & x < 3 \text{ or } x > 7 \\ -(x-5)^2 + 4, & 3 \le x \le 7 \end{cases}.$$

Therefore $f'(x) = \begin{cases} 2(x-5), & x < 3 \text{ or } x > 7 \\ -2(x-5), & 3 < x < 7 \end{cases}$

$f'(-2) = -14$ and $f'(12) = 14$.

From the graph :

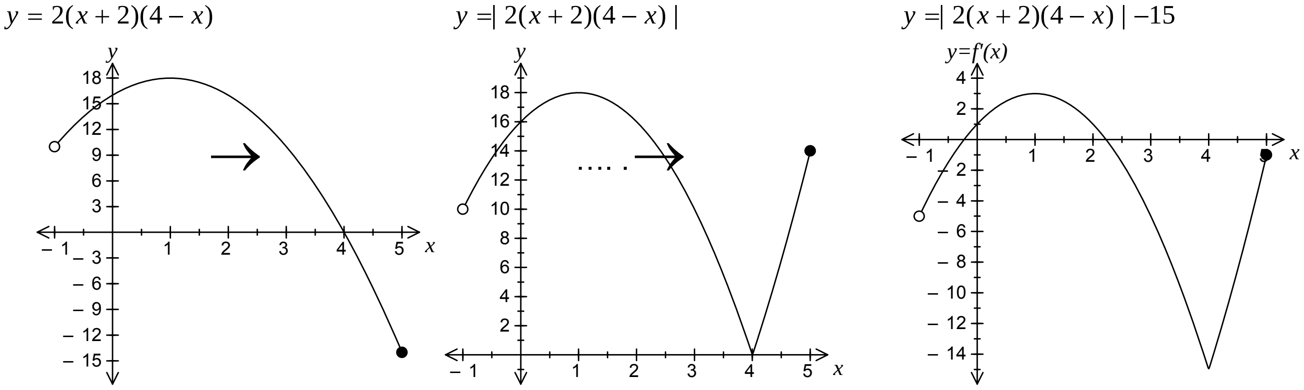
$-14 < y < -4$, $-4 < y < 4$ and $4 < y < 14$.



The answer is E.

QUESTION 11

A graph of $f(x) = |2(x+2)(4-x)| - 15$ can be drawn by translating the graph of $y = |2(x+2)(4-x)|$ down by 15 units:



$f(-1) = -5$ and $f(5) = -1$.

The salient point is at $(4, -15)$.

The turning point is at $(1, 3)$.

From the graph of $y = f(x)$ it is therefore clear that $-15 < y < 3$.

The answer is B.

QUESTION 12

Chain rule: Let $u = f(cx)$

$$\Rightarrow \frac{du}{dx} = cf'(cx) \quad \text{and} \quad y = \sin(\log_e u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot cf'(cx)$$

Use the Chain rule to find $\frac{dy}{du}$: Let $w = \log_e u$

$$\Rightarrow \frac{dw}{du} = \frac{1}{u} \quad \text{and} \quad y = \sin w$$

$$\frac{dy}{du} = \frac{dy}{dw} \cdot \frac{dw}{du} = \cos w \cdot \frac{1}{u} = \frac{\cos(\log_e u)}{u} = \frac{\cos(\log_e [f(cx)])}{f(cx)}$$

Therefore $\frac{dy}{dx} = \frac{\cos(\log_e [f(cx)])}{f(cx)} \cdot cf'(cx) = \frac{c\cos(\log_e [f(cx)])\, f'(cx)}{f(cx)}$

The answer is D.

QUESTION 13

Using three left rectangles of width $\frac{2-1}{3}=\frac{1}{3}$:

$$\int_1^2 ax^2+b\,dx \approx \frac{1}{3}\left[f(1)+f\left(\frac{4}{3}\right)+f\left(\frac{5}{3}\right)\right]=\frac{1}{3}\left[(a+b)+\left(\frac{16a}{9}+b\right)+\left(\frac{25a}{9}+b\right)\right]$$

$$=\frac{1}{3}\left[\left(\frac{9a}{9}+b\right)+\left(\frac{16a}{9}+b\right)+\left(\frac{25a}{9}+b\right)\right]=\frac{1}{3}\left[\frac{50a}{9}+3b\right]=\frac{1}{3}\left[\frac{50a+27b}{9}\right].$$

The answer is D.

QUESTION 14

$f'(x)=a\cos(x)-b\sin(x)$

Solve $f'(x)=0$ to get x-coordinates of stationary points:

$0=a\cos(x)-b\sin(x)$

$\Rightarrow a\cos(x)=b\sin(x)$

$\Rightarrow \frac{\sin(x)}{\cos(x)}=\frac{a}{b}$

$\Rightarrow \tan(x)=\frac{a}{b}$

Substitute $x=-\frac{\pi}{4}$: $\tan\left(-\frac{\pi}{4}\right)=\frac{a}{b}$

$\Rightarrow -1=\frac{a}{b} \qquad \therefore a=-b$

Therefore $f(x)=a\sin(x)-a\cos(x)=a[\sin(x)-\cos(x)]$.

It can be seen by drawing a graph that $y=\sin(x)-\cos(x)$ has a maximum turning point at $x=-\frac{\pi}{4}$. Therefore $f(x)=a[\sin(x)-\cos(x)]$ has a

- Maximum turning point at $x=-\frac{\pi}{4}$ if $a>0$ (a dilates the graph $y=\sin(x)-\cos(x)$ from the x-axis).
- Minimum turning point at $x=-\frac{\pi}{4}$ if (a dilates the graph $y=\sin(x)-\cos(x)$ from the x-axis and reflects in the x-axis).

It is therefore required that $a=-b$ and $a<0$.

The answer is A.

QUESTION 15

$$2\cos\left(\frac{x}{a}\right)+\sqrt{3}=0$$

$$\Rightarrow \cos\left(\frac{x}{a}\right)=-\frac{\sqrt{3}}{2}$$

Basic Angle: $Cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

$\Rightarrow \frac{x}{a}=\pi-\frac{\pi}{6}+2n\pi$ or $\frac{x}{a}=\pi+\frac{\pi}{6}+2n\pi$

$\Rightarrow \frac{x}{a}=\frac{5\pi}{6}+2n\pi$ or $\frac{x}{a}=\frac{7\pi}{6}+2n\pi$

$\Rightarrow x=\frac{5a\pi}{6}+2na\pi$ or $x=\frac{7a\pi}{6}+2na\pi$

$n=-1$: $x=\frac{5a\pi}{6}-2na\pi=-\frac{7a\pi}{6}$ or $x=\frac{7a\pi}{6}-2na\pi=-\frac{5a\pi}{6}$

$n=1$: $x=\frac{5a\pi}{6}+2na\pi=\frac{17a\pi}{6}$ or $x=\frac{7a\pi}{6}+2na\pi=\frac{19a\pi}{6}$

The answer is B.

QUESTION 16

x-coordinate of intersection points: $f(x)=g(x)$

$$\Rightarrow \frac{1}{x}-3=-ax$$

$$\Rightarrow 1-3x=-ax^2$$

$$\Rightarrow ax^2-3x+1=0 \qquad \text{…. (1)}$$

Discriminant: $\Delta=(-3)^2-4(a)(1)=9-4a$

Two distinct solutions: $\Delta>0\Rightarrow 9-4a>0\Rightarrow a<\frac{9}{4}$ $\Delta>0\Rightarrow 9-4a>0\Rightarrow a<\frac{9}{4}$.

However, if $a=0$ equation (1) becomes $-3x+1=0$ which has only one solution.

Two distinct solutions: $a<\frac{9}{4}$, $a\neq 0$.

The answer is E.

QUESTION 17

Use inverse normal: $\Pr(z < z_a) = 0 \cdot 9332 \Rightarrow z_a = 1 \cdot 5$

$\Pr(z > z_b) = 0 \cdot 841345$

$\Rightarrow \Pr(z < z_b) = 0 \cdot 158655$

$\Rightarrow z_b = -1$

Substitute into $Z = \dfrac{X - \mu}{\sigma}$: $1 \cdot 5 = \dfrac{a - \mu}{\sigma}$

$\Rightarrow \sigma = \dfrac{a - \mu}{1 \cdot 5}$ …. (1)

$-1 = \dfrac{b - \mu}{\sigma}$

$\Rightarrow -\sigma = b - \mu$ …. (2)

Add equations (1) and (2) together: $0 = \dfrac{a - \mu}{1 \cdot 5} + b - \mu$

$\Rightarrow 0 = a - \mu + 1 \cdot 5b - 1 \cdot 5\mu$

$\Rightarrow \dfrac{5}{2}\mu = a + \dfrac{3b}{2} = \dfrac{2a + 3b}{2}$

$\Rightarrow \mu = \dfrac{2a + 3b}{5}$

Substitute $\mu = \dfrac{2a + 3b}{5}$ into equation (2): $\sigma = \dfrac{2a - 2b}{5}$.

The answer is E.

QUESTION 18

$$\int_0^b ax^3 dx = 1$$

$$\left[\frac{ax^4}{4}\right]_0^b = 1$$

$$\Rightarrow \frac{ab^4}{4} = 1$$

$\Rightarrow ab^4 = 4$ …. (1)

$$E(X^2) = \int_0^b x^2(ax^3)\ dx = \int_0^b ax^5 dx = \left[\frac{ax^6}{6}\right]_0^b = \frac{ab^6}{6}$$

Therefore $\frac{ab^6}{6} = \frac{3}{8} \Rightarrow ab^6 = \frac{9}{4}$ …. (2)

Substitute equation (1) into equation (2): $4b^2 = \frac{9}{4} \Rightarrow b = \frac{3}{4}$ (since $b > 0$).

The answer is E.

QUESTION 19

Let X be the random variable *amount of kitty litter in a bag (kg)*.

X ~ Normal($\mu = 1, \sigma = 0 \cdot 1$).

Pr(X > 0.95) = 0.6915.

Let Y be the random variable *number of bags that have more than 0.95 kg of kitty litter*.

Y ~ Binomial(n = 40, p = 0.6915).

E(Y) = np = (40)(0.6915) = 27.66.

The answer is B.

QUESTION 20

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\Rightarrow (3)^2 = E(X^2) - (4)^2$$

$$\Rightarrow 9 = E(X^2) - 16$$

$$\Rightarrow E(X^2) = 25$$

The answer is A.

QUESTION 21

Sum of the probabilities is equal to 1: $\frac{k}{2}+\frac{k}{4}+k+\frac{k}{4}+\frac{k}{2}=1$

$$\Rightarrow k=\frac{2}{5}$$

x^2	0	1	4	9	16
x	0	1	2	3	4
$\Pr(X=x)$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{5}$

$$E(X)=(0)\left(\frac{1}{5}\right)+(1)\left(\frac{1}{10}\right)+(2)\left(\frac{2}{5}\right)+(3)\left(\frac{1}{10}\right)+(4)\left(\frac{1}{5}\right)=2$$

$$E(X^2)=(0)\left(\frac{1}{5}\right)+(1)\left(\frac{1}{10}\right)+(4)\left(\frac{2}{5}\right)+(9)\left(\frac{1}{10}\right)+(16)\left(\frac{1}{5}\right)=\frac{29}{5}$$

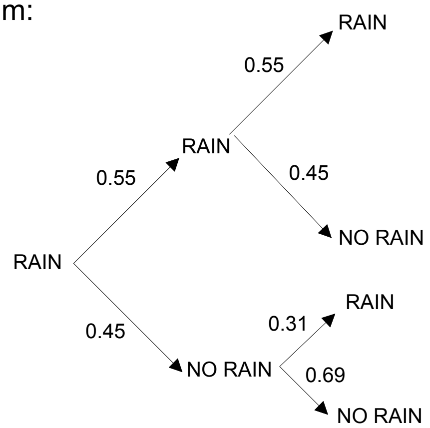
$$Var(X)=E(X^2)-[E(X)]^2=\frac{29}{5}-(2)^2=\frac{9}{5}$$

Therefore $sd(X)=\sqrt{\frac{9}{5}}\approx 1\cdot 34$

The answer is B.

QUESTION 22

Draw a tree diagram:



Pr(No rain on Saturday) = $(0\cdot 55)(0\cdot 45)+(0\cdot 45)(0\cdot 69)=0\cdot 5580$.

The answer is A.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

a. $V = Cross\ sectional\ area \times L$

$V = A \times 8$

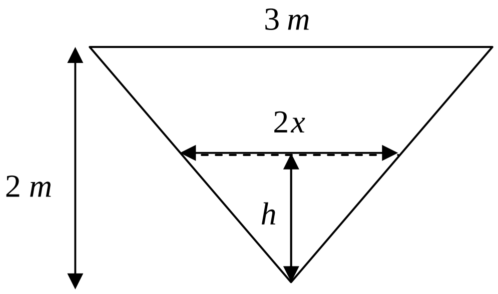
$\therefore V = 8xh\ m^3$

$A = \frac{1}{2} \times b \times h$

$A = \frac{1}{2} \times 2x \times h$

$A = xh\ m^2$

Using similar triangles:



$\frac{3}{2x} = \frac{2}{h}$

$3h = 4x$

$x = \frac{3h}{4}\ m$

$\therefore V = 8\left(\frac{3h}{4}\right)h = 6h^2\ m^3$

b. (i) $\frac{dh}{dt} = \frac{dv}{dt} \times \frac{dh}{dv}$

$Given\ \frac{dv}{dt} = +0.05m^3 / \min$

$V = 6h^2$

$\frac{dV}{dh} = 12h$

$\therefore \frac{dh}{dv} = \frac{1}{12h}$

$\therefore \frac{dh}{dt} = 0.05 \times \frac{dh}{dv}$

$= 0.05 \times \frac{1}{12h}$

$= \frac{1}{240h}\ m / \min$

(ii) Find $\frac{dh}{dt}$ when $V = \frac{Volume\ Max}{2}$

Maximum V occurs when $h = 2$, i.e. $V = 6(2)^2 = 24$

Half full: $V = 12m^3$

Find corresponding value of h: $6h^2 = 12$

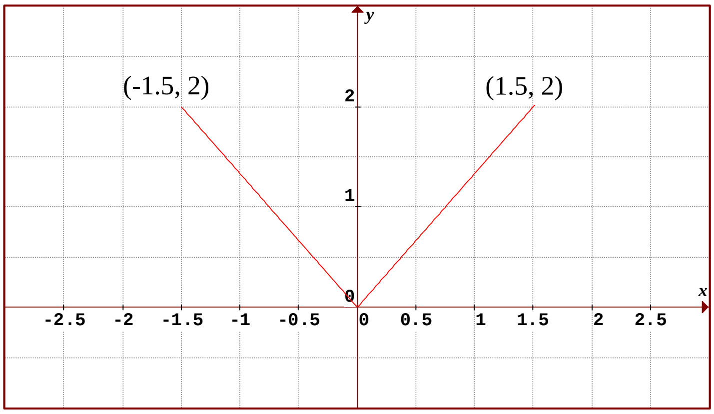
$h^2 = 2$

$h = \sqrt{2}m$

$\therefore$ Find $\frac{dh}{dt}$ when $h = \sqrt{2}\ m$:

$$\frac{dh}{dt} = \frac{1}{240h} m/\min = \frac{1}{240\sqrt{2}} m/\min \qquad (\approx 0.0029 m/\min)$$

c. (i) $D = \{x : -1.5 \le x \le 1.5\}$



For $y > 0$:

$y = mx + c$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-2}{0-1.5} = \frac{4}{3}$

$0 = \frac{4}{3}(0) + c$

$\therefore c = 0$

$y = \frac{4}{3}x$

For $y < 0$: This curve is the reflection of the portion of the curve $y = \frac{4}{3}x$ that falls below the X axis in the X axis i.e. $y = -\left(\frac{4}{3}x\right)$.

Hence the equation describing AOB is $y = \frac{4}{3}|x|$.

(ii) As triangle $-POP$ is isosceles then length $d(OP) = \frac{1}{2}d(-PP) = 0.5$

$\therefore P_x = 0.5$

$y = \frac{4}{3}x = \frac{4}{3} \times \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

$P = \left(\frac{1}{2}, \frac{2}{3}\right)$

d. (i) $AP: y - y_1 = m(x - x_1)$

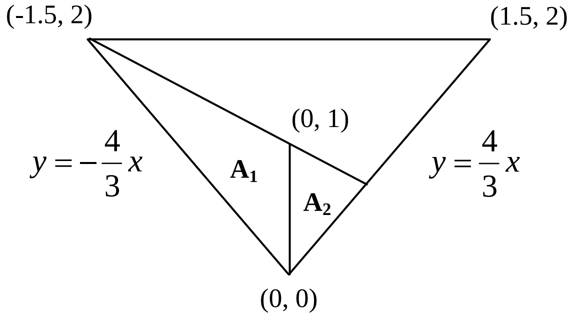
Let $(x_1, y_1) = (-1.5, 2)$

$(x_2, y_2) = \left(\frac{1}{2}, \frac{2}{3}\right)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{2}{3} - 2}{\frac{1}{2} - -1.5} = -\frac{2}{3}$

$y - 2 = -\frac{2}{3}(x + 1.5)$

$y = -\frac{2x}{3} + 1$



$A_1 = \int_{-1.5}^{0} \left(\frac{-2x}{3} + 1\right) - \left(-\frac{4}{3}x\right) dx$

$= \int_{-1.5}^{0} \left(1 + \frac{2x}{3}\right) dx$

$= \left[x + \frac{2x^2}{6}\right]_{-1.5}^{0}$

$= \left[x + \frac{x^2}{3}\right]_{-1.5}^{0}$

$= (0) - \left(-1.5 + \frac{3}{4}\right)$

$= \frac{3}{4}\, m^2$

$A_2 = \int_0^{\frac{1}{2}} \left(-\frac{2x}{3} + 1\right) - \left(\frac{4}{3}x\right) dx$

$= \int_0^{\frac{1}{2}} \left(-\frac{2x}{3} + 1 - \frac{4}{3}x\right) dx$

$= \int_0^{\frac{1}{2}} (1 - 2x)\, dx$

$= \left[x - x^2\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2} - \frac{1}{4}\right)$

$= \frac{1}{4}$

Total Area = $\frac{3}{4} + \frac{1}{4} = 1\, m^2$

(ii) $V = A \times L = 1 \times 8 = 8\, m^3$

QUESTION 2

a. $n = 8$

$p = \dfrac{3}{5}$

$E(x) = 8 \times \dfrac{3}{5} = \dfrac{24}{5}$

b. (i) $p = \dfrac{3}{5}, \quad n = 8$

$$\Pr(X = 5) = \binom{8}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^3 = \frac{108864}{390625} \approx 0.279$$

OR binompdf $\left(8, \dfrac{3}{5}, 5\right) = 0.279$

(ii) $\Pr(x \geq 2) = 1 - \Pr(X = 0, 1)$

$$= 1 - \left[\binom{8}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^7\right] = 0.991$$

OR $1 - binomcdf\left(8, \dfrac{3}{5}, 1\right) = 0.991$

(iii) $\Pr(X = 5 / X \geq 2) = \dfrac{\Pr(X = 5)}{\Pr(X \geq 2)} = \dfrac{0.279}{0.991} = 0.281$

c. $\Pr(x \geq 1) \geq 0.95$

$1 - \Pr(x = 0) \geq 0.95$

$\Pr(x = 0) \leq 0.05$

$\binom{n}{0}\left(\dfrac{3}{5}\right)^0\left(\dfrac{2}{5}\right)^n \leq 0.05$

$\left(\dfrac{2}{5}\right)^n \leq 0.05$

$\log_e\left(\dfrac{2}{5}\right)^n \leq \log_e(0.05)$

$n\log_e\left(\dfrac{2}{5}\right) \leq \log_e(0.05)$

$n \geq \dfrac{\log_e(0.05)}{\log\left(\dfrac{2}{5}\right)}$

$n \geq 3.27$

$\therefore n = 4$

d.

Profit ($)	1300	-800
$\Pr(P = p)$	0.4	0.6

$$\therefore E(P) = (1300\times 0.4) + (-800\times 0.6)$$
$$= \$40$$

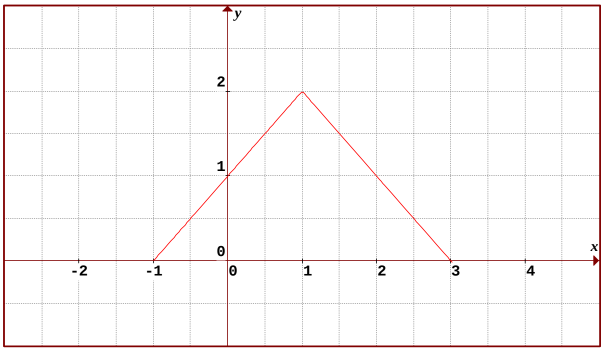
e.

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-1| & \text{if } -1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$y = f(x)$ is the graph of $y = 2 - |x-1|$ dilated by a factor of $\frac{1}{4}$ from the x-axis.

$$2 - |x-1| = \begin{cases} 3 - x & \text{if } x \ge 1. \\ x + 1 & \text{if } x < 1. \end{cases}$$

Therefore:



$$\Pr(X \ge a) = \frac{3}{4}$$

$$\frac{1}{4}\int_a^3 2 - |x-1|\, dx = \frac{3}{4}$$

$$\Rightarrow \int_a^3 2 - |x-1|\, dx = 3$$

a lies between $-1 < a < 1$ as $\Pr(X \ge a) = \frac{3}{4}$.

By symmetry, the median value of X is 1.

Therefore:

$$\int_a^1 x + 1\, dx + \int_a^1 3 - x\, dx = 3$$

$$\left[\frac{x^2}{2} + x\right]_a^1 + \left[3x - \frac{x^2}{2}\right]_1^3 = 3$$

$$\frac{7}{2}-\frac{a^2}{2}-a=3$$

$$\Rightarrow a^2+2a-1=0$$
$$\Rightarrow a=-1\pm\sqrt{2}$$

But $0<a<1$ therefore $a=-1+\sqrt{2}$.

(ii) $Var(X)=E(X^2)-[E(X)]^2$

By symmetry: $E(X)=1$

By definition:

$$E\left(X^2\right)=\int_{-\infty}^{+\infty} x^2 f(x)\,dx=\frac{1}{4}\int_{-1}^{3} x^2\left(2-|x-1|\right)dx$$

$$=\frac{1}{4}\left(\int_{-1}^{1} x^2(x+1)\,dx+\int_{1}^{3} x^2(3-x)\,dx\right)$$

$$=\frac{1}{4}\left(\int_{-1}^{1} x^3+x^2\,dx+\int_{1}^{3} 3x^2-x^3\,dx\right)=\frac{1}{4}\left(\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}+\left[x^3-\frac{x^4}{4}\right]_{1}^{3}\right)$$

$$=\frac{5}{3}$$

Therefore: $Var(X)=\frac{5}{3}-[1]^2=\frac{2}{3}$

f. (i) $kf(2y-1)=kf\left(2\left[y-\frac{1}{2}\right]\right)$.

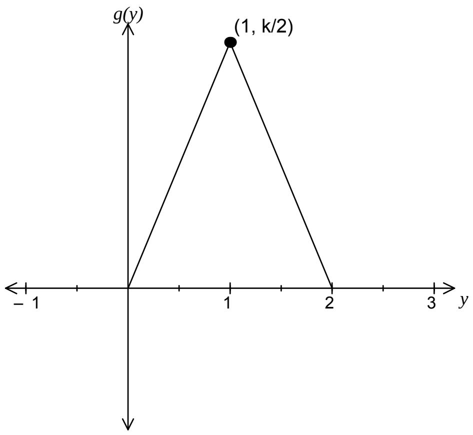
Therefore the graph of $g(y)$ can be obtained from the graph of $f(x)$ using the following transformations:

Dilation by a factor of $\frac{1}{2}$ from the y-axis,

Translation by $\frac{1}{2}$ units from the y-axis,

Dilation by a factor of k from the x-axis.

It follows that $\alpha=0$ and $\beta=2$.



(ii) The area under the graph is required to equal 1.

Using the formula for area of a triangle: $1 = \frac{1}{2}(2)\left(\frac{k}{2}\right) = \frac{k}{2}$

$\Rightarrow k = 2$

QUESTION 3

a. $\min = 20 - (4 \times 1) = 16^0 C$

$\max = 20 + (4 \times 1) = 24^0 C$

b. (i) Let $T_1 = 24^0 C$

$$20 - 4\cos\left(\frac{\pi(t-1)}{12}\right) = 24$$

$$-4\cos\left(\frac{\pi(t-1)}{12}\right) = 4$$

$$\cos\left(\frac{\pi(t-1)}{12}\right) = -1$$

$$\therefore \left(\frac{\pi(t-1)}{12}\right) = \pi$$

$$t - 1 = 12$$

$$t = 13 \quad \therefore 1pm$$

(ii) $T_1 = 24^0 C$ at $t = 13 \pm Period$

$$t = 13 \pm \frac{2\pi}{\frac{\pi}{12}}$$

$$t = 13 \pm (24hrs \times no.cycles)$$

$\therefore T_1 = 24^0 C$ at $t = 13 + 24n$, where n represents the number of cycles and $n \in Z^{+} U \{0\}$

i.e. At 1pm every day.

c. (i) $T_2 = A + Bsin(Ct + D)$

If $T_2 - T_1 = 0$ then $T_2 = T_1$

$$20 - 4\cos\left(\frac{\pi(t-1)}{12}\right) = A + B\sin(Ct + D)$$

Need to identify what transformations would be required to convert the cosine curve to a sine curve.

For curves to be equal, the amplitude, period and vertical translation must be the same.

$\therefore$ Sine curve must have: $Amp = 4$
$Vert\ Trans = 20$
Reflection in x axis

$\therefore$ Equation becomes $20 - 4\sin\left(\frac{\pi}{12}(t + c)\right) = A + B\sin(Ct + D)$

$\therefore A = 20$
$B = -4$
$C = \frac{\pi}{12}$

(ii) If c was equal to -1

i.e. $20 - 4\sin\left(\frac{\pi}{12}(t-1)\right)$, the first maximum would occur at $t = 19$ hours.

i.e. $20 - 4\sin\left(\frac{\pi}{12}(t-1)\right) = 24$

$$\sin\left(\frac{\pi}{12}(t-1)\right) = -1$$

$$\frac{\pi}{12}(t-1) = \frac{3\pi}{2}$$

t = 19 hours

Use this to identify the translation (horizontal) required to move the first sin max (at $t = 19$) to the maxima on the cos curve (at $t = 13$).

i.e. Need to move sin 6 units to **left.**

i.e. $(t - 1 + 6) = (t + 5)$

$\therefore$ Sine curve equation is: $20-4\sin\left(\frac{\pi}{12}(t+5)\right)$

Expand: $20-4\sin\left(\frac{\pi}{12}t+\frac{5\pi}{12}\right)$

Equate: $D=\frac{5\pi}{12}$

Alternatively:

Move sin curve
24-6 hours = 18 hours
to right

$\therefore (t-1-18)$

$\downarrow$

$(t-19)$

d. Let $T_3=T_2$ or $T_3=T_1$

$\downarrow$

Safer as both equations have been given and are error free.

$$\frac{8}{\sqrt{3}}\cos\left(\frac{\pi(t-1)}{12}\right)\sin\left(\frac{\pi(t-1)}{12}\right)+20=20-4\cos\left(\frac{\pi(t-1)}{12}\right)$$

$$\frac{8}{\sqrt{3}}\cos\left(\frac{\pi}{12}(t-1)\right)\sin\left(\frac{\pi}{12}(t-1)\right)=-4\cos\left(\frac{\pi}{12}(t-1)\right)$$

$$\frac{8}{\sqrt{3}}\cos\left(\frac{\pi}{12}(t-1)\right)\sin\left(\frac{\pi}{12}(t-1)\right)+4\cos\left(\frac{\pi}{12}(t-1)\right)=0$$

$$\cos\left(\frac{\pi}{12}(t-1)\right)\left\{\frac{8}{\sqrt{3}}\sin\left(\frac{\pi}{12}(t-1)\right)+4\right\}=0$$

Case 1: $\cos\left(\frac{\pi}{12}(t-1)\right)=0$

$\Rightarrow \frac{\pi}{12}(t-1)=\frac{\pi}{2},\frac{3\pi}{2}$

$\Rightarrow t-1=6,18 \Rightarrow t=7,19$

Case 2: $\frac{8}{\sqrt{3}}\sin\left(\frac{\pi}{12}(t-1)\right)+4=0$

$\sin\left(\frac{\pi}{12}(t-1)\right)=-\frac{\sqrt{3}}{2}$

$\Rightarrow \frac{\pi}{12}(t-1)=\frac{4\pi}{3},\frac{5\pi}{3}$

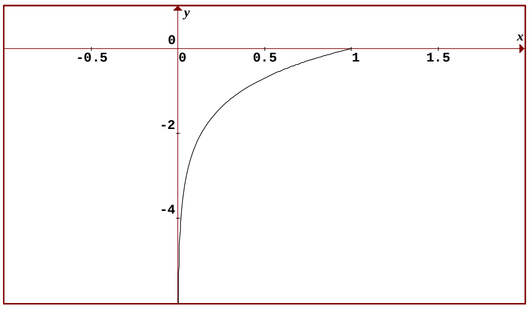
$\Rightarrow t-1=16, 20 \Rightarrow t=17, 21$

Therefore, the first two times are $t=7,17$.

e. (i) fog is defined if $r_g \le d_f$

$d_g = S$ $\qquad$ $d_f = (0,1]$

max $r_g = [-1,1]$ $\qquad$ $r_f = (-\infty, 0]$



$r_g \le d_f$

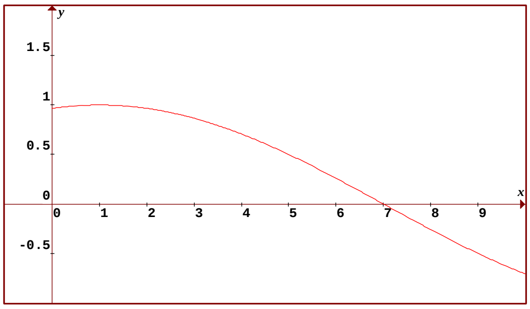
$(0,1] = (0,1]$

$\downarrow$

This is the max possible range of g.
Find the corresponding domain - S.

Let $\cos\left(\dfrac{\pi(t-1)}{12}\right) = 0$

$\therefore \dfrac{\pi(t-1)}{12} = \dfrac{\pi}{2}$



$t-1 = \dfrac{12\pi}{2\pi}$

$t-1=6$

$t=7$

$\therefore S = \{x : 0 \le t < 7\}$

(ii) $f[g(t)] = \log_e\left[\cos\left(\dfrac{\pi(t-1)}{12}\right)\right]$

$d_{fog} = d_g = [0,7)$

QUESTION 4

a. (i) When $x = 0, y = 5$

$\therefore 5 = ae^{\circ}$

$\therefore a = 5$

(ii) $y = ae^{kx} = 5e^{kx}$

Substitute $(25, 17.5)$:

$5e^{25k} = 17.5$

$$k = \frac{1}{25}\log_e\left(\frac{17.5}{5}\right)$$

$\therefore k \approx 0.050$

b. (i) $y_2 = \dfrac{b}{x+c}$

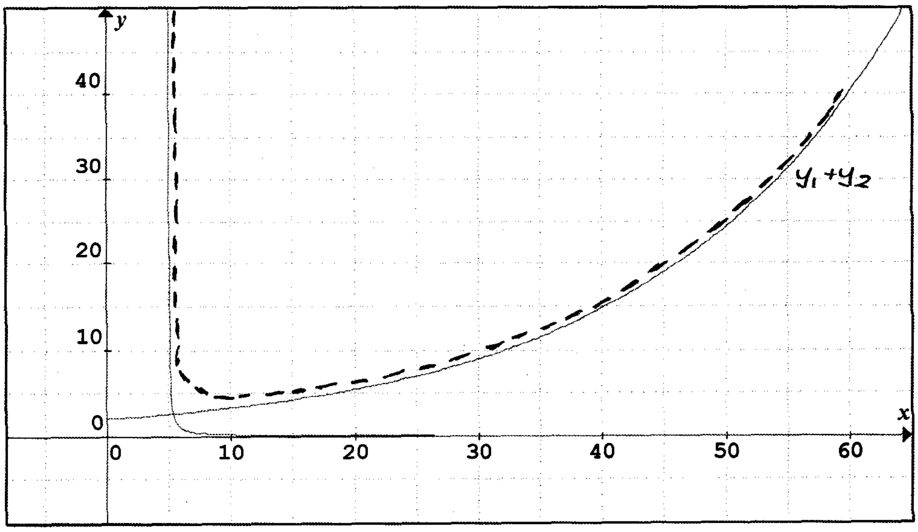
Let $x + c = 0 \rightarrow$ Equation asymptote

$x = -c$

$x = 5$

$\therefore c = -5$

(ii)



Can only add functions in common domain, therefore, domain: $5 < x\ (km/hr) \leq 60$

c. (i) $y = 5e^{0.05x} + \frac{b}{x-5}$

$= 5e^{0.05x} + b(x-5)^{-1}$

$\frac{dy}{dx} = 0.05 \times 5e^{+0.05x} - b(x-5)^{-2}$

$= 0.25e^{0.05x} - \frac{b}{(x-5)^2}$

(ii) For a local minimum:

At $x = 6$, gradient must be negative.
At $x = 14$, gradient must be positive.

At $x = 6$, $\frac{dy}{dx} < 0$:

Find $\frac{dy}{dx}$ at $x = 6$: $0.25e^{(0.05\times 6)} - \frac{b}{(6-5)^2}$

$= 0.3375 - b$

i.e. $0.3375 - b < 0$

$b > 0.3375$

At $x = 14$, $\frac{dy}{dx} > 0$:

Find $\frac{dy}{dx}$ at $x = 14$: $0.25e^{(0.05\times 14)} - \frac{b}{(14-5)^2}$

$= 0.5034 - \frac{b}{81}$

$0.5034 - \frac{b}{81} > 0$

$b < 40.778$

Minimum exists if $0.3375 < b < 40.778$

d. (i) $xy = c^2$

$\therefore y = \dfrac{c^2}{x} = c^2 x^{-1}$

$(x_1, y_1) = \left(cp, \dfrac{c}{p}\right)$

$m_p \rightarrow$ Find $\dfrac{dy}{dx}$ at $x = cp$

$\dfrac{dy}{dx} = -c^2 x^{-2} = \dfrac{-c^2}{x^2}$

At $x = cp$, $\dfrac{dy}{dx} = \dfrac{-c^2}{(cp)^2} = \dfrac{-c^2}{c^2 p^2} = \dfrac{-1}{p^2}$

Equation of the tangent: $y - y_1 = m(x - x_1)$

$y - \dfrac{c}{p} = \dfrac{-1}{p^2}(x - cp)$

$\dfrac{yp^2 - cp}{p^2} = \dfrac{-x + cp}{p^2}$

$\therefore yp^2 - cp = -x + cp$

$yp^2 + x = 2cp$

(ii) At $Q \rightarrow$ Same logic applies, but replace (x, y_1) with $\left(cq, \dfrac{c}{q}\right)$ and

$m = \dfrac{-c^2}{x^2} = \dfrac{-c^2}{(cq)^2} = \dfrac{-1}{q^2}$

$\therefore yq^2 + x = 2cq$

(iii) T represents the point of intersection of the two curves.

$$yp^2 + x = 2cp \qquad\qquad yq^2 + x = 2cq$$

$$yp^2 = 2cp - x \qquad\qquad yq^2 = 2cq - x$$

$$y = \frac{2cp - x}{p^2} \qquad\qquad y = \frac{2cq - x}{q^2}$$

Let $y = y$:

$$\frac{2cp - x}{p^2} = \frac{2cq - x}{q^2}$$

$$(2cp - x)q^2 = (2cq - x)^2\ p^2$$

$$2cpq^2 - xq^2 = 2cqp^2 - xp^2$$

$$xp^2 - xq^2 = 2cqp^2 - 2cpq^2$$

$$x\left(p^2 - q^2\right) = 2cpq(p - q)$$

$$x = \frac{2cpq(p - q)}{\left(p^2 - q^2\right)}$$

$$x = \frac{2cpq(p - q)}{(p - q)(p + q)} = \frac{2cpq}{(p + q)}$$

Substitute $x = \dfrac{2cpq}{p + q}$ into $y = \dfrac{2cp - x}{p^2}$:

$$\therefore y = \frac{2c}{p + q}$$