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Student Name……………………………………

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

TRIAL EXAMINATION 2

2009

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section 1 and Section 2.
Section 1 consists of 22 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 27 of this exam.
Section 2 consists of 4 extended-answer questions.
Section 1 begins on page 2 of this exam and is worth 22 marks.
Section 2 begins on page 11 of this exam and is worth 58 marks.
There is a total of 80 marks available.
All questions in Section 1 and Section 2 should be answered.
Diagrams in this exam are not to scale except where otherwise stated.
Where more than one mark is allocated to a question, appropriate working must be shown.
Students may bring one bound reference into the exam.
Students may bring a CAS calculator into the exam.
A formula sheet can be found on page 26 of this exam.

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SECTION 1

Question 1

The discrete random variable X has the following probability distribution.

x	-2	-1	0	1
$\Pr(X = x)$	0.3	0.2	0.4	0.1

The mean of this distribution is

A. -1.1
B. -0.9
C. $-0\ .7$
D. $-0\ 5$
E. -0.175

Question 2

Let $f:[0,a] \to R, f(x) = \sin(2x - \pi)$. If the inverse function f^{-1} exists then the largest value that a can take is

A. $\frac{\pi}{4}$
B. 1
C. $\frac{\pi}{2}$
D. 2
E. π

Question 3

If $y = |1 - x|$, then the rate of change of y with respect to x at $x = 2$ is

A. -2
B. -1
C. 0
D. 1
E. 2

Question 4

The simultaneous linear equations

$$5x+(a-3)y=1$$
$$ax+2y=a$$

where $a \in R$, will have no solutions for

A. $a=-2$
B. $a=5$
C. $a \in R\backslash\{-2\}$
D. $a \in R\backslash\{-2,5\}$
E. $a \in \{-2,5\}$

Question 5

At a particular school, ten per cent of Year 11 students have unpaid library fines. If 8 Year 11 students are selected at random, the probability that at least 2 of them have an unpaid library fine would be closest to

A. 0.1869
B. 0.2669
C. 0.3792
D. 0.7331
E. 0.8722

Question 6

The average value of the function $y=\frac{1}{x+2}-3$ over the interval $[1,3]$ is

A. $-\frac{1}{2}\log_e\left(\frac{3}{2}\right)+3$

B. $-\frac{1}{2}\log_e(3)+5$

C. $\frac{1}{2}\log_e\left(\frac{5}{3}\right)-3$

D. $\frac{1}{2}\log_e(3)-1$

E. $\frac{1}{2}\log_e(5)+1$

Question 7

If $\int_0^4 f(x) = 3$, then $\int_0^4 (2 - 5f(x))\, dx$ is equal to

A. -7
B. -9
C. -11
D. -13
E. -15

Question 8

Let $f: \left(\frac{\pi}{12}, \frac{2\pi}{3}\right) \to R, f(x) = 4\cos(2x)$.

The range of f is

A. $[-4, 2\sqrt{3})$
B. $[-4, 2\sqrt{3}]$
C. $(-2\sqrt{3}, 2\sqrt{3})$
D. $[-2, 2\sqrt{3})$
E. $(-2, 2\sqrt{3}]$

Question 9

The random variable X has a normal distribution with mean 10 and standard deviation 2. The probability that X is less than 5, given that it is less than 10, is closest to

A. 0.0062
B. 0.0124
C. 0.4969
D. 0.5
E. 0.9938

Question 10

For the graph of $y = x^4 - 5x^2 + 4$, the values of x for which the gradient of the graph is positive are closest to

A. $x \in (-1 \cdot 58, 0)$
B. $x \in (-1 \cdot 58, \infty)$
C. $x \in (-\infty, -1 \cdot 58) \cup (0, 1 \cdot 58)$
D. $x \in (-1 \cdot 58, 0) \cup (1 \cdot 58, \infty)$
E. $x \in (-\infty, 2) \cup (-1, 1) \cup (2, \infty)$

Question 11

The average rate of change of the function $f(x) = \log_e(2x+1)$ between $x = 1$ and $x = 2$ is

A. $\log_e\left(\frac{3}{5}\right)$

B. $\log_e\left(\frac{5}{3}\right)$

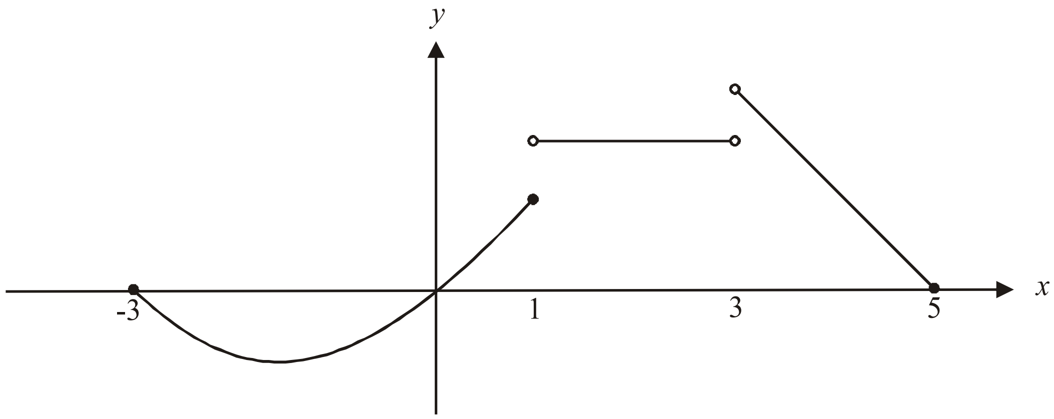
C. $\log_e(2)$

D. $\log_e(3)$

E. $\log_e(15)$

Question 12

The graph of the function h is shown below.



Which one of the following statements **is true** about the function h?

A. h is a continuous function for $x \in [-3,5]$

B. h exists for $x \in [-3,5]$

C. $h(x) > 0$ for $x \in [-3,5]$

D. $h(x)$ exists at $x = 1$

E. $h'(x)$ exists at $x = 1$

Question 13

The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

The image of the curve with equation $y = x^2 + 1$ is given by

A. $y = \dfrac{x^2+1}{4}$

B. $y = \dfrac{(x-2)^2}{4} - 1$

C. $y = \dfrac{1}{16}(2-x)^2 - 2$

D. $y = 4(2-x)^2 - 2$

E. $y = 4(2-x)^2 + 1$

Question 14

The volume of a spherical object is increasing at the rate of 5cm^3/hour.
The rate, in cm/hour, at which the radius of the sphere is increasing is given by

A. $\dfrac{15}{4\pi r^3}$

B. $\dfrac{5}{4\pi r^2}$

C. $\dfrac{1}{20\pi r^2}$

D. $\dfrac{500\pi}{3}$

E. $20\pi r^2$

Question 15

Let $g:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R, g(x)=\tan(2x)$.

The graph of $y=g(x)$ is transformed by a dilation by a factor of 3 from the x-axis followed by a reflection in the x-axis.

The resulting function h, is given by

A. $h:\left(-\frac{3\pi}{4},\frac{3\pi}{4}\right)\to R,\ h(x)=-3\tan(2x)$

B. $h:\left(-\frac{3\pi}{4},\frac{3\pi}{4}\right)\to R,\ h(x)=-\frac{1}{3}\tan(2x)$

C. $h:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R, h(x)=-3\tan(2x)$

D. $h:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R,\ h(x)=-\frac{1}{3}\tan(2x)$

E. $h:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R,\ h(x)=\frac{1}{3}\tan(2x)$

Question 16

If a random variable X has probability density function

$$f(x)=\begin{cases}\frac{x^2}{k} & x\in[0,1]\\ 0 & \text{otherwise}\end{cases}$$

then the value of k is

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. 1

E. 3

Question 17

If $f(x)=\log_e(3x)$ for $x \in R^+$ and $2f(x)=f(y)$ then y is equal to

A. 3
B. $3x$
C. $6x$
D. $3x^2$
E. $9x^2$

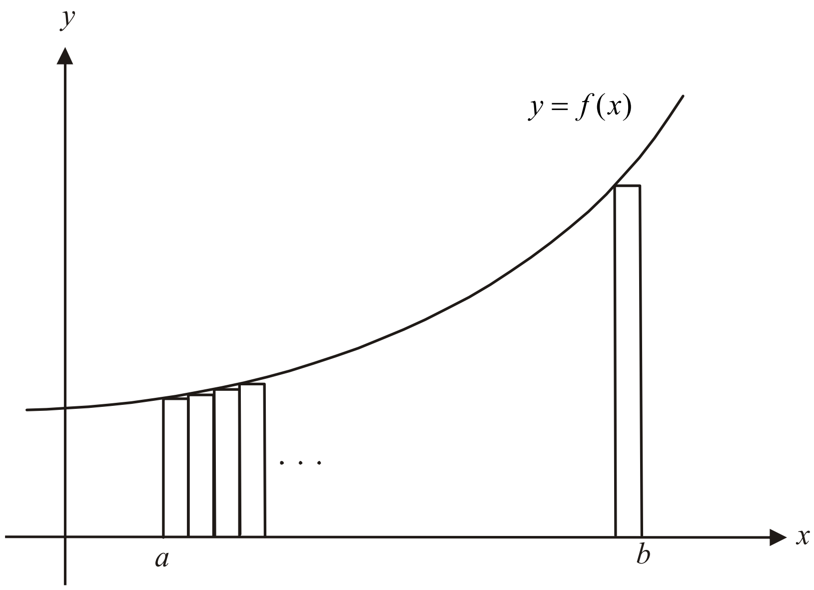
Question 18

The weights of the members of a junior swimming squad are normally distributed with a mean of 46kg and a standard deviation of 3.2kg. Ten percent of these children are not allowed to enter an endurance event because their body weight is too low.
The minimum weight; in kg, of a child permitted to enter the endurance event is closest to

A. 41.899
B. 42.121
C. 42.155
D. 42.800
E. 50.101

Question 19

An approximation to an area under a curve between $x=a$ and $x=b$ is to be found by summing the area of n rectangles of width h units that lie under the curve $y=f(x)$ between $x=a$ and $x=b$ as shown below.

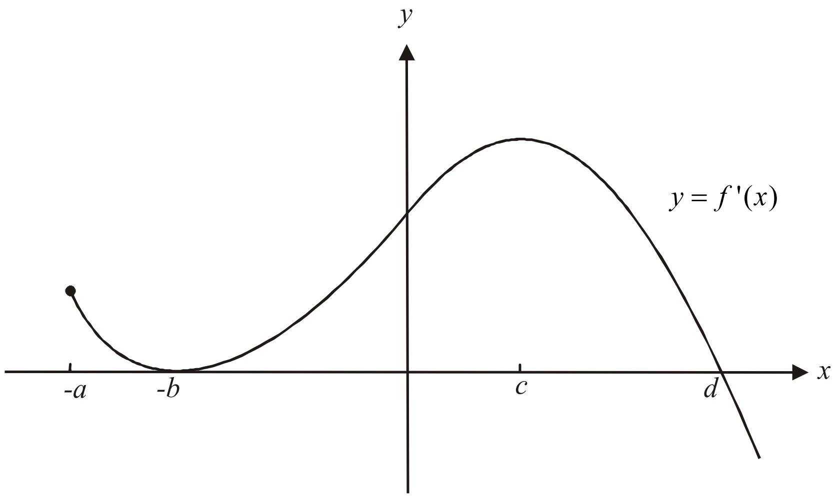


The approximation will be most accurate if the value of

A. n is small
B. a is small
C. b is small
D. h is small
E. $f(x)$ is small

Question 20

The graph of the derivative function f' for $x \in [-a, \infty)$ is shown below.



On the graph of the function *f*, a stationary point of inflection would occur at

A. $x = -a$
B. $x = -b$
C. $x = 0$
D. $x = c$
E. $x = d$

Question 21

The function *f* where $f(x) = \dfrac{1}{\sqrt{x-1}}$ has a maximal domain.

Let $g: (a, \infty) \to R, g(x) = x - 1$.

If $f(g(x))$ exists then *a* could equal

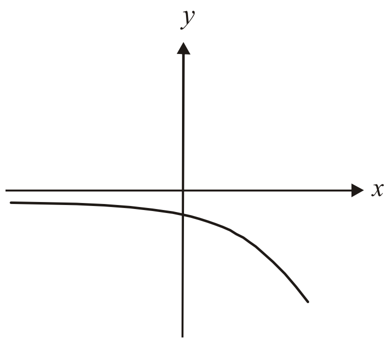
A. -2
B. -1
C. 0
D. 1
E. 2

Question 22

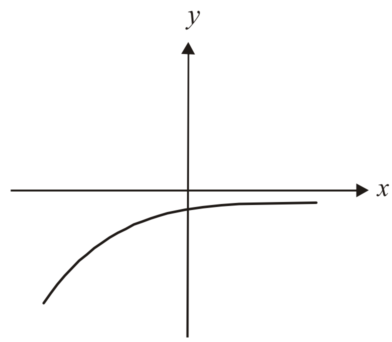
Let $y = ae^{bx} + c$, where $a < 0,\ b < 0$ and $c < 0$.

Which one of the following could show the graph of this function?

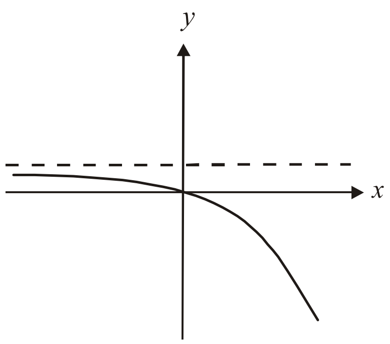
A.



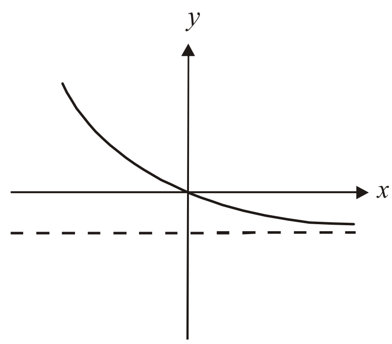
B.



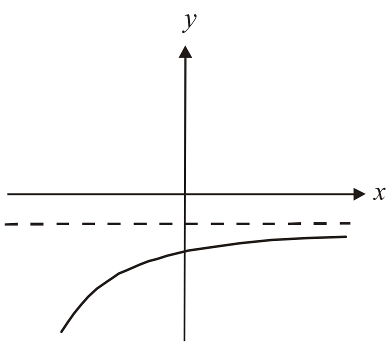
C.



D.



E.

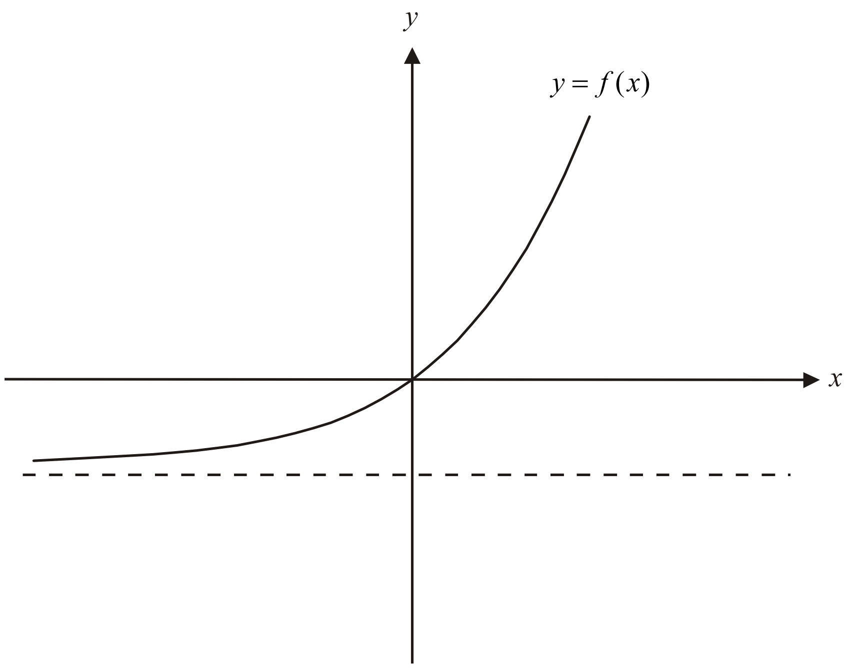


SECTION 2

Answer all questions in this section.

Question 1

The graph of the function $f: R \to R, f(x) = e^{ax} - b$ has a horizontal asymptote and passes through the origin.
The graph of $y = f(x)$ is shown below.



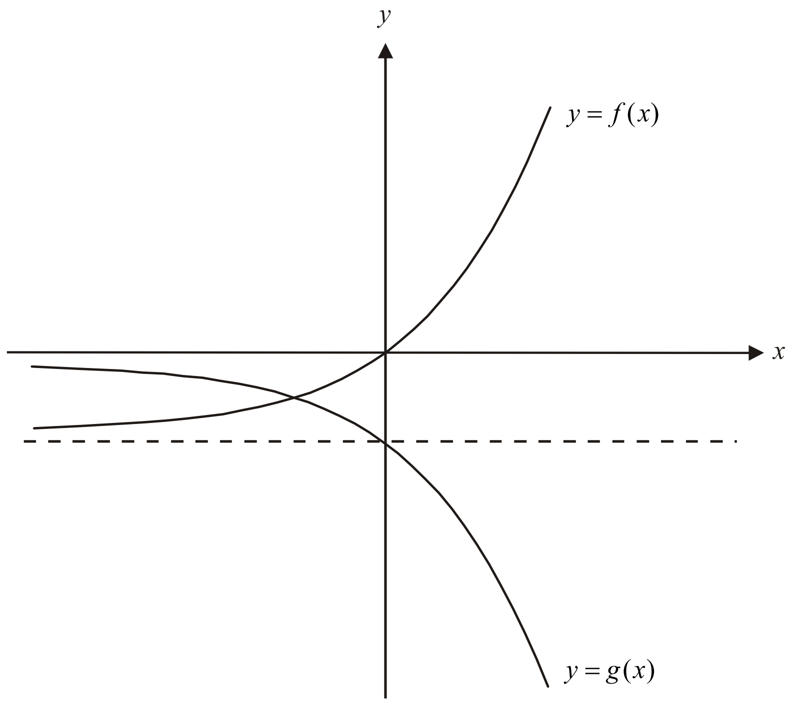
a. Show that $b = 1$.

1 mark

b. Given that $f'(1) = 2e^2$, show that $a = 2$.

2 marks

The graph of the function $y = f(x)$ undergoes two transformations. The graph of the original function $y = f(x)$, together with the graph of the final function $y = g(x)$, are shown below.

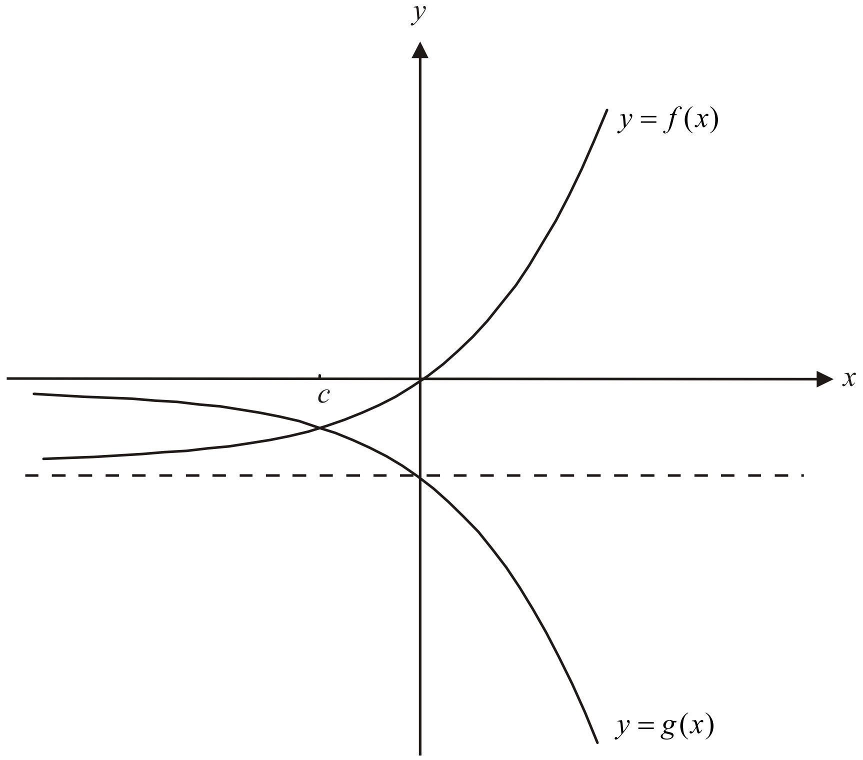


c. **i.** Describe the two transformations that the graph of $y = f(x)$ has undergone.

ii. Show that $g(x) = -e^{2x}$.

2 + 2 = 4 marks

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point where $x = c$ as shown below.



d. Show that $c = -\frac{1}{2}\log_e(2)$.

2 marks

e. Write down, **but do not evaluate**, an expression involving $g(x)$ and $f(x)$ that gives the area enclosed between the graphs of $y = f(x)$ and $y = g(x)$, and the lines $x = -1$ and $x = 0$.

2 marks

f. **i.** Show that $f\left(\frac{u+v}{2}\right)=e^{(u+v)}-1$ where u and v are real numbers.

ii. Hence show that $\left(f\left(\frac{u+v}{2}\right)\right)^2=f(u+v)-2f\left(\frac{u+v}{2}\right)$.

1 + 3 = 4 marks

Total 15 marks

Question 2

Rafael can either ride his bike to school or catch a bus. His decision as to how he gets to school one day is independent of his decision the next day.
Over time it works out that he rides his bike to school sixty percent of the time.

a. **i.** What is the probability; correct to four decimal places, that Rafael rides his bike to school on the next 5 consecutive days?

ii. What is the probability that Rafael rides his bike to school on exactly 2 of the next 5 days?

$1+2=3$ marks

Rafael's friend Jordan either walks to school or goes by car. If Jordan walks to school one day then the probability that he walks the next day is 0.4. If Jordan goes by car one day then the probability that he walks the next day is 0.3.
On Monday, the first day of term, Jordan walked to school.

b. **i.** What is the probability that Jordan walked to school on the next 4 days?

ii. What is the probability that Jordan walked to school on exactly 1 of the next 3 days?

iii. What is the probability, correct to four decimal places, that Jordan walked to school on the tenth day of term?

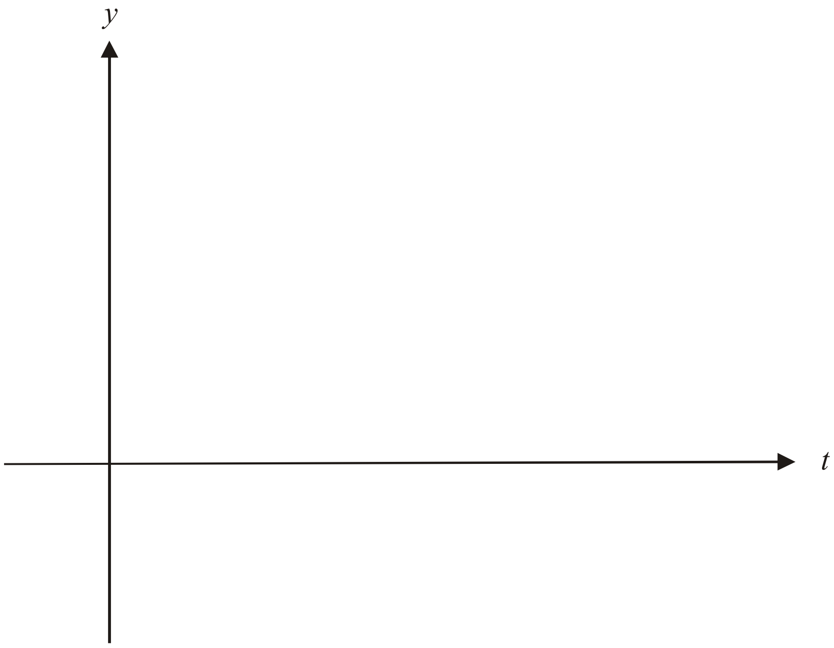
iv. What percentage of days will Jordan walk to school over the long term? Express your answer correct to 2 decimal places.

$1+3+3+1=8$ marks

When Rafael rides his bike to school, the time; t minutes, it takes him is a continuous random variable with a probability density function given by

$$f(t)=\begin{cases}\frac{1}{1000}(t+20), & \text{if } 20\le t\le 40\\ 0 & \text{otherwise}\end{cases}$$

c. **i.** Sketch the graph of the discontinuous function $y=f(t)$ on the set of axes below. Label endpoints appropriately.



ii. What is the mode of this distribution?

____

iii. Find the median time, to the nearest minute, that Rafael takes to ride to school.

____

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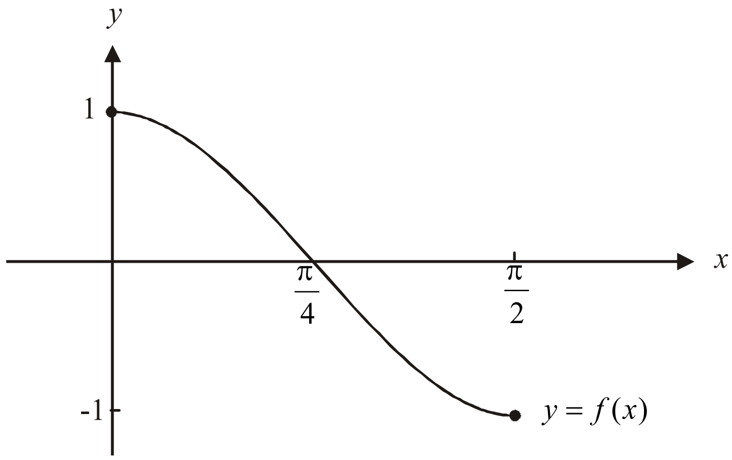
$2+1+3=6$ marks
Total 17 marks

Question 3

The graph of the function $f:\left[0,\frac{\pi}{2}\right]\to R, f(x)=\cos(2x)$ is shown below.



a. Find $f'(x)$.

1 mark

b. Find the coordinates of the point(s) where the gradient of the tangent to the graph of $y=f(x)$ is – 1. Express the coordinates as exact values.

2 marks

A tangent to the graph of $y = f(x)$ has an x-intercept of $\frac{\pi + \sqrt{3}}{6}$ and a y-intercept of $\frac{\sqrt{3}\pi + 3}{6}$.

c. **i.** Show that the gradient of this tangent is $-\sqrt{3}$.

ii. Using your answer to part **a.** find the values of x where the function f has a gradient of $-\sqrt{3}$.

iii. Hence find the coordinates of the point of tangency.

1 + 2 + 2 = 5 marks

d. Find the maximum and minimum values of $|f(x)|$.

2 marks

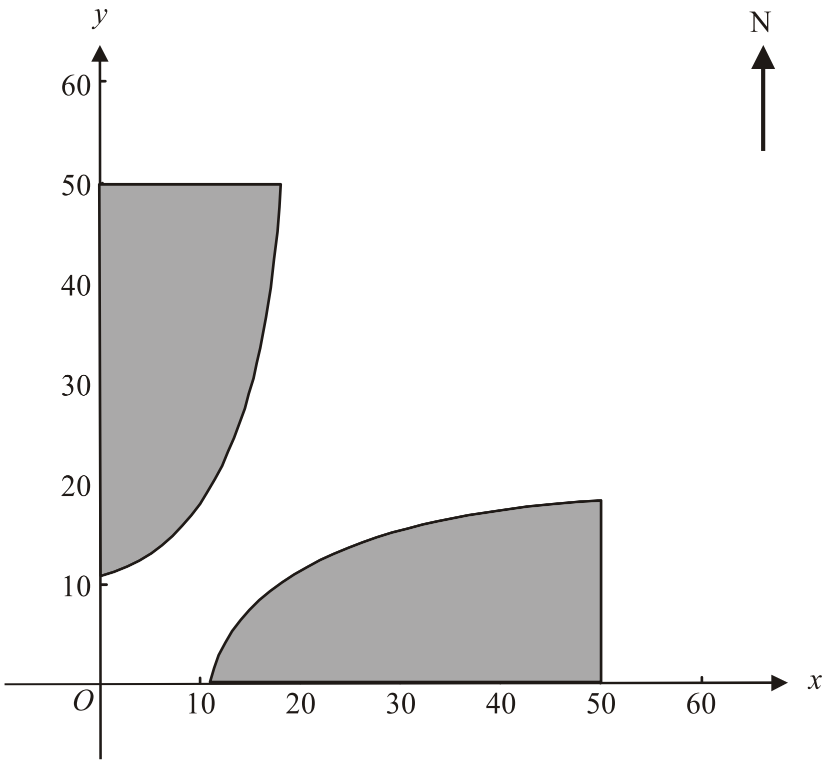
e. Let $g: R \to R, g(x) = \cos(2x)$.
Find the general solution for x of the equation $g(x) = 0 \cdot 5$.

2 marks

Total 12 marks

Question 4

Victoria James is a spy. She is attempting to flee enemy territory and begins her escape at the point $O(0,0)$ shown on the diagram below.



The x-axis runs in an east-west direction. Part of the ground she must run through is floodlit and this floodlit area is shaded in the diagram above.

The floodlit area to the east of $O(0,0)$ is enclosed by the x-axis, the line $x = 50$ and the function

$$f:[a,50] \to R,\ f(x) = 5\log_e(x-10).$$

The unit of measurement is the metre.

a. Show that $a = 11$

1 mark

The floodlit area to the north of $O(0,0)$ is enclosed by the y-axis, the line $y = 50$ and the graph of the function f^{-1}; the inverse function of f.

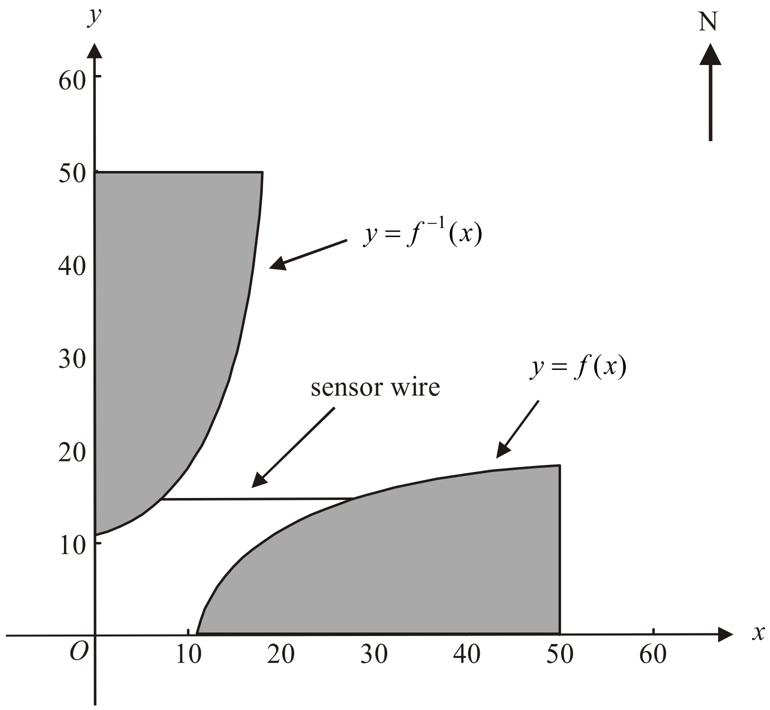
b. Find the rule and the domain of f^{-1}. Express values correct to 1 decimal place where appropriate.

2 marks

c. Find the total area; to the nearest square metre, of the ground that is floodlit.

2 marks

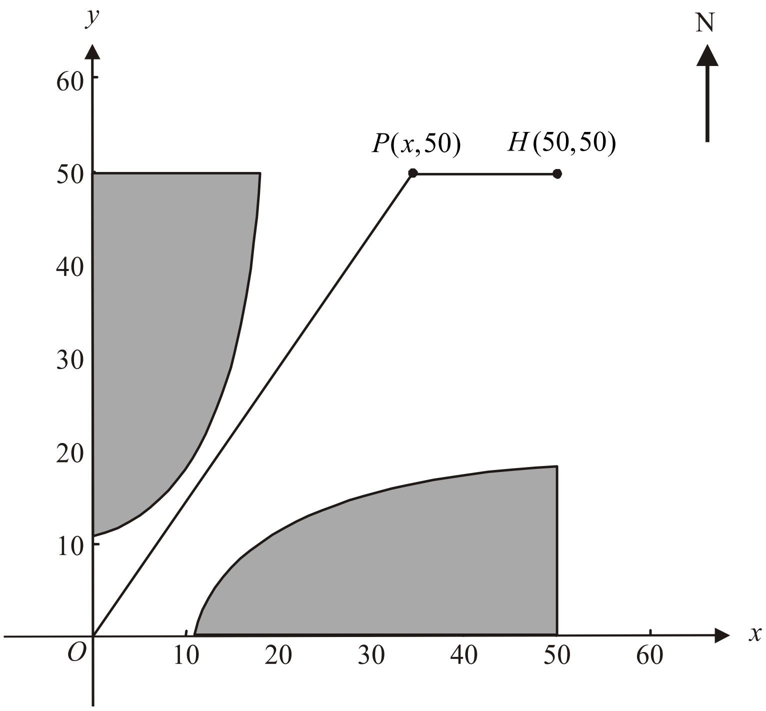
Victoria moves in a straight line from $O(0,0)$ and without knowing, passes over a sensor wire that runs in an east-west direction along the line $y = 15$ in the area that is **not** floodlit.
The point where she passes over the sensor wire is given by $(b,15)$.



d. Given that Victoria can move through the floodlit area if necessary, find the possible values of b correct to 2 decimal places where appropriate.

2 marks

Victoria continues to move in a straight line from her starting point at $O(0,0)$ until she is at a point $P(x,50)$. From this point she moves due east to a waiting helicopter at $H(50,50)$.



The time *T*, in seconds, taken by Victoria to move from *O* to *H* via *P* is given by

$$T = \sqrt{x^2 + 2500} + \frac{50 - x}{2}, \; x \in [0,50].$$

e. **i.** Find the value of *x*, correct to 2 decimal places, for which Victoria reaches the helicopter in the minimum time.

___

___

___

ii. Hence find the minimum value of *T* correct to 1 decimal place.

___

___

$2 + 1 = 3$ marks

Assume that Victoria moves along the path that takes the minimum time.

f. Explain whether or not taking this path will mean Victoria has to pass through any floodlit area.

2 marks

One minute after Victoria escapes from the point $O(0,0)$, guard dogs are released from the same point. They run in a straight line at 7 m/s towards the helicopter at $H(50,50)$.

Assume that Victoria will take the minimum time possible to reach the helicopter at *H* having passed through point *P,* and that the helicopter will take off at the instant she reaches it.

g. Explain whether or not Victoria escapes the dogs.

2 marks
Total 14 marks

Mathematical Methods and Mathematical Methods CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$

$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$

$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$

$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$

$\int \frac{1}{x}\,dx=\log_e|x|+c$

$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$

$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$

$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$

$\int \cos\,(ax)\,dx=\frac{1}{a}\sin(ax)+c$

$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$

product rule: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h)\approx f(x)+hf'(x)$

Probability

$\Pr(A)=1-\Pr(A')$

$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$

$\Pr(A/B)=\frac{\Pr(A\cap B)}{\Pr(B)}$

mean: $\mu=E(X)$

variance: $\mathrm{var}(X)=\sigma^2=E((X-\mu)^2)=E(X^2)-\mu^2$

probability distribution		mean	variance
discrete	$\Pr(X=x)=p(x)$	$\mu=\Sigma x\,p(x)$	$\sigma^2=\Sigma(x-\mu)^2\,p(x)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty}x\,f(x)dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2\,f(x)\,dx$

MATHEMATICAL METHODS (CAS)

TRIAL EXAMINATION 2

MULTIPLE- CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ■ C D E

The answer selected is B. Only one answer should be selected.

1.	A	B	C	D	E	12.	A	B	C	D	E
2.	A	B	C	D	E	13.	A	B	C	D	E
3.	A	B	C	D	E	14.	A	B	C	D	E
4.	A	B	C	D	E	15.	A	B	C	D	E
5.	A	B	C	D	E	16.	A	B	C	D	E
6.	A	B	C	D	E	17.	A	B	C	D	E
7.	A	B	C	D	E	18.	A	B	C	D	E
8.	A	B	C	D	E	19.	A	B	C	D	E
9.	A	B	C	D	E	20.	A	B	C	D	E
10.	A	B	C	D	E	21.	A	B	C	D	E
11.	A	B	C	D	E	22.	A	B	C	D	E