INSIGHT
Trial Exam Paper

2009
MATHEMATICAL METHODS
Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

SECTION 1

Question 1

The function $f:[0,2\pi]\to R,\ f(x)=1-3\sin(2x-\frac{\pi}{2})$ has a range and period respectively of

A. $[0,3]$ and 2

B. $[-1,3]$ and 2π

C. **$[-2,4]$ and π**

D. $[-3,3]$ and π

E. $[-2,4]$ and 2π

Answer is C

Solution

- The lowest value of the function is $1-3=-2$.
- The highest value of the function is $1+3=4$.
- The period of the function is given by $\frac{2\pi}{n}=\frac{2\pi}{2}=\pi$.

Question 2

The range of the function $f(x)=e^{|x|}-3$ is

A. **$[-2,\infty)$**

B. R^+

C. $(-2,0]$

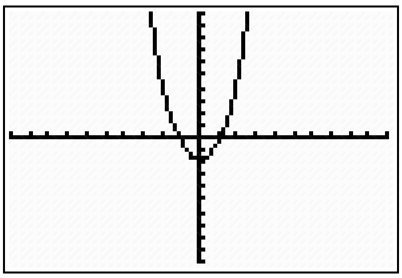
D. $(-2,0)$

E. $(-3,\infty)$

Answer is A

Solution

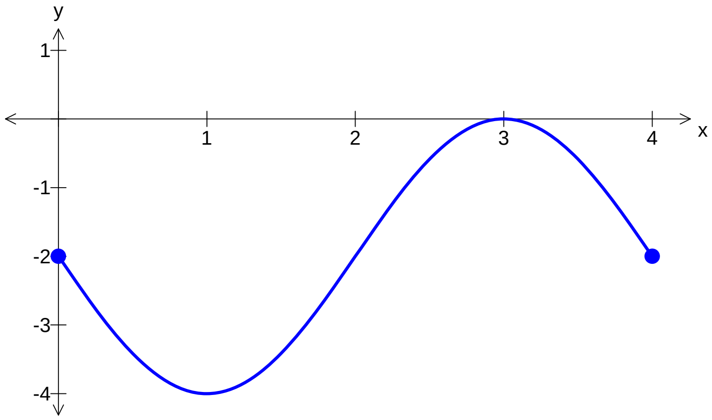
The graph of the function looks like—



The lowest value of the function occurs at the y-intercept (0, -2), so the range is $[-2,\infty)$.

Question 3

The diagram below shows one cycle of the graph of a circular function.



A possible equation for the function whose graph is shown is

A. $y = 2 - 2\sin(4x)$

B. $y = -2 - 2\sin(4x)$

C. $y = 4 - 2\sin(4\pi x)$

D. $\boldsymbol{y = -2 - 2\sin(\frac{\pi}{2}x)}$

E. $y = -2 - 2\sin(\frac{1}{2}x)$

Answer is D

Solution

The graph shows an upside down sin curve with an amplitude of 2, a centre value of –2 and a period of 4 units (i.e. $4 = \frac{2\pi}{n}$, so $n = \frac{\pi}{2}$).

Question 4

One cycle of the graph of the function with the equation $y = \tan(ax)$ has successive vertical asymptotes at $x = \frac{3}{8}$ and $x = \frac{5}{8}$.

A possible value for a is

A. 2

B. 2π

C. 4

D. $\boldsymbol{4\pi}$

E. 8

Answer is D

Solution

The difference between the asymptotes is $\frac{2}{8} = \frac{1}{4}$ units, meaning the period is $\frac{1}{4}$. The period of a tan function, $y = \tan(bx)$ is given by $\frac{\pi}{b}$ so $a = 4\pi$ units.

Question 5

$\log_e\left(\frac{3}{e^{5x}}\right)$ is equal to

A. $\frac{3}{5x}$

B. $\frac{\log_e(3)}{5x}$

C. $\log_e(3) - \log_e(5x)$

D. $3 - \log_e(5x)$

E. $\boldsymbol{\log_e(3) - 5x}$

Answer is E

Solution

Using the log laws we get

$$\begin{aligned}\log_e\left(\frac{3}{e^{5x}}\right) &= \log_e(3) - \log_e(e^{5x}) \\ &= \log_e(3) - 5x\log_e(e) \\ &= \log_e(3) - 5x\end{aligned}$$

Question 6

If $e^x = 1 + 6e^{-x}$, then x is equal to

A. $3, -2$

B. $\mathbf{\log_e 3}$

C. $\log_e 3, -\log_e 2$

D. $-\log_e 3,\ \log_e 2$

E. $-3,\ 2$

Answer is B

Solution

The equation is a trinomial and needs to be firstly written in the usual form.

$e^x = 1 + 6e^{-x}$

so $e^x - 1 - 6e^{-x} = 0$

multiply through by e^{x} gives

$e^{2x} - e^x - 6 = 0$

let $a = e^x$ gives

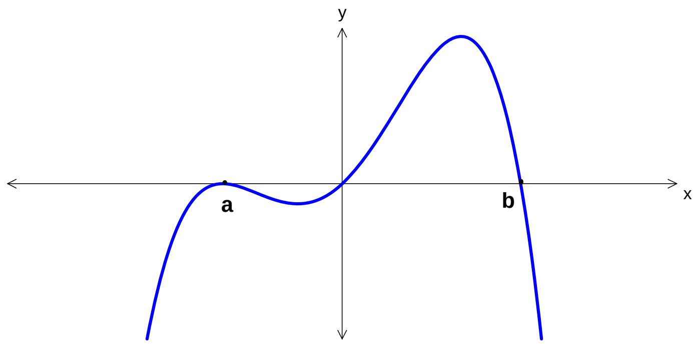
$a^2 - a - 6 = 0$

so $(a-3)(a+2) = 0$

which gives $a = 3$ *or* $a = -2$

so $x = \log_e 3$ is the only solution.

Question 7



The graph shown could be that of a function f with the equation

A. $y = -x(x+a)^2(x-b)$

B. $y = -x(x+a)(x-b)$

C. $y = x(x-a)^2(x-b)$

D. $y = -x(x+a)^2(b-x)$

E. $\boldsymbol{y = x(x-a)^2(b-x)}$

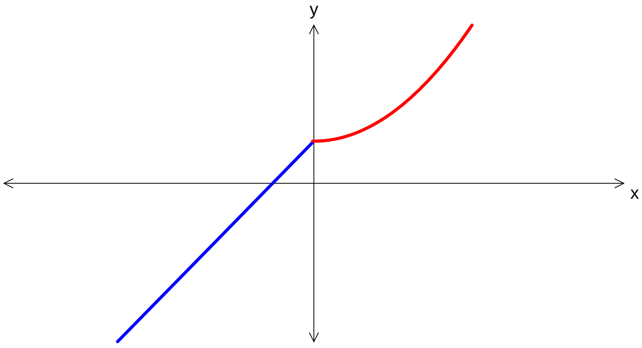
Answer is E

Solution

- As both ends of the graph point in the same direction (in this case, down) this means the overall power of the equation is an even power, so alternative B is out.
- Both ends point down so it has to be a $-x^{even}$ so alternatives C and D are out.
- The graph has x-intercepts at $(a,0)$ *and* $(b,0)$ so this means that the equation must have factors of $(x-a)$ *and* $(x-b)$ $(or\ (b-x))$ — alternative E has this.

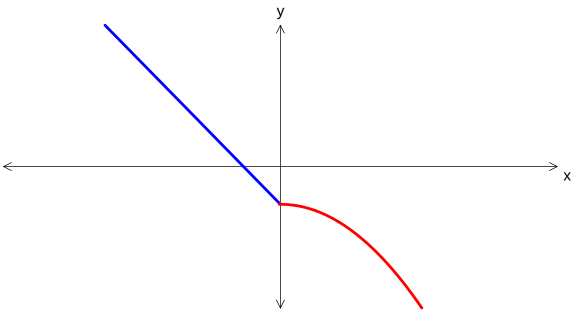
Question 8

The graph of the function with the equation $y = f(x)$ is shown below.

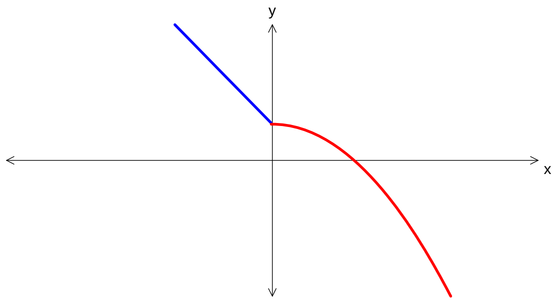


Which of the following is most likely to be the graph of the inverse function?

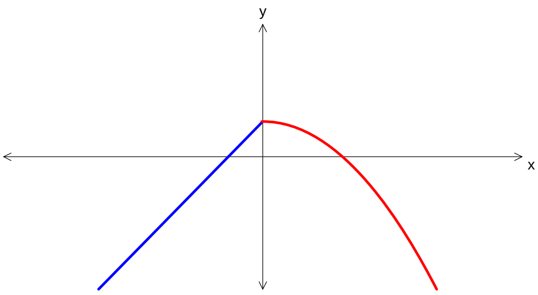
A.



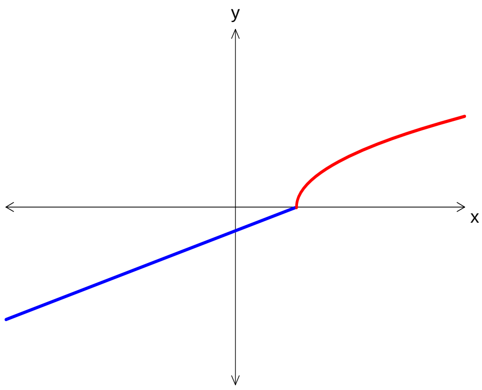
B.



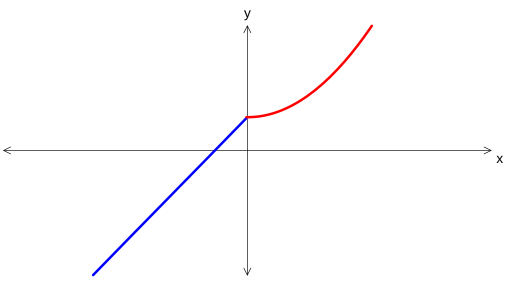
C.



D.



E.



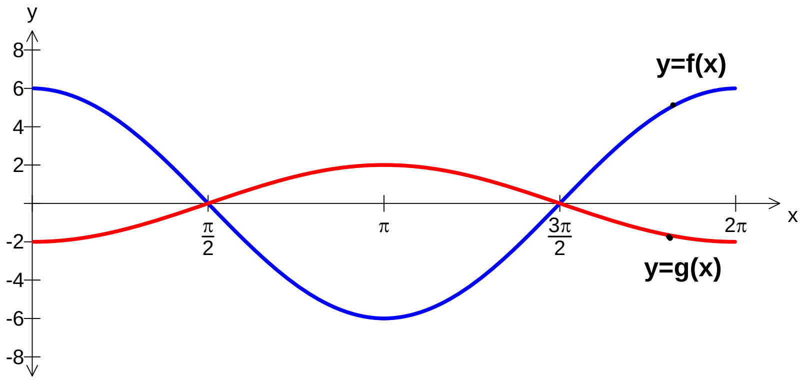
Answer is D

Solution

The inverse function is the reflection of the graph of the function in the line $y = x$.

Question 9

The diagram below shows the graphs of two circular functions, *f* and *g*.



The graph of the function with the equation $y = f(x)$ is transformed into the graph of the function with the equation $y = g(x)$ by

A. a dilation by a scale factor of $\frac{1}{3}$ from the *y*- axis followed by a reflection in the *x*-axis.

B. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *x*-axis.

C. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *y*-axis.

D. **a dilation by a scale factor of $\frac{1}{3}$ from the x- axis followed by a reflection in the x-axis.**

E. a dilation by a scale factor of 3 from the *x*- axis followed by a reflection in the *y*-axis.

Answer is D

Solution

Graph has been reflected in x-axis and the amplitude has decreased from 6 to 2, i.e. by a factor of $\frac{1}{3}$ in the y-direction or from the x-axis.

Question 10

The function defined by $f : A \to R,\ f(x) = e^{(x-b)^2}$, $b \in R$, will have an inverse function for all values of *b,* if its domain *A* is

A. R

B. $R \setminus \{b\}$

C. $\boldsymbol{[b, \infty)}$

D. R^+

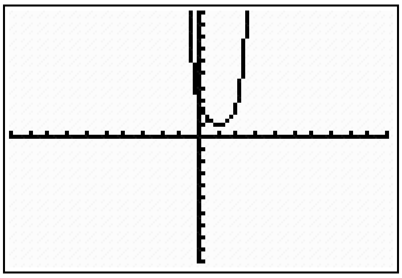
E. $[-b, \infty)$

Answer is C

Solution

To have an inverse function, the function needs to be one-to-one.

A graph of one member of the family of curves with $b = 1$, shows



with the graph being symmetrical about $x = 1$, so the graph is one-to-one for either $(-\infty, b]\ or\ [b, \infty)$ – therefore the answer is alternative C.

Question 11

If $f(x) = e^{2x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ then $g(f(1))$ is

A. $\boldsymbol{\dfrac{1}{e}}$

B. $-e$

C. $\sqrt{e}$

D. 1

E. -1

Answer is A

Solution

The composite function $g(f(x))$ is formed by replacing the x term in $g(x)$ with the function of $f(x)$. So $g(f(x)) = \dfrac{1}{\sqrt{e^{2x}}}$, therefore $g(f(1)) = \dfrac{1}{\sqrt{e^2}} = \dfrac{1}{e}$.

Question 12

If $y = \log_e(\sin(3x))$ then $\dfrac{dy}{dx}$ is equal to

A. $\boldsymbol{\dfrac{3\cos(3x)}{\sin(3x)}}$

B. $\dfrac{1}{\sin(3x)}$

C. $\dfrac{1}{3\cos(3x)}$

D. $3\tan(3x)$

E. $-3\tan(3x)$

Answer is A

Solution

Using the chain rule $\dfrac{d}{dx}(\log_e(\sin(3x))) = \dfrac{1}{\sin(3x)} \times 3\cos(3x) = \dfrac{3\cos(3x)}{\sin(3x)}$.

Question 13

The equation of the normal to the curve with equation $y = x\cos(x)$, at the point on the curve with *x*-coordinate 2π is

A. $y = \dfrac{1}{x\sin(x)}$

B. $y = -x$

C. $y = x$

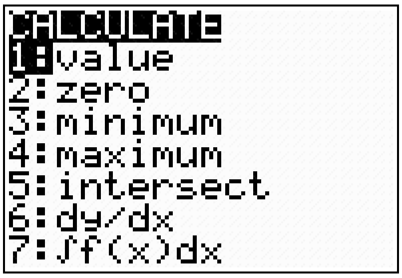
D. $\boldsymbol{y = -x + 4\pi}$

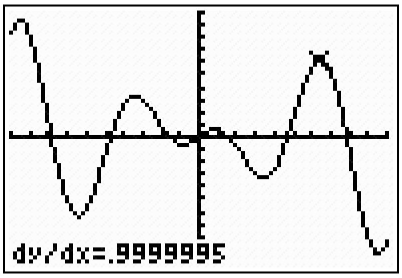
E. $y = x - 4\pi$

Answer is D

Solution

The gradient of the tangent can be found by finding $\dfrac{dy}{dx}$ at $x = 2\pi$





so $\dfrac{dy}{dx} = 1$, which means the gradient of the normal is -1, only alternatives B and D are valid at this stage. The normal must also pass through the point on the curve when $x = 2\pi$. At $x = 2\pi$, $y = 2\pi$ therefore $y = -x + 4\pi$ meets this requirement, so the answer is D.

Question 14

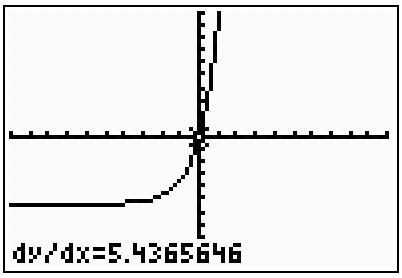
If $y = 2e(e^x - 1)$ then the rate of change of y with respect to x when $x = 0$ is

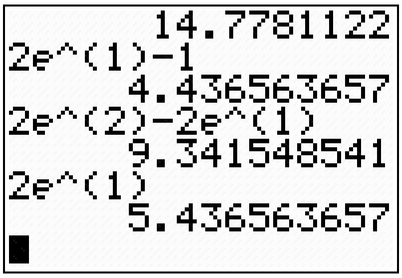
A. $2e^2$
B. 0
C. $2e - 1$
D. $2e^2 - 2e$
E. $\mathbf{2e}$

Answer is E

Solution

The rate of change means $\frac{dy}{dx}$. We can find the approximate rate of change from the calculator using the $\frac{dy}{dx}$ from the calc menu and then match up the decimal approximations of the alternatives.





The decimal version of alternative E is equal to $\frac{dy}{dx}$.

Question 15

Using the approximation formula, $f(x+h) \approx f(x) + hf'(x)$ where $f(x) = x^3$ with $x = 4$, an approximate value for 4.3^3 is given by

A. $\mathbf{f(4) + 0.3f'(4)}$
B. $f(4) + 0.0027f'(4)$
C. $f(64) + 0.3f'(64)$
D. $f(64) + 0.0027f'(64)$
E. $f'(4.3)$

Answer is A

Solution

Using the approximation formula $x = 4$ and $h = 0.3$.

Question 16

If $f'(x) = 5e^{2x}$ and c is a real constant, then $f(x)$ is equal to

A. $\mathbf{\frac{5}{2}e^{2x} + c}$

B. $10e^{2x} + c$

C. $\frac{5}{2}e^{3x} + c$

D. $\frac{5}{2}e^{x^2} + c$

E. $5e^{x^2} + c$

Answer is A

Solution

The antiderivative of e^{ax} is $\frac{1}{a}e^{ax}$. So the antiderivative of $5e^{2x}$ is $\frac{5}{2}e^{2x}$+c.

Question 17

Let p be a function defined on the interval $[3, 5]$ and q a function such that $q'(x) = p(x)$, for all $x \in [3, 5]$

$\int_3^5 p(x)\,dx$ is equal to

A. $q(x) + c$

B. $p(5) - p(3)$

C. ***q (5) – q (3)***

D. $q'(5) - q'(3)$

E. $p(x) + c$

Answer is C

Solution

Need to replace $p(x)$ with $q'(x)$ to get

$$\int_3^5 p(x)\ dx = \int_3^5 q'(x)\ dx$$

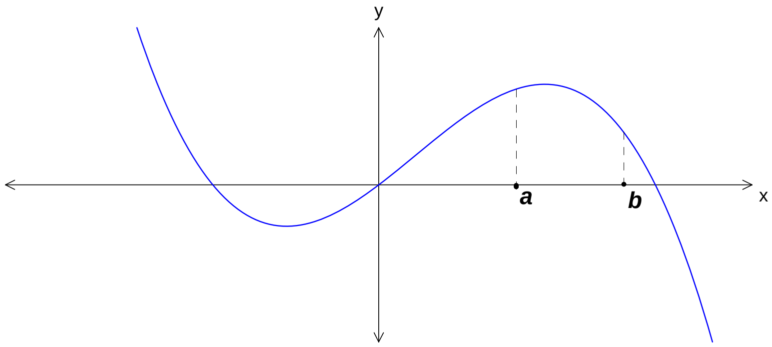
$$= \left[q(x)\right]_3^5$$

$$= q(5) - q(3)$$

SECTION 1 – continued
TURN OVER

Question 18

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$.

On the interval (a, b), the graph of g will have a

A. maximum turning point.
B. minimum turning point.
C. negative gradient.
D. **positive gradient.**
E. stationary point of inflection.

Answer is D

Solution

The graph drawn is the graph of the derivative, i.e. the graph of the gradient function. The graph has a positive value over this domain and does not have an x-intercept or change signs (therefore no turning points and stationary points of inflexion over this interval).

Question 19

The number of defective batteries in a box of batteries ready for sale is a random variable with a binomial distribution with mean 8.1 and standard deviation 0.9.

If a battery is drawn at random from the box, the probability that it is not defective is

A. **0.1**
B. 0.9
C. 0.09
D. 0.3
E. 0.7

Answer is A

Solution

The distribution is binomial so the mean is given by $\mu = np$ and the standard deviation is $\sigma = \sqrt{npq}$ so

so

$8.1 = np$

$0.9 = \sqrt{npq} \quad \Rightarrow 0.81 = npq$

$\Rightarrow 0.81 = 8.1q$ replacing np with 8.1

$\Rightarrow q = 0.1$ and as $p = 1 - q,\ p = 0.9$

so the probability of a defective battery is 0.9 and a non-defective battery 0.1.

Question 20

Let X be a normally distributed random variable with mean μ and standard deviation σ.
Which one of the following is not always true?

A. $\Pr(X > \mu) = 0.5$

B. $\Pr(X > a) = 1 - \Pr(X < a)$

C. $\Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68$

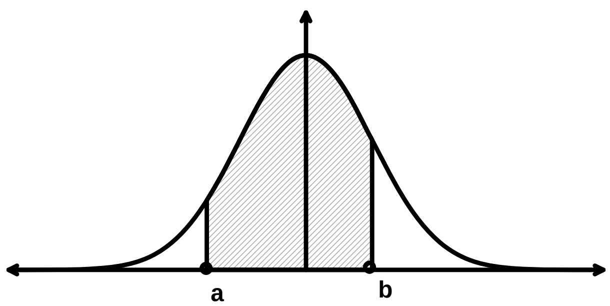
D. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

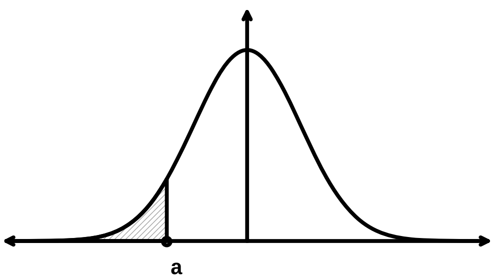
E. $\mathbf{Pr(}\boldsymbol{a < X < b}\mathbf{) = Pr(}\boldsymbol{X > b}\mathbf{) - Pr(}\boldsymbol{X > a}\mathbf{)}$

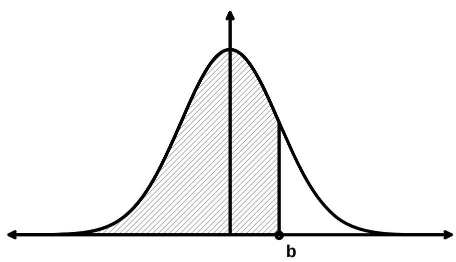
Answer is E

Solution

$\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a)$ and this can be seen from a series of normal distribution diagrams.







SECTION 1 – continued
TURN OVER

Question 21

The random variable X has the following probability distribution.

x	1	2	3
$\Pr(X=x)$	a	b	0.1

If the mean of X is 1.3, then the values of a and b respectively are—

A. 0.5, 0.4

B. **0.8, 0.1**

C. 0.1, 0.8

D. 0.3, 0.7

E. 0.4, 0.5

Answer is B

Solution

Since the distribution is a pdf the sum of the probabilities is 1.

i.e.,

$a+b+0.1=1$

$a+b=0.9$

The mean of the pdf

$E(X)=\sum xp(x)=1\times a+2\times b+3\times 0.1=a+2b+0.3=1.3$

$so\ \ a+2b=1$

solving simultaneously gives $b=0.1$ *and* $a=0.8$.

Question 22

Juicy Giant orange juice is packed in small glass bottles labelled as containing 250 *ml*. The packing process produces bottles that are normally distributed with a standard deviation of 3 *ml*. In order to guarantee that only 1% of bottles are under-volume, the actual mean volume, in *ml*, would be required to be closest to

A. 256

B. **257**

C. 258

D. 250.01

E. 247

Answer is B

Solution

$X \sim N(\mu, \sigma = 3)$

$\Pr(X < 250) = 0.01$

converting to standard normal gives

$\Pr(Z < \frac{250-\mu}{3}) = 0.01.$

Using a program called ALLPROBX or similarly gives

$\frac{250-\mu}{3} = -2.326$

$250 - \mu = -6.978$

$\mu = 256.978.$

END OF SECTION 1
TURN OVER

SECTION 2

Question 1

Grace shops each day at one of two supermarkets, Costless and Spendway. Her choice of supermarket each day depends only on which supermarket she has shopped at on the previous day. If she shops at Costless one day then the chance of her shopping at Costless the next day is $\frac{3}{5}$ while if she shops at Spendway one day then the chance of her shopping at Spendway the next day is $\frac{2}{3}$.

a. If she shops at Spendway one day, what is the chance she shops at Costless the next day?

1 mark

Solution

The information states that the $\Pr(S\ then\ S) = \frac{2}{3}$ so the $\Pr(S\ then\ C) = \frac{1}{3}$.

Mark allocation

- 1 mark for answer of $\frac{1}{3}$.

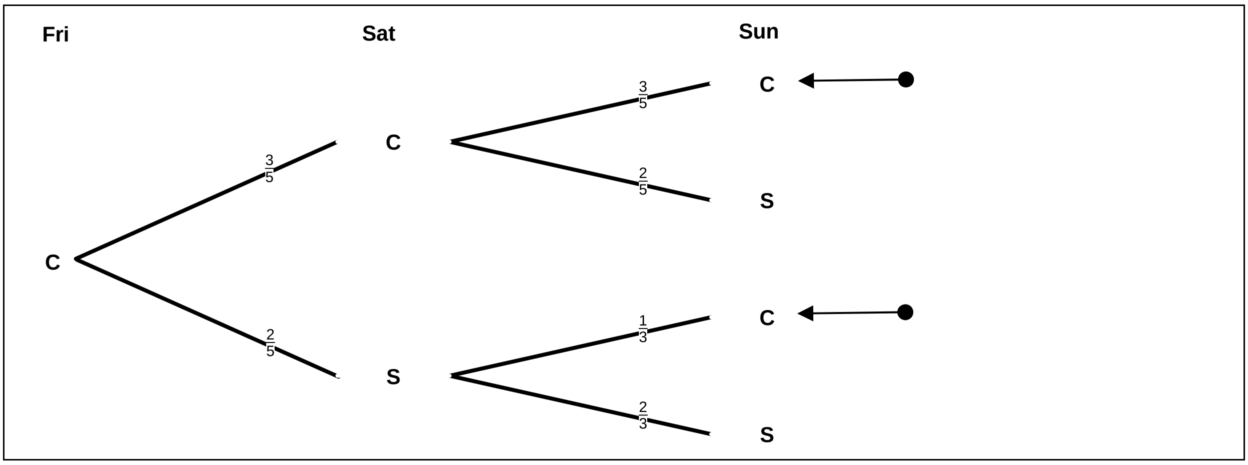
b. Suppose she shops at Costless one Friday.

i. What is the exact probability she shops at Costless the next Sunday?

3 marks

Solution

Using a tree diagram gives



so the probability of shopping at Costless next Sunday comes from two of the branches $\Pr(CCC)$ *and* $\Pr(CSC)$.

$$\Pr(shopping\ at\ Costless) = \frac{3}{5}\times\frac{3}{5}+\frac{2}{5}\times\frac{1}{3} = \frac{37}{75}.$$

Mark allocation

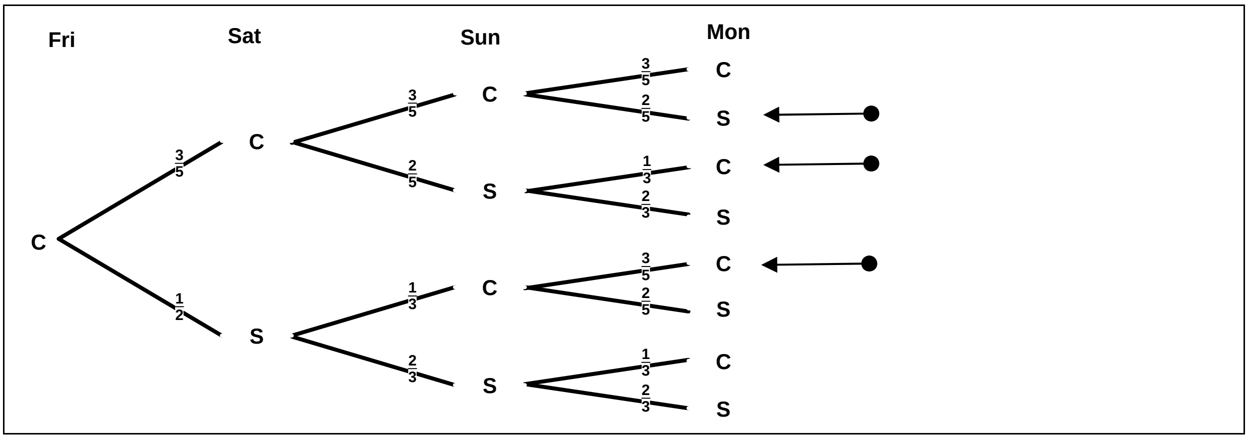
- 1 method mark for drawing a tree diagram.
- 1 answer mark for identifying the correct branches.
- 1 answer mark for correct answer.

ii. What is the exact probability she shops at Costless on exactly two of the next three days?

2 marks

Solution

Extend the tree diagram to give three days in total.



$$\Pr(2\ days\ at\ Costless) = \Pr(CCCS) + \Pr(CCSC) + \Pr(CSCC)$$
$$= \frac{3}{5}\times\frac{3}{5}\times\frac{2}{5}+\frac{3}{5}\times\frac{2}{5}\times\frac{1}{3}+\frac{2}{5}\times\frac{1}{3}\times\frac{3}{5}$$
$$= \frac{38}{125}$$

Mark allocation

- 1 method mark for extending the tree diagram.
- 1 mark for answer.

The time, t in minutes, she spends in the supermarket is independent of the type of supermarket she shops at and is a random variable with probability density function

$$f(t) = \begin{cases} at(60-t) & if \;\; 0 \le t \le 60 \\ 0 & otherwise \end{cases}$$

c. Show that $a = \dfrac{1}{36000}$.

2 marks

Solution

To be a probability density function the area under the graph needs to equal 1.

i.e. $\displaystyle\int_0^{60} at(60-t)\,dt = 1$

so set

$$\int_0^{60} at(60-t)\,dt = 1$$

$$\begin{aligned} LHS = \int_0^{60} at(60-t)\,dt &= a\int_0^{60} t(60-t)\,dt \\ &= a\int_0^{60} (60t-t^2)\,dt \\ &= a\left[\frac{60t^2}{2} - \frac{t^3}{3}\right]_0^{60} \\ &= a\left(\frac{60(60^2)}{2} - \frac{60^3}{3}\right) \\ &= a \times 36000 \end{aligned}$$

so

$$36000a = 1$$

$$\Rightarrow a = \frac{1}{36000}$$

Mark allocation

- 1 method mark for setting the integral with correct terminals and dt equal to 1.
- 1 answer mark for correctly antidifferentiating, leading to the given answer.

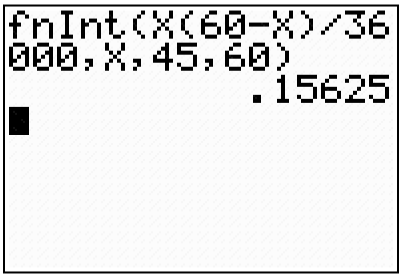
d. What is the probability, correct to 3 decimal places, that Grace spends longer than 45 minutes shopping in the supermarket?

2 marks

Solution

$$\Pr(\textit{more than } 45 \text{ mins}) = \int_{45}^{60} \frac{1}{36000} t(60-t)\ dt$$

using the calculator this integral equals $0.15625 = 0.156$ correct to 3 decimal places.



Mark allocation

- 1 method mark for setting up the integral (terminals correct and dt).
- 1 answer mark for correct answer.

e. It is known that on one particular day Grace spends longer than 45 minutes shopping in the supermarket. What is the probability, correct to 3 decimal places, that she spends less than 50 minutes shopping in the supermarket?

2 marks

Solution

This is a conditional probability question. We want

$$\Pr(\textit{less than } 50 \text{ mins, } \textit{given more than } 45 \text{ mins}) = \frac{\Pr(T < 50 \cap T > 45)}{\Pr(X > 45)}$$

$$= \frac{\Pr(45 < T < 50)}{\Pr(T > 45)}$$

$$\Pr(45 < T < 50) = \int_{45}^{50} \frac{1}{36000} t(60-t)\ dt = 0.082176$$

Found using the fInt facility again on the calculator.

So $\Pr(T < 50 \mid T > 45) = \dfrac{0.082176}{0.15625} = 0.5259 = 0.526$

Mark allocation

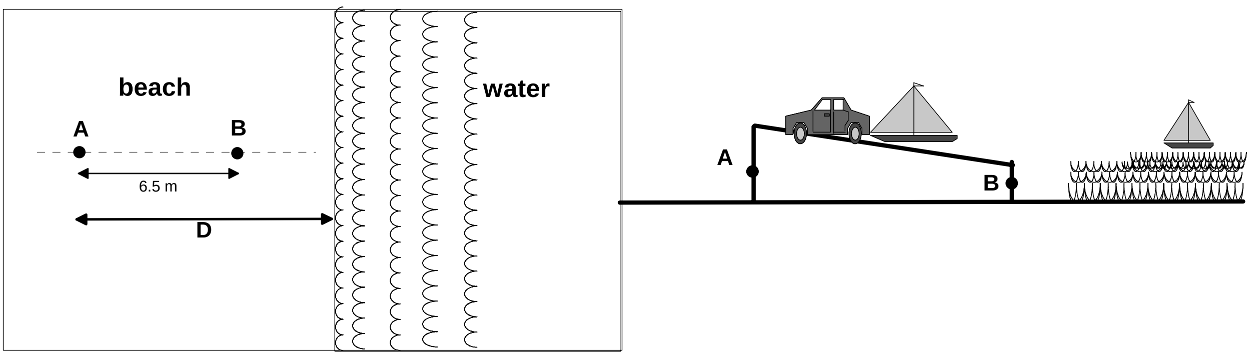
- 1 method mark for recognising the conditional probability and interpreting correctly.
- 1 answer mark for correct answer to 3 decimal places (need to work to more than 3 decimal places and then round off).

Total 12 marks

Question 2

Keith likes to holiday at Point Roadknight beach where the waves roll on to the beach at regular intervals.

The diagrams below show the beach with two markers, *A* and *B* at the start and end of the boat ramp, where the base of marker *A* is 6.5 metres further up the beach than the base of marker *B*. The line *AB* is perpendicular to the water's edge.

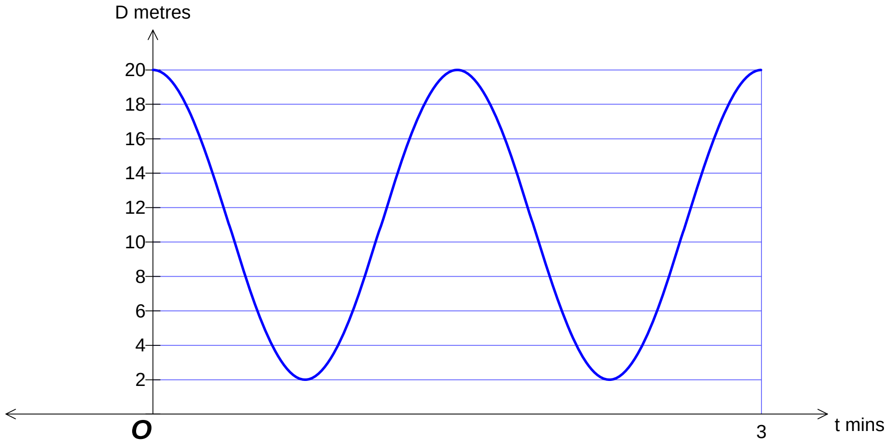


View from above **Side view**

Keith records the horizontal distance of the water's edge from the top marker *A*. He calculates that on one particular day the distance *D* metres of the water's edge from the top marker *A* is a function of time *t* (in minutes from when he starts to observe the waves). It can be modelled exactly by the equation

$$D = a\cos(bt) + c$$

where a, b and c are positive constants. The graph of *D* as a function of time *t* is shown below:



a. State the maximum and minimum distances of the water's edge from marker *A* on this day.

2 marks

Solution

The maximum distance from A is 20 metres and the minimum 2 metres.

Mark allocation

- 1 answer mark for maximum.
- 1 answer mark for minimum.

b. Find the number of waves that hit the beach in one hour on this day.

2 marks

Solution

There are 2 waves in 3 minutes, so in

$$\begin{aligned} 60 \text{ min}s &= 20 \times 3 \text{mins} \\ &= 20 \times 2 \ \textit{waves} \\ &= 40 \ \textit{waves}. \end{aligned}$$

Mark allocation

- 1 method mark for a recognizing that there are 2 waves in 3 minutes or calculating the period.
- 1 mark for answer.

c. Find the values of *a*, *b* and *c*.

3 marks

Solution

Graph is centred around 11, so $c = 11$.

The amplitude is 9 units so $a = 9$.

There are 2 waves in 3 minutes so the period is $\frac{3}{2} = \frac{2\pi}{b}$, so $b = \frac{4\pi}{3}$.

Mark allocation

- 1 answer mark for a.
- 1 answer mark for b.
- 1 answer mark for c.

d. **i.** Write down an equation, the solution of which gives the times when the marker at *B* is in the water on this day.

1 mark

Solution

Marker *B* is in the water when the water is less than 6.5 metres from *A*, i.e. when $D < 6.5$, so $9\cos\frac{4\pi t}{3} + 11 < 6.5$ is the equation that gives the times when marker *B* is in the water.

Mark allocation

- 1 answer mark for answer.

ii. Find the exact percentage of time that the marker at B is in the water in one cycle.

3 marks

Solution

$9\cos\frac{4\pi t}{3}+11<6.5$

$9\cos(\frac{4\pi t}{3})<-4.5$

$\cos(\frac{4\pi t}{3})<\frac{-4.5}{9}$

$\cos(\frac{4\pi t}{3})<\frac{-1}{2}$

reference angle is $\frac{\pi}{3}$ and the required angle is in the second and third quadrant so

$\frac{2\pi}{3}$ *and* $\frac{4\pi}{3}$. This gives

$\frac{4\pi t}{3}=\frac{2\pi}{3},\frac{4\pi}{3}$

$t=\frac{1}{2}$, *or* 1

So the marker is in the water for between $\frac{1}{2}<t<1$ minutes, or for $\frac{1}{2}$ minutes in total.

$\frac{1}{2}$ minutes each cycle means $\frac{\frac{1}{2}}{\frac{3}{2}}\times 100\%=33\frac{1}{3}\%$.

Mark allocation

- 1 method mark for finding reference angle of $\frac{\pi}{3}$.
- 1 method mark for finding $\frac{1}{2}<t<1$.
- 1 answer mark for correct percentage.

Due to winds, tides and currents, on some days the waves come further up the beach and are closer together. Keith observes that on such a day the distance of the water's edge from marker A can be described by the equation

$D=(8+S)\cos(\frac{8\pi t}{3})+11$ where S metres is the seasonal tidal factor which varies with the factors described above. S is normally distributed with a mean of 2 and a standard deviation of 0.3.

e. On a particular day the waves just reach the top of the boat ramp at marker A. Find the value of S on this day.

2 marks

Solution

To just reach the marker at A, $D = 0$ so this means the amplitude $8 + S = 11$ which gives $S = 3$.

Mark allocation

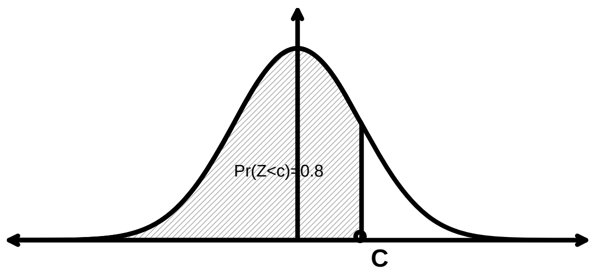
- 1 method mark for recognising amplitude equals 11.
- 1 answer mark for correct answer.

f. Marcus decides to go fishing from the boat ramp at Point Roadknight beach, however he does not know the values of S. How many metres up the boat ramp from marker B should he stand so that he has an 80% chance of not getting wet from the waves? Answer correct to two decimal places.

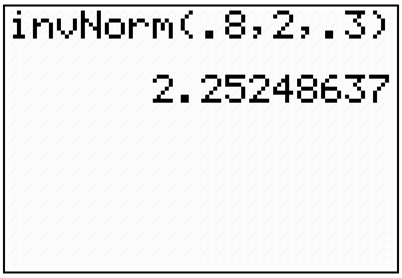
3 marks

Solution

$\mu = 2 \text{ and } \sigma = 0.3$



Using $invnorm(0.8, 2, 0.3)$ gives $S = 2.2525$



with $S = 2.2525$ the water goes to within 0.7475 metres of marker A, so Marcus should position himself 5.7525 = 5.75 metres up from marker B.

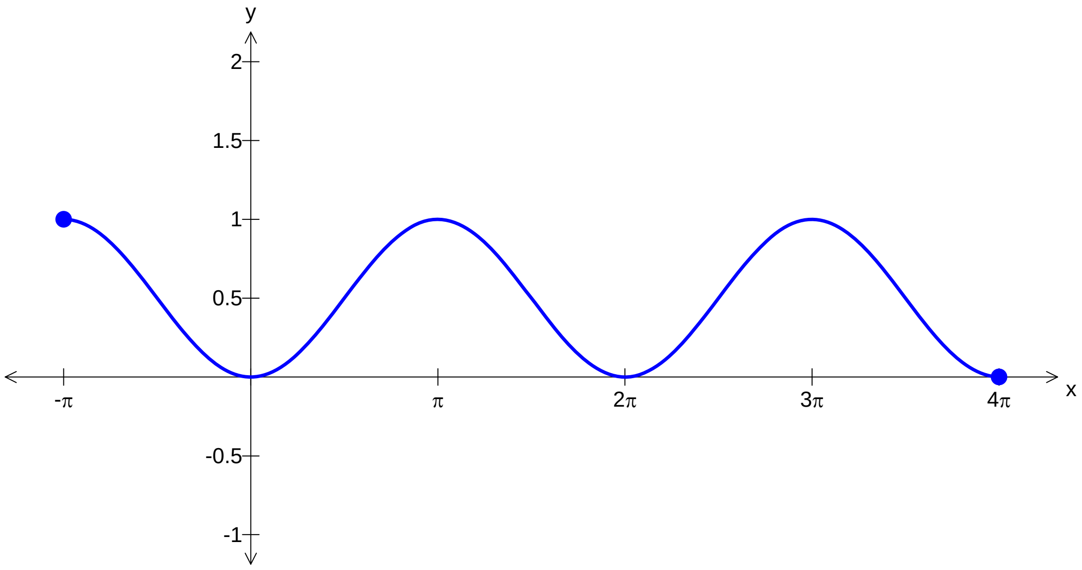
Mark allocation

- 1 method mark for using invnorm.
- 1 answer mark for 0.7475 m from marker A.
- 1 answer mark for 5.75 m from marker B.

Total 16 marks

Question 3

The graph of $f:[-\pi,4\pi]\to R,\ f(x)=\sin^2\left(\dfrac{x}{2}\right)$ is shown below.



a. **i.** Use calculus to find $f'(\frac{3\pi}{2})$ and $f'(\frac{5\pi}{2})$.

3 marks

Solution

Using the chain to differentiate

$f(x)=\sin^2\left(\dfrac{x}{2}\right)$ *gives* $f'(x)=2\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)\times\dfrac{1}{2}=\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)$

At $x=\dfrac{3\pi}{2},\ f'(x)=\sin\left(\dfrac{3\pi}{4}\right)\cos\left(\dfrac{3\pi}{4}\right)=\dfrac{1}{\sqrt{2}}\times\dfrac{-1}{\sqrt{2}}=\dfrac{-1}{2}$

and similarly for $x=\dfrac{5\pi}{2},\ f'(x)=\dfrac{1}{2}$.

Mark allocation

- 1 answer mark for correct derivative.
- 1 answer mark for $f'(\frac{3\pi}{2})$.
- 1 answer mark for $f'(\frac{5\pi}{2})$.

ii. Find the exact equations of the **normals** to the curve at the points where $x = \frac{3\pi}{2}$ and $x = \frac{5\pi}{2}$.

4 marks

Solution

At $x = \frac{3\pi}{2}$, $f'(x) = \frac{-1}{2}$ so the gradient of the normal is 2.

At $x = \frac{3\pi}{2}$, $f(x) = \frac{1}{2}$, so using $y = mx + c$ to find the equation of the normal gives

$$y = 2x + c$$
$$\frac{1}{2} = 2 \times \frac{3\pi}{2} + c$$
$$c = -3\pi + \frac{1}{2}$$
$$\Rightarrow y = 2x - 3\pi + \frac{1}{2}.$$

Similarly for $x = \frac{5\pi}{2}$, with $f'(x) = \frac{1}{2}$, *so* $m = -2$ *and* $f(\frac{5\pi}{2}) = \frac{1}{2}$.

Using

$$y = -2x + c$$
$$\frac{1}{2} = -2 \times \frac{5\pi}{2} + c$$
$$c = 5\pi + \frac{1}{2}$$
$$\Rightarrow y = -2x + 5\pi + \frac{1}{2}$$

Mark allocation

- 1 mark for finding the gradient of both normals.
- 1 mark for finding $f(\frac{3\pi}{2}) = \frac{1}{2}$ and $f(\frac{5\pi}{2}) = \frac{1}{2}$.
- 1 mark for finding equation of normal for $x = \frac{3\pi}{2}$.
- 1 mark equation of normal for $x = \frac{5\pi}{2}$.

b. The graph of $y = f(x)$ is transformed to give the graph of $y = f(x) + b$. Find the value of b, correct to 3 decimal places, such that the graph of the normal at $x = \frac{3\pi}{2}$ and the graph of the normal at $x = \frac{5\pi}{2}$ intersect on the *x*-axis.

3 marks

Solution

Need to find the point of intersection of the two normals.

Using simultaneous equations, a program or graphically, find the point of intersection.

$$\Rightarrow y = -2x + 5\pi + \frac{1}{2}$$

$$\Rightarrow y = 2x - 3\pi + \frac{1}{2}$$

adding the two equations gives

$$2y = 2\pi + 1$$

$$y = \pi + \frac{1}{2} = 3.642$$

If the graph is moved down 3.642 units then the normals will intersect on the *x*-axis, so $b = -3.642$.

Mark allocation

- 1 method mark for a valid attempt at finding point of intersection.
- 1 answer mark for correct point of intersection.
- 1 answer mark for value of *b*.

Let $g:[-\pi, 4\pi] \to R,\ g(x) = \sin^2\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) + 1$.

c. **i.** Use calculus to find $g'(x)$.

2 marks

Solution

Using the chain rule gives

$$g'(x) = \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) \times \frac{1}{2}$$

$$= \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(\frac{x}{2}\right)$$

Mark allocation

- 1 method mark for using chain rule.
- 1 answer mark for correct derivative.

ii. Solve the equation $g'(x)=0$ for $x \in [-\pi,\ 4\pi]$. Give exact values.

3 marks

Solution

Let $g'(x)=0$ gives

$$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)-\frac{1}{2}\sin\left(\frac{x}{2}\right)=0$$

$$\sin\left(\frac{x}{2}\right)\left(\cos\left(\frac{x}{2}\right)-\frac{1}{2}\right)=0$$

$$\sin\left(\frac{x}{2}\right)=0 \qquad or \qquad \left(\cos\left(\frac{x}{2}\right)+\frac{1}{2}\right)=0$$

$$\left(\frac{x}{2}\right)=0,\pi,\ 2\pi,\ or\ \ \cos\left(\frac{x}{2}\right)=\frac{1}{2}$$

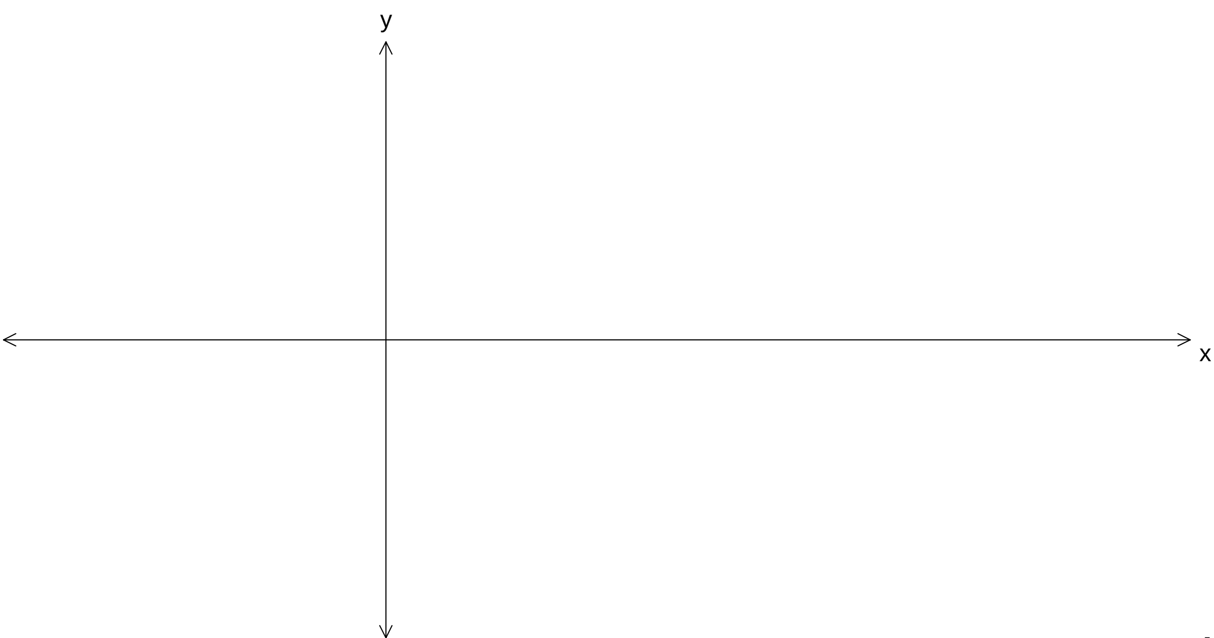
$$x=0,\ 2\pi,4\pi \qquad or \qquad \frac{x}{2}=\frac{\pi}{3},\frac{5\pi}{3},\frac{-\pi}{3}$$

$$x=\frac{2\pi}{3},\frac{10\pi}{3},\frac{-2\pi}{3}$$

Mark allocation

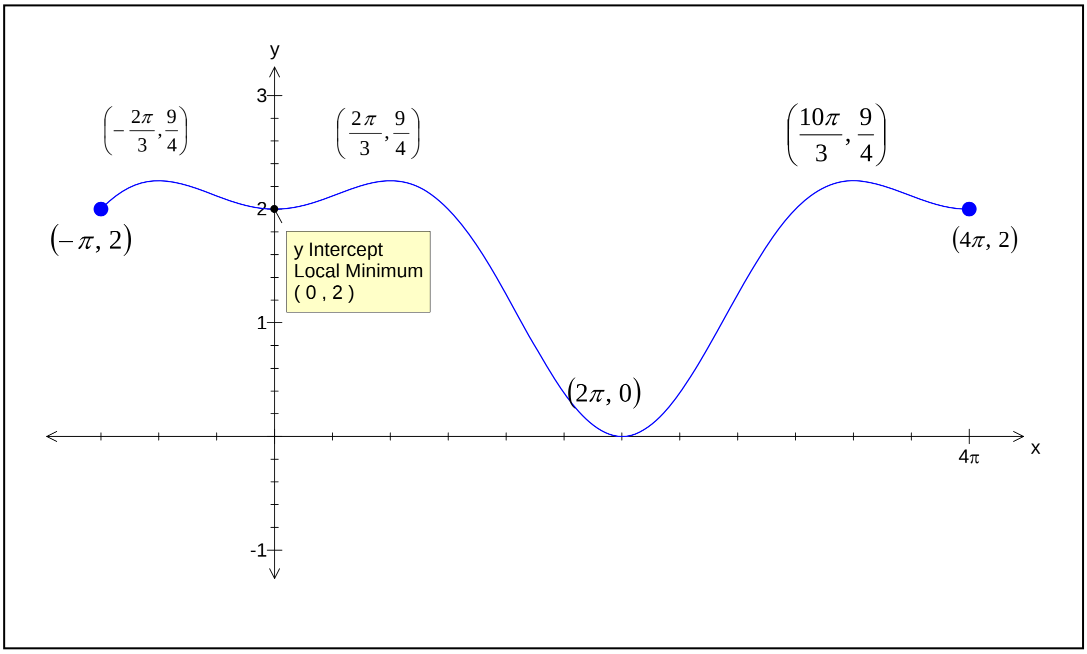
- 1 method mark for separating equation into two parts.
- 1 answer mark for $x=0,\ 2\pi,4\pi$.
- 1 answer mark for $x=\frac{2\pi}{3},\frac{10\pi}{3},\frac{-2\pi}{3}$.

iii. Sketch the graph of $y=g(x)$ on the axes below, labelling intercepts, stationary points and endpoints with exact coordinates.



2 marks

Solution



Mark allocation

- 1 mark for correct shape over correct domain.
- 1 mark for all points labelled and correct.

Total 17 marks

Question 4

Let $f : R \rightarrow R,\ f(x) = 9e^{-x} - 5$.

a. **i.** State the range of the function, f.

1 mark

Solution

Range of the function is $(-5, \infty)$.

Mark allocation

- 1 mark for correct answer.

SECTION 2 – Question 4 – continued

ii. Find the rule and the domain of the inverse function, f^{-1}**.**

3 marks

Solution

To find the inverse x and y need to be interchanged and then the equation is rearranged to make y the subject.

$x \Leftrightarrow y$

$x = 9e^{-y} - 5$

$x + 5 = 9e^{-y}$

$\frac{x+5}{9} = e^{-y}$

$\log_e(\frac{x+5}{9}) = -y$

$y = -\log_e(\frac{x+5}{9}) = \log_e(\frac{9}{x+5})$

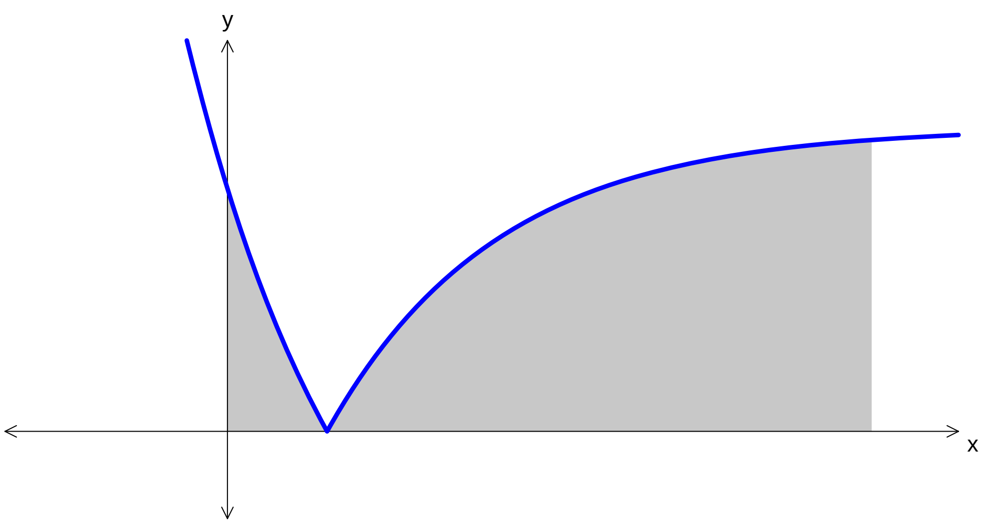
domain of the inverse = range of original = $(-5, \infty)$.

Mark allocation

- 1 method mark for swapping x and y and attempting to find inverse.
- 1 answer mark for correct rule for inverse.
- 1 answer mark for domain.

1+3= 4 marks

A panel of a stained glass window has a section of lead outline described by the graph with the equation $g(x) = |9e^{-x} - 5|$ and is shown below. The shaded area is red-coloured glass.



b. Find the exact value of the *x*-intercept.

2 marks

Solution

x-intercept occurs when $y = 0$.

$$9e^{-x} - 5 = 0$$

$$9e^{-x} = 5$$

$$e^{-x} = \frac{5}{9}$$

$$-x = \log_e(\frac{5}{9})$$

$$x = \log_e(\frac{9}{5})$$

Mark allocation

- 1 method mark for setting equation equal to 0.
- 1 answer mark for answer.

c. The red-coloured glass is bounded by the area under the graph, the x-axis, the y-axis and the line $x = 4$. Find the exact area of the red-coloured glass.

4 marks

Solution

$$area = \int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \, dx + \int_{\log_e(\frac{9}{5})}^{4} -(9e^{-x} - 5) \, dx$$

$$= \left[-9e^{-x} - 5x\right]_0^{\log_e(\frac{9}{5})} + \left[9e^{-x} + 5x\right]_{\log_e(\frac{9}{5})}^{4}$$

$$= (-9\left(\frac{5}{9}\right) - 5\log_e(\frac{9}{5})) - (-9 - 0) + (9e^{-4} + 20) - (9\left(\frac{5}{9}\right) + 5\log_e(\frac{9}{5}))$$

$$= -5 - 5\log_e(\frac{9}{5}) + 9 + 9e^{-4} + 20 - 5 - 5\log_e(\frac{9}{5})$$

$$= 19 - 10\log_e(\frac{9}{5}) + 9e^{-4}$$

Mark allocation

- 1 method mark for setting up the correct integral.
- 1 answer mark for correctly antidifferentiating .
- 1 answer mark for simplifying $e^{-\log_e(\frac{9}{5})}$.
- 1 answer mark for correct answer.

d. Find the value of a, correct to 3 decimal places, such that the area of the red-coloured glass enclosed by the graph , the x-axis, the y-axis and the line $x = a$ is equal to 35 square units.

3 marks

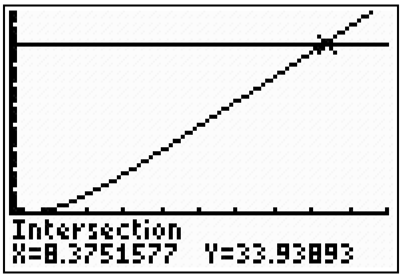
Solution

$$\int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \; dx = 1.06107$$

so $let \int_{\log_e(\frac{9}{5})}^{a} 5 - 9e^{-x} \; dx = 35 - 1.06107 = 33.93893$

using graphs set up

$y_1 = fInt(5 - 9e^{-x}, x, \log_e(9/5), x)$
$y_2 = 33.93893$



so $a = 8.375$ gives an area of 35 units.

Mark allocation

- 1 answer mark for finding $\int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \; dx = 1.06107$.
- 1 method mark for writing equation as integral with a, as $\int_{\log_e(\frac{9}{5})}^{a} 5 - 9e^{-x} \; dx = 33.93893$.
- 1 answer mark for correct answer.

Total 13 marks

END OF SOLUTIONS BOOK