Mathematical Association of Victoria
Trial Exam 2009

MATHEMATICAL METHODS / MATHEMATICAL METHODS (CAS)

STUDENT NAME ____

Written Examination 1

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	**9**	**40**

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied
- Question and answer book of 10 pages, with a detachable sheet of miscellaneous formulas at the back
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f(x)=\left(e^{2x}-3\right)^4$. Evaluate $f'(0)$.

3 marks

Question 2

Solve $2\log_e(x-2)-\log_e(x)=0$ for x, where $x>2$.

3 marks

Question 3

Solve $2\sin\left(2x-\frac{\pi}{6}\right)=1$ for $-\pi<x<\pi$.

3 marks

Question 4

Consider the function $g:[-3,\infty)\to R$, with rule $g(x)=2(x+3)^2-1$.

a. Find the **rule** of the inverse function, g^{-1}.

b. Find the **domain** of the inverse function, g^{-1}.

2 +1 = 3 marks

Question 5

Let $h(x)=\dfrac{1}{\tan(x)}+x$.

a. Show that $h'(x)=-\dfrac{1}{\sin^2(x)}+1$.

b. **Hence** find $\int\left(\dfrac{1}{\sin^2(x)}dx\right)$.

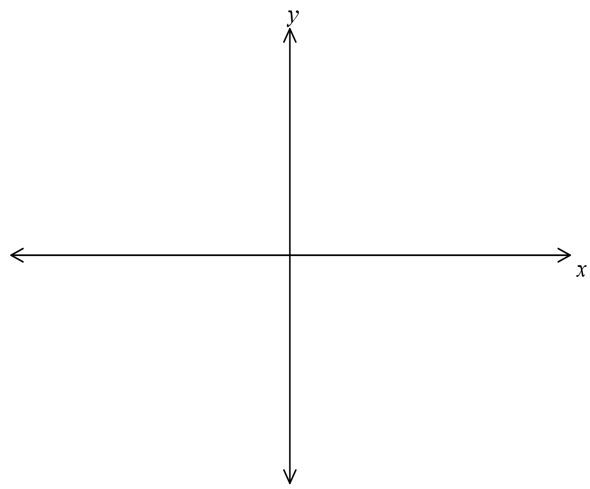
2 + 3 = 5 marks

Question 6

Consider the function $f : R\backslash\{1\} \rightarrow R,\ f(x) = 2 - \dfrac{2}{(x-1)^2}$.

a. Find the **coordinates** of the x-axis intercepts of the graph of f.

b. On the axes below, sketch the graph of $y = |f(x)|$. Label all axes intercepts with their coordinates. Label each asymptote with its equation.

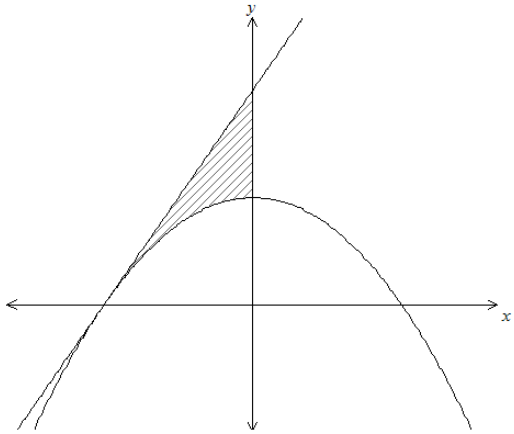


c. Write down the domain of the derivative function, $\dfrac{d}{dx}\left(|f(x)|\right)$.

1 + 3 + 1 = 5 marks

Question 7

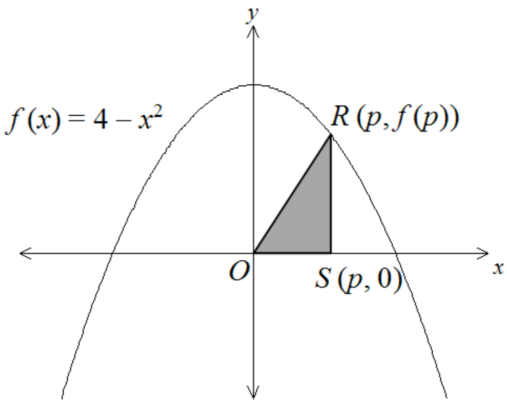
The graph of $f: R \to R,\ f(x) = 4 - x^2$ and the tangent to the graph of f, where it crosses the negative x-axis, are shown.



a. Find the equation of the tangent to the graph of f where it crosses the negative x-axis.

b. Find the area of the shaded region.

c. Consider the triangle *ORS*, where vertex *R* is on the graph of *f*, with coordinates $(p, f(p))$, and vertices *O* and *S* are on the *x*-axis, with coordinates $(0, 0)$ and $(p, 0)$, respectively.



If $p \in [-2, 2]$, find the value(s) of *p* for which the area of triangle *ORS* is a maximum.

2 + 3 + 3 = 8 marks

Question 8

A spherical balloon is being inflated at a rate of 10 cm^3/s. The balloon will burst when the surface area reaches 3600π cm^3.

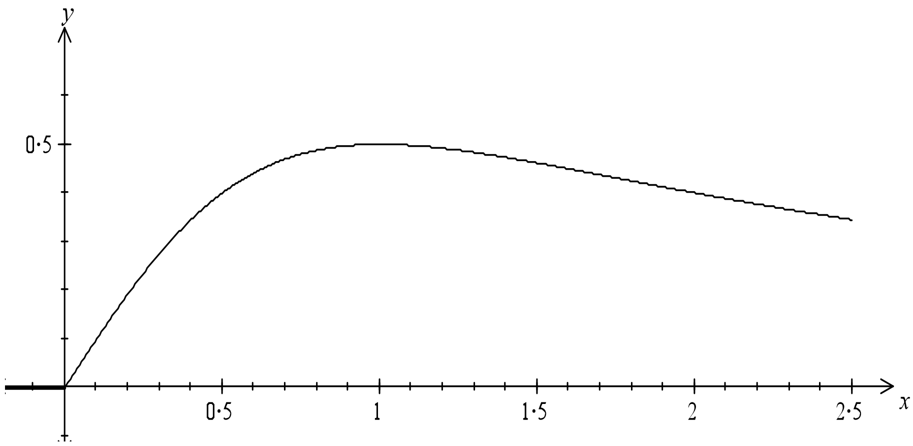
a. The surface area of a sphere of radius *r* is given by $4\pi r^2$. Show that the radius of the balloon is 30 cm at the instant when the balloon bursts.

b. Find the rate at which the radius of the balloon is changing, in cm per second, at the instant when the balloon bursts.

1 + 3 = 4 marks

Question 9

A continuous random variable X has a probability density function $f(x) = \begin{cases} \dfrac{x}{x^2+1} & \text{for } 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$, where a is a real constant. Part of the graph of f, where $0 \le x \le a$, is shown below.



a. Given that $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\log_e\left(x^2+1\right) + c$, where c is a real constant, find the value of a.

2 marks

b. Find the mode of X.

c. Find $\Pr(0.25 \le X \le 0.75)$. Put your answer in the form $\log_e\left(\dfrac{a}{\sqrt{b}}\right)$ where a and b are positive integers.

2 + 2 + 2 = 6 marks

END OF QUESTION AND ANSWER BOOK

Mathematical Methods and Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$$

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$ $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

mean: $\mu = \mathrm{E}(X)$ variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		**mean**	**variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET

MAV Trial Examination Papers 2009
Mathematical Methods / Mathematical Methods (CAS)
Examination 1 - SOLUTIONS

Question 1

Using the chain rule, **1M**

$f'(x) = 8e^{2x}\left(e^{2x} - 3\right)^3$ **1A**

$f'(0) = 8e^{0}\left(e^{0} - 3\right)^3$

$= 8 \times (-2)^3$

$= -64$ **1A**

Question 2

$2\log_e(x-2) - \log_e(x) = 0$

$\log_e\left(\frac{(x-2)^2}{x}\right) = 0$ **1M**

$\frac{(x-2)^2}{x} = 1$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$ **1M**

$(x-4)(x-1) = 0$

$x = 1$ or $x = 4$

$x = 4$ as $x > 2$ **1A**

Question 3

$2\sin\left(2x - \frac{\pi}{6}\right) = 1$

$\sin\left(2x - \frac{\pi}{6}\right) = \frac{1}{2}$

$2x - \frac{\pi}{6} = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{6} + 2\pi,\ \frac{5\pi}{6} + 2\pi \ldots$ **1M**

$2x = \frac{\pi}{3},\ \pi,\ \frac{\pi}{3} + 2\pi,\ \pi + 2\pi \ldots$

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{6} + \pi, \frac{\pi}{2} + \pi \ldots$

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2} \ldots$ **1M**

However, $-\pi < x < \pi$. By symmetry,

$x = -\frac{5\pi}{6},\ -\frac{\pi}{2},\ \frac{\pi}{6}, \frac{\pi}{2}$ **1A**

Question 4

a. To find the rule of an inverse function, interchange the x and y values and make y the subject of the equation.

$x = 2\left(y+3\right)^2 - 1$

$\left(y+3\right)^2 = \frac{x+1}{2}$

$y+3=\sqrt{\frac{x+1}{2}}$ **1M**

(Reject the negative solution because the domain of g is $[-3,\infty)$.)

$y=\sqrt{\frac{x+1}{2}}-3$

$g^{-1}(x)=\sqrt{\frac{x+1}{2}}-3$ **1A**

b. The domain of g^{-1} is the range of g. Domain of g^{-1} is $[-1,\infty)$. **1A**

Question 5

a. $h(x)=\frac{1}{\tan(x)}+x=\frac{\cos(x)}{\sin(x)}+x$

$h'(x)=\frac{-\sec^2(x)}{\tan^2(x)}+1$ using the quotient rule. **2M**

$h'(x)=-\frac{1}{\sin^2(x)}+1$, as required.

Alternatively

$h(x)=\frac{1}{\tan(x)}+x=(\tan(x))^{-1}+x$

$h'(x)=\frac{-\sec^2(x)}{\tan^2(x)}+1$ using the chain rule. **2M**

$h'(x)=-\frac{1}{\sin^2(x)}+1$, as required.

b. From part **a.,**

$\int\left(-\frac{1}{\sin^2(x)}+1\right)dx=\frac{1}{\tan(x)}+x+C_1$ **1M**

$\int\left(-\frac{1}{\sin^2(x)}\right)dx+x+C_2=\frac{1}{\tan(x)}+x+C_1$

$\int\left(-\frac{1}{\sin^2(x)}\right)dx=\frac{1}{\tan(x)}+x-x+C_3$ **1M**

Where $C_3=C_1-C_2$

$\int\left(\frac{1}{\sin^2(x)}\right)dx=-\frac{1}{\tan(x)}+C_3$ **1A**

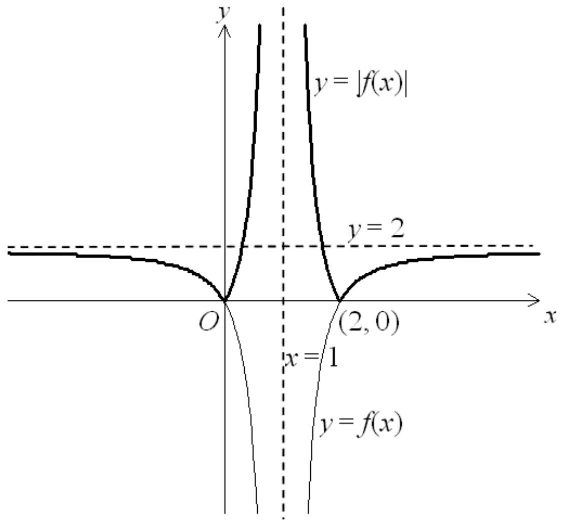
Question 6

a. To find x-axes intercepts, $0 = 2 - \dfrac{2}{(x-1)^2}$

$$\frac{2}{(x-1)^2} = 2$$
$$(x-1)^2 = 1$$
$$x - 1 = \pm\sqrt{1}$$
$$x = 2 \text{ or } x = 0$$

The coordinates are (0, 0) or (2, 0) **1A**

b. Note that the required graph is $y = |f(x)|$.



Correct shape (including cusps) **1A**
Correct asymptotes labelled **1A**
Axes intercepts labelled **1H**

c. The domain of the derivative function is $R \setminus \{0, 1, 2\}$.

Alternatively, $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$. **1A**

Question 7

a. The graph of f crosses the negative x-axis at (–2, 0). The gradient of the tangent at this point is $f'(-2) = 4$. **1M**

Using $y - y_1 = m(x - x_1)$ for the equation of a straight line, the equation of the tangent is

$$y - 0 = 4(x - (-2))$$
$$y = 4x + 8$$ **1A**

b. $A = \int_{-2}^{0} \left((4x+8) - (4 - x^2)\right) dx$ **1M**

$$A = \int_{-2}^{0} \left(x^2 + 4x + 4\right) dx$$

$A = \left[\dfrac{x^3}{3} + 2x^2 + 4x\right]_{-2}^{0}$ **1M**

$A=\left[0-\left(\frac{(-2)^3}{3}+2\times(-2)^2+4\times(-2)\right)\right]$

$A=\frac{8}{3}$ square units **1A**

c. $A=\frac{1}{2}$base × height

$A=\frac{1}{2}p\left(4-p^2\right)$ **1M**

$A=2p-\frac{p^3}{2}$

For maximum area, $\frac{dA}{dp}=0$

$2-\frac{3p^2}{2}=0$ **1M**

$\frac{3p^2}{2}=2$

$p^2=\frac{4}{3}$

$p=\pm\frac{2}{\sqrt{3}}=\pm\frac{2\sqrt{3}}{3}$ **1A**

Maximum area when $p=\frac{2\sqrt{3}}{3}$ or $p=-\frac{2\sqrt{3}}{3}$

Question 8

a. $4\pi r^2=3600\pi$

$r^2=900$

$r=\sqrt{900}=30$ Radius is 30 cm, as required. **1M**

b. Require $\frac{dr}{dt}$ when $r=30$, given that $\frac{dV}{dt}=10$ cm^3/s.

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$ **1M**

For these related rates

$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$

Substitute $\frac{dV}{dr}=4\pi r^2$ and $\frac{dV}{dt}=10$ cm^3/s.

$\frac{dr}{dt}=\frac{1}{4\pi r^2}\times 10$ **1M**

At $r=30$,

$$\frac{dr}{dt}=\frac{1}{4\pi(30)^2}\times 10$$

$$\frac{dr}{dt}=\frac{10}{3600\pi}=\frac{1}{360\pi}$$ **1A**

The rate of change in radius is $\frac{1}{360\pi}$ cm/s.

Question 9

a. $\int_{-\infty}^{\infty}\left(\frac{x}{1+x^2}\right)dx=\int_{0}^{a}\left(\frac{x}{1+x^2}\right)dx=1$

$\left[\frac{1}{2}\log_e(1+x^2)\right]_0^a=1$ **1M**

$\frac{1}{2}\left(\log_e(1+a^2)-\log_e(1)\right)=1$

$\log_e(1+a^2)=2$

$1+a^2=e^2$

$a=\sqrt{e^2-1}$ as $a>0$ **1A**

b. The mode is the x-value at the turning point.

$\frac{d}{dx}\left(\frac{x}{1+x^2}\right)=0$

$\frac{1+x^2-2x^2}{(1+x^2)^2}=0$ **1M**

$\frac{1-x^2}{(1+x^2)^2}=0$

$1-x^2=0$ as $1+x^2\neq 0$

$x=1$ as $x>0$ **1A**

c. $\Pr(0.25\leq X\leq 0.75)=\int_{0.25}^{0.75}\left(\frac{x}{1+x^2}\right)dx$ **1M**

$=\left[\frac{1}{2}\log_e(1+x^2)\right]_{0.25}^{0.75}$

$=\frac{1}{2}\left(\log_e\left(\frac{25}{16}\right)-\log_e\left(\frac{17}{16}\right)\right)$

$=\frac{1}{2}\left(\log_e\left(\frac{25}{17}\right)\right)$

$=\log_e\left(\frac{5}{\sqrt{17}}\right)$ **1A**
