MAV Trial Examination Papers 2009
Mathematical Methods / Mathematical Methods (CAS)
Examination 1 - SOLUTIONS

Question 1

Using the chain rule, **1M**

$f'(x) = 8e^{2x}\left(e^{2x} - 3\right)^3$ **1A**

$f'(0) = 8e^{0}\left(e^{0} - 3\right)^3$

$\quad = 8 \times (-2)^3$

$\quad = -64$ **1A**

Question 2

$2\log_e(x-2) - \log_e(x) = 0$

$\log_e\left(\dfrac{(x-2)^2}{x}\right) = 0$ **1M**

$\dfrac{(x-2)^2}{x} = 1$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$ **1M**

$(x-4)(x-1) = 0$

$x = 1$ or $x = 4$

$x = 4$ as $x > 2$ **1A**

Question 3

$2\sin\left(2x - \dfrac{\pi}{6}\right) = 1$

$\sin\left(2x - \dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$2x - \dfrac{\pi}{6} = \dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{\pi}{6} + 2\pi,\ \dfrac{5\pi}{6} + 2\pi\ ...$ **1M**

$2x = \dfrac{\pi}{3},\ \pi,\ \dfrac{\pi}{3} + 2\pi,\ \pi + 2\pi\ ...$

$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{\pi}{6} + \pi, \dfrac{\pi}{2} + \pi\ ...$

$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}\ ...$ **1M**

However, $-\pi < x < \pi$. By symmetry,

$x = -\dfrac{5\pi}{6},\ -\dfrac{\pi}{2},\ \dfrac{\pi}{6}, \dfrac{\pi}{2}$ **1A**

Question 4

a. To find the rule of an inverse function, interchange the *x* and *y* values and make *y* the subject of the equation.

$x = 2\left(y+3\right)^2 - 1$

$\left(y+3\right)^2 = \dfrac{x+1}{2}$

$y+3=\sqrt{\frac{x+1}{2}}$ **1M**

(Reject the negative solution because the domain of g is $[-3,\infty)$.)

$y=\sqrt{\frac{x+1}{2}}-3$

$g^{-1}(x)=\sqrt{\frac{x+1}{2}}-3$ **1A**

b. The domain of g^{-1} is the range of g. Domain of g^{-1} is $[-1,\infty)$. **1A**

Question 5

a. $h(x)=\frac{1}{\tan(x)}+x=\frac{\cos(x)}{\sin(x)}+x$

$h'(x)=\frac{-\sec^2(x)}{\tan^2(x)}+1$ using the quotient rule. **2M**

$h'(x)=-\frac{1}{\sin^2(x)}+1$, as required.

Alternatively

$h(x)=\frac{1}{\tan(x)}+x=(\tan(x))^{-1}+x$

$h'(x)=\frac{-\sec^2(x)}{\tan^2(x)}+1$ using the chain rule. **2M**

$h'(x)=-\frac{1}{\sin^2(x)}+1$, as required.

b. From part **a.,**

$\int\left(-\frac{1}{\sin^2(x)}+1\right)dx=\frac{1}{\tan(x)}+x+C_1$ **1M**

$\int\left(-\frac{1}{\sin^2(x)}\right)dx+x+C_2=\frac{1}{\tan(x)}+x+C_1$

$\int\left(-\frac{1}{\sin^2(x)}\right)dx=\frac{1}{\tan(x)}+x-x+C_3$ **1M**

Where $C_3=C_1-C_2$

$\int\left(\frac{1}{\sin^2(x)}\right)dx=-\frac{1}{\tan(x)}+C_3$ **1A**

Question 6

a. To find x-axes intercepts, $0 = 2 - \dfrac{2}{(x-1)^2}$

$$\frac{2}{(x-1)^2} = 2$$

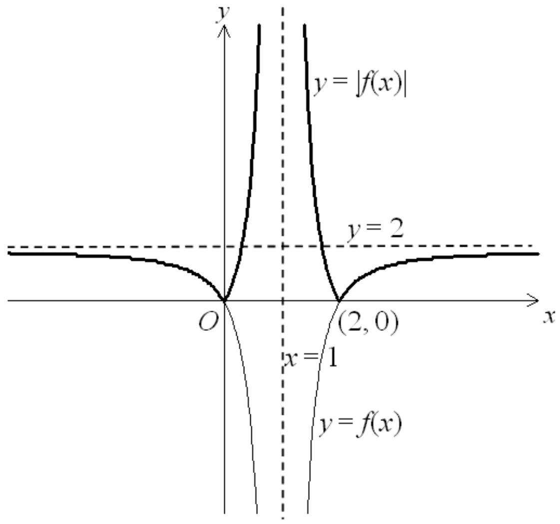
$$(x-1)^2 = 1$$

$$x - 1 = \pm\sqrt{1}$$

$$x = 2 \text{ or } x = 0$$

The coordinates are (0, 0) or (2, 0) **1A**

b. Note that the required graph is $y = |f(x)|$.



Correct shape (including cusps) **1A**
Correct asymptotes labelled **1A**
Axes intercepts labelled **1H**

c. The domain of the derivative function is $R \setminus \{0, 1, 2\}$.

Alternatively, $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$. **1A**

Question 7

a. The graph of f crosses the negative x-axis at (–2, 0). The gradient of the tangent at this point is $f'(-2) = 4$. **1M**

Using $y - y_1 = m(x - x_1)$ for the equation of a straight line, the equation of the tangent is

$$y - 0 = 4(x - (-2))$$

$$y = 4x + 8$$ **1A**

b. $A = \int_{-2}^{0} \left((4x+8) - (4 - x^2)\right) dx$ **1M**

$$A = \int_{-2}^{0} \left(x^2 + 4x + 4\right) dx$$

$A = \left[\dfrac{x^3}{3} + 2x^2 + 4x\right]_{-2}^{0}$ **1M**

$$A=\left[0-\left(\frac{(-2)^3}{3}+2\times(-2)^2+4\times(-2)\right)\right]$$

$$A=\frac{8}{3}\ \text{square units}$$ **1A**

c. $A=\frac{1}{2}\text{base}\times\text{height}$

$$A=\frac{1}{2}p\left(4-p^2\right)$$ **1M**

$$A=2p-\frac{p^3}{2}$$

For maximum area, $\frac{dA}{dp}=0$

$$2-\frac{3p^2}{2}=0$$ **1M**

$$\frac{3p^2}{2}=2$$

$$p^2=\frac{4}{3}$$

$$p=\pm\frac{2}{\sqrt{3}}=\pm\frac{2\sqrt{3}}{3}$$ **1A**

Maximum area when $p=\frac{2\sqrt{3}}{3}$ or $p=-\frac{2\sqrt{3}}{3}$

Question 8

a. $4\pi r^2=3600\pi$

$r^2=900$

$r=\sqrt{900}=30$ Radius is 30 cm, as required. **1M**

b. Require $\frac{dr}{dt}$ when $r=30$, given that $\frac{dV}{dt}=10\ \text{cm}^3\text{/s}$.

$$V=\frac{4}{3}\pi r^3$$

$$\frac{dV}{dr}=4\pi r^2$$ **1M**

For these related rates

$$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$$

Substitute $\frac{dV}{dr}=4\pi r^2$ and $\frac{dV}{dt}=10\ \text{cm}^3\text{/s}$.

$$\frac{dr}{dt}=\frac{1}{4\pi r^2}\times 10$$ **1M**

At $r=30$,

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2} \times 10$$

$$\frac{dr}{dt} = \frac{10}{3600\pi} = \frac{1}{360\pi}$$ **1A**

The rate of change in radius is $\frac{1}{360\pi}$ cm/s.

Question 9

a. $\int_{-\infty}^{\infty} \left(\frac{x}{1+x^2}\right) dx = \int_0^a \left(\frac{x}{1+x^2}\right) dx = 1$

$$\left[\frac{1}{2}\log_e(1+x^2)\right]_0^a = 1$$ **1M**

$$\frac{1}{2}\left(\log_e(1+a^2) - \log_e(1)\right) = 1$$

$$\log_e(1+a^2) = 2$$

$$1 + a^2 = e^2$$

$a = \sqrt{e^2 - 1}$ as $a > 0$ **1A**

b. The mode is the x-value at the turning point.

$$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = 0$$

$$\frac{1+x^2-2x^2}{(1+x^2)^2} = 0$$ **1M**

$$\frac{1-x^2}{(1+x^2)^2} = 0$$

$1 - x^2 = 0$ as $1 + x^2 \neq 0$

$x = 1$ as $x > 0$ **1A**

c. $\Pr(0.25 \le X \le 0.75) = \int_{0.25}^{0.75} \left(\frac{x}{1+x^2}\right) dx$ **1M**

$$= \left[\frac{1}{2}\log_e(1+x^2)\right]_{0.25}^{0.75}$$

$$= \frac{1}{2}\left(\log_e\left(\frac{25}{16}\right) - \log_e\left(\frac{17}{16}\right)\right)$$

$$= \frac{1}{2}\left(\log_e\left(\frac{25}{17}\right)\right)$$

$$= \log_e\left(\frac{5}{\sqrt{17}}\right)$$ **1A**
