THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

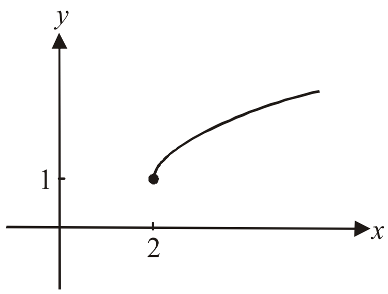
MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2010

SECTION 1 – Multiple-choice answers

1. E	9. B	17. C
2. E	10. E	18. C
3. D	11. C	19. A
4. C	12. A	20. B
5. D	13. A	21. D
6. C	14. B	22. A
7. D	15. E	
8. D	16. A	

SECTION 1 – Multiple-choice solutions

Question 1



Method 1
Sketch the graph of $y = 1 + \sqrt{x-2}$.
The maximal domain is $x \in [2, \infty)$
The answer is E.

Method 2
The function $y = 1 + \sqrt{x-2}$ is only defined for R if $x - 2 \geq 0$; that is for $x \geq 2$.
The maximal domain is $x \in [2, \infty)$.
The answer is E.

Question 2
Method 1 – by hand
$f(x) = ax^3 + bx^2 + cx + d$
$f(1) = 4$ so $a + b + c + d = 4$
$f(-2) = 10$ so $-8a + 4b - 2c + d = 10$
$f'(x) = 3ax^2 + 2bx + c$
$f'(1) = 4$ so $3a + 2b + c = 4$
$f'(3) = 36$ so $27a + 6b + c = 36$

So $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 2 & 1 & 0 \\ 27 & 6 & 1 & 0 \end{bmatrix}$

The answer is E.

Method 2 – using CAS

Define $f(x)=ax^3+bx^2+cx+d$

$f(1)=4 \qquad a+b+c+d=4$

$f(-2)=10 \qquad -8a+4b-2c+d=10$

$\frac{d}{dx}(f(x))=4|x=1 \qquad 3a+2b+c=4$

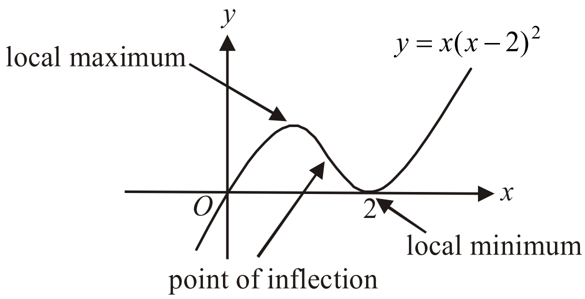
$\frac{d}{dx}(f(x))=36|x=3 \qquad 27a+6b+c=36$

$$\text{So} \quad A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 2 & 1 & 0 \\ 27 & 6 & 1 & 0 \end{bmatrix}$$

The answer is E.

Question 3

Do a quick sketch.



There is no **stationary** point of inflection.
The local maximum and local minimum each occur at a point where $x>0$.
There is a point of inflection.
There are only two stationary points.
The answer is D.

Question 4

Given $\int_2^4 f(x)dx=3$

$\int_2^4 (1-5f(x))dx$

$=\int_2^4 1\,dx-5\int_2^4 f(x)dx$

$=[x]_2^4-5\times 3$

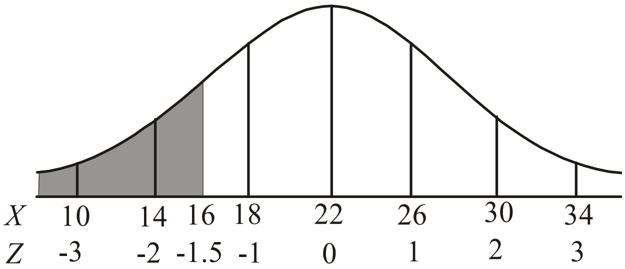
$=4-2-15$

$=-13$

The answer is C.

Question 5

$$z=\frac{x-\mu}{\sigma}$$
$$z=\frac{16-22}{4}$$
$$=\frac{-6}{4}$$
$$=-1.5$$

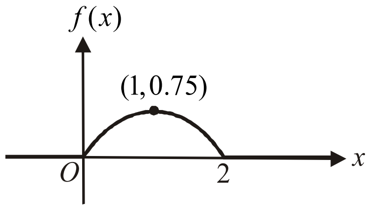


$$\Pr(X<16)=\Pr(Z<-1.5)$$
$$=\Pr(Z>1.5) \text{ due to the symmetry of the normal curve}$$

The answer is D.

Question 6

The mode is the most frequently occurring value. We want to find the value of X where $f(x)$ is a maximum.
Do a quick sketch.



The maximum is 0.75 and this occurs when $x=1$.
So the mode is 1.
The answer is C.

Question 7

$$\Pr(X<0.5|X<1)$$
$$=\frac{\Pr(X<0.5\cap X<1)}{\Pr(X<1)} \quad \text{(conditional probability formula)}$$
$$=\frac{\Pr(X<0.5)}{0.5}$$
$$=\frac{0.15625}{0.5}$$
$$=0.3125 \quad \text{(correct to 4 decimal places)}$$

Note that $\Pr(X<1)=0.5$ because the graph is that of an inverted parabola with an axis of symmetry given by $x=1$ and the function's domain is $0\le x\le 2$.

Note also that

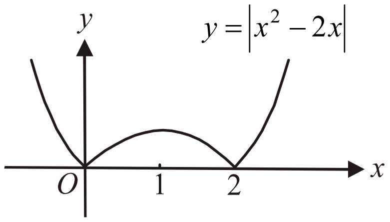
$$\Pr(X<0.5)=\int_0^{0.5}\frac{-3x}{4}(x-2)dx$$
$$=0.15625$$

The answer is D.

Question 8

$$f(x)=\left|x^2-2x\right|$$
$$=\left|x(x-2)\right|$$

Sketch the graph of $y=f(x)$.



The gradient of the graph is positive for $x\in(0,1)\cup(2,\infty)$.
Note at $x=0$ and $x=2$ the graph has cusps (i.e. pointy bits) and the gradient function is not defined at these points.
At $x=1$, there is a local maximum where $f'(x)=0$.
The answer is D.

Question 9

Let (x', y') be the image point.

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}-3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}\pi\\2\end{bmatrix}$$
$$=\begin{bmatrix}-3x\\2y\end{bmatrix}-\begin{bmatrix}\pi\\2\end{bmatrix}$$
$$=\begin{bmatrix}-3x-\pi\\2y-2\end{bmatrix}$$

$x'=-3x-\pi$

$3x=-x'-\pi$

$x=\dfrac{-x'-\pi}{3}$

$y'=2y-2$

$2y=y'+2$

$y=\dfrac{y'+2}{2}$

$y=\cos(3x)$
becomes
$$\frac{y'+2}{2}=\cos(-x'-\pi)$$
$$y'+2=2\cos(-(x'+\pi))$$
$$y'=2\cos(-(x'+\pi))-2$$
The image equation is $y=2\cos(-(x+\pi))-2$.
The answer is B.

Question 10

The graph of $y = g(x)$ has been translated 1 unit up and 2 units to the left. The image of the point (2,7) under this transformation is (0,8) so this image point is in an equivalent position to the image graph as the point (2,7) is to the graph of $y = g(x)$.
Therefore the gradient of the tangent will be the same; that is 3.

Now, $\quad y - y_1 = m(x - x_1)$

becomes $y - 8 = 3(x - 0)$

$$y = 3x + 8$$

is the required equation of the tangent.
The answer is E.

Question 11

<u>Method 1</u> – using CAS

$$\frac{d}{dx}\left(\frac{f(x)}{f(x)+1}\right) = \frac{\frac{d}{dx}(f(x))}{(f(x)+1)^2}$$

The answer is C.

<u>Method 2</u> – by hand

$$g(x) = \frac{f(x)}{f(x)+1}$$

$$g'(x) = \frac{(f(x)+1) \times f'(x) - f'(x) \times f(x)}{(f(x)+1)^2} \quad \text{Quotient rule}$$

$$= \frac{f'(x)\{f(x)+1 - f(x)\}}{(f(x)+1)^2}$$

$$= \frac{f'(x)}{(f(x)+1)^2}$$

The answer is C.

Question 12

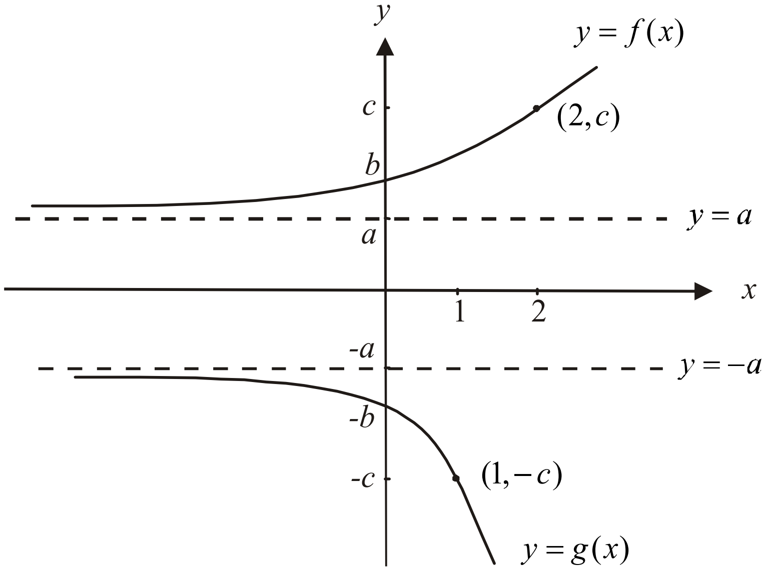
This is a binomial distribution with $n = 10$ and $p = 0.2$.

$\Pr(X > 2) = \text{binom Cdf } (10,\ 0.2,\ 3,\ 10)$

$\qquad = 0.3222...$

The answer is A.

Question 13



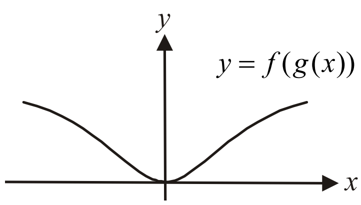
The graph of $y = f(x)$ has been reflected in the x-axis.
Note that the point $(0,b)$ becomes the point $(0,-b)$.
The graph has been dilated by a factor of $\frac{1}{2}$ from the y-axis (i.e. squashed). Note that the point $(2,c)$ becomes the point $(1,-c)$ which includes the reflection.
There is no translation up or down or left or right.
The transformation is given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The answer is A.

Question 14

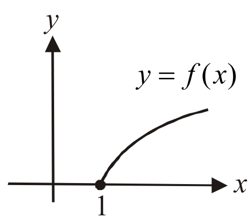
<u>Method 1</u>
Sketch the graph of $f(g(x)) = \log_e(x^2 + 1)$.



So $r_{f(g(x))} = [0, \infty)$.
The answer is B.

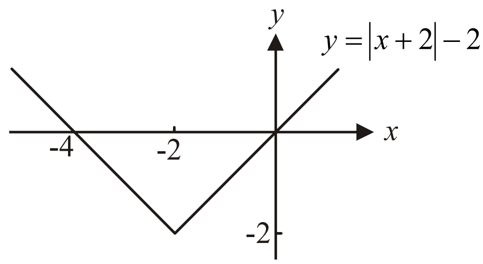
<u>Method 2</u>

$g(x) = x^2 + 1$ $\quad d_g = R$ and $r_g = [1, \infty)$
$f(x) = \log_e(x)$ $\quad$ For $f(g(x))$, $d_f = r_g = [1, \infty)$.
So, $r_f = r_{f(g(x))} = [0, \infty)$.
The answer is B.



Question 15

Do a quick sketch.



The rate of change of y with respect to x is the gradient of the function or $\frac{dy}{dx}$.
At $x = 0$, the gradient is 1. Option A is incorrect.
At $x = -4$, the gradient is – 1. Option B is incorrect.
The gradient is negative for $x \in (-\infty, -2)$.
Note at $x = -2$, the gradient is not defined.
Option C is incorrect.
For $x \in (-2, 0)$, the gradient is positive so option D is incorrect.
Option E is correct because for $x \in (-\infty, -2)$ the gradient is negative, for $x \in (-2, \infty)$ the gradient is positive and for $x = -2$ the gradient is undefined. The gradient is never equal to zero.
The answer is E.

Question 16

Method 1 – using CAS

$$y = 2 + \frac{1}{x-5}$$

Inverse is $y = 5 + \frac{1}{x-2}$.

$$d_g = R \setminus \{5\} \quad r_g = R \setminus \{2\}$$

So $d_{g^{-1}} = R \setminus \{2\}$

$$g^{-1} : R \setminus \{2\} \to R, \ g^{-1}(x) = 5 + \frac{1}{x-2}$$

The answer is A.

Method 2 – by hand

$$y = 2 + \frac{1}{x-5}$$

Swap x and y for inverse

$$x = 2 + \frac{1}{y-5}$$

Rearrange

$$x - 2 = \frac{1}{y-5}$$

$$y - 5 = \frac{1}{x-2}$$

$$y = 5 + \frac{1}{x-2}$$

$$d_g = R \setminus \{5\} \quad r_g = R \setminus \{2\}$$

So $d_{g^{-1}} = R \setminus \{2\}$

$$g^{-1} : R \setminus \{2\} \to R, \ g^{-1}(x) = 5 + \frac{1}{x-2}$$

The answer is A.

Question 17

We want to find $\frac{dV}{dt}$.

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ (chain rule)

Now $\frac{dh}{dt} = 0.02$ (given)

Also, $V = \frac{1}{2} \times \pi r^2 h$

$= \frac{1}{2} \times \pi \times 2^2 \times h$

$= 2\pi h$

$\frac{dV}{dh} = 2\pi$

So $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

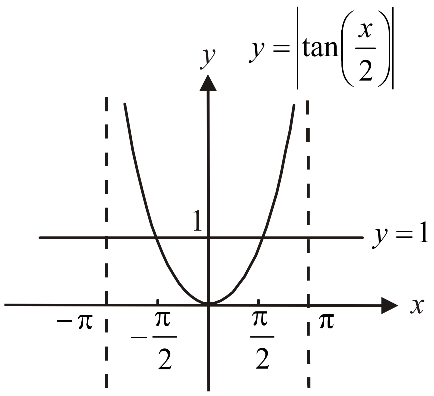
becomes $= 2\pi \times 0.02$

$= 0.04\pi \text{ m}^3$ per hour

The answer is C.

Question 18

Do a quick sketch.

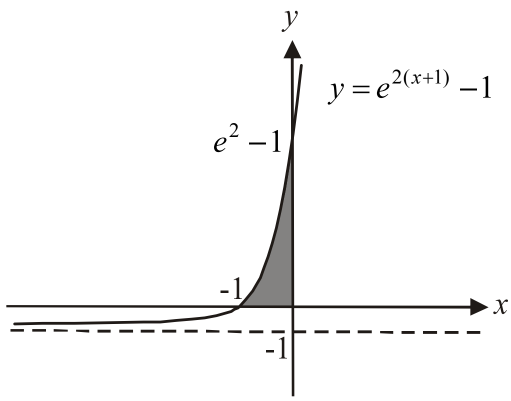


There are two solutions; $x = \pm\frac{\pi}{2}$

The answer is C.

Question 19

Do a quick sketch.



$$\text{Area} = \int_{-1}^{0} (e^{2(x+1)} - 1)dx$$

The answer is A.

Question 20

For option A the events $\{1,2,3,4\}$ and $\{5,6,7,8\}$ are mutually exclusive but not independent.

If $X = \{1,2,3,4\}$ and $Y = \{5,6,7,8\}$ then $\Pr(X) = \frac{1}{2}$ and $\Pr(Y) = \frac{1}{2}$ and $\Pr(X \cap Y) = 0$.

For independent events $\Pr(X \cap Y) = \Pr(X) \times \Pr(Y)$. Since $0 \neq \frac{1}{4}$ X and Y are not independent.

For option B, if $X = \{1,2,3,4\}$ and $Y = \{1,4,5,8\}$

$$\Pr(X) = \frac{1}{2} \text{ and } \Pr(Y) = \frac{1}{2}$$

$$\Pr(X \cap Y) = \frac{1}{4}$$

For independent events

$$\begin{aligned}\Pr(X \cap Y) &= \Pr(X) \times \Pr(Y)\\ \frac{1}{4} &= \frac{1}{2} \times \frac{1}{2}\\ &= \frac{1}{4}\end{aligned}$$

So X and Y are independent.

The condition for independent events; that $\Pr(X \cap Y) = \Pr(X) \times \Pr(Y)$, does not apply in options C, D or E.

The answer is B.

Question 21

$$\begin{aligned}f(x) &= \sqrt{x}\\ f(x+h) &\approx f(x) + hf'(x)\\ f(5-0.3) &\approx f(5) - 0.3f'(5)\end{aligned}$$

The answer is D.

Question 22

The graph of $y = h(x)$ is the gradient function. From the graph of $y = h(x)$, we know that there are stationary points to be located at $x = 0$ and $x = a$.

The gradient is negative for $x < 0$ and $x > a$. The gradient is positive for $0 < x < a$.

Only option A fulfils these requirements.

The answer is A.

SECTION 2

Question 1

a. Define $f(x) = 3|x| - x^3$

Solve $f(x) = 0$ for x

$x = 0$ or $x = \sqrt{3}$

So $a = \sqrt{3}$

(1 mark)

b. Graph $y = f(x)$

The local maximum occurs at $(1,2)$.

(1 mark)

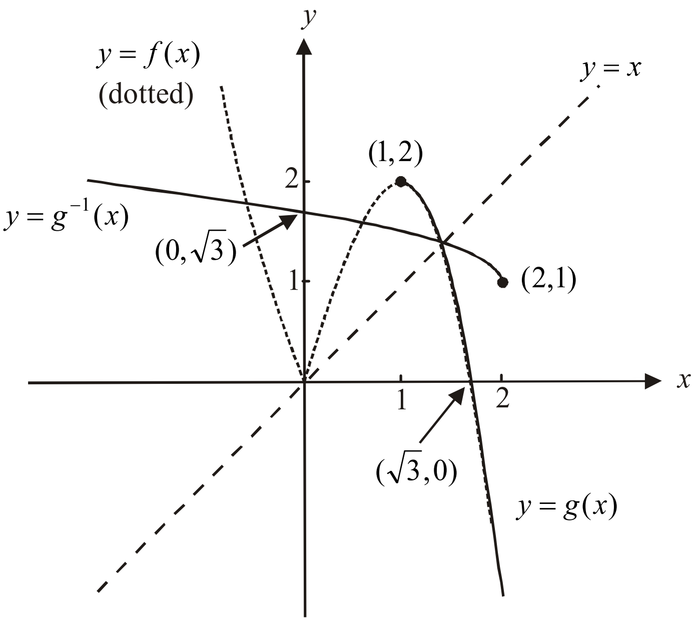
c. Using the answer from part **b.**, for $x > 0$, the graph is strictly decreasing for $x \in [1, \infty)$.

(1 mark)

d. **i.** $c = 1$

(1 mark)

ii.



(1 mark) – correct endpoint
(1 mark) – correct shape including correct intercept

e. Area $= \int_0^{\sqrt{3}} f(x)dx$ **(1 mark)**

$= \frac{9}{4}$ or 2.25 square units

(1 mark)

(Note when a question is worth 2 marks some working, eg. the first line, must be shown.)

f. **i.** $f(x) = 3|x| - x^3$

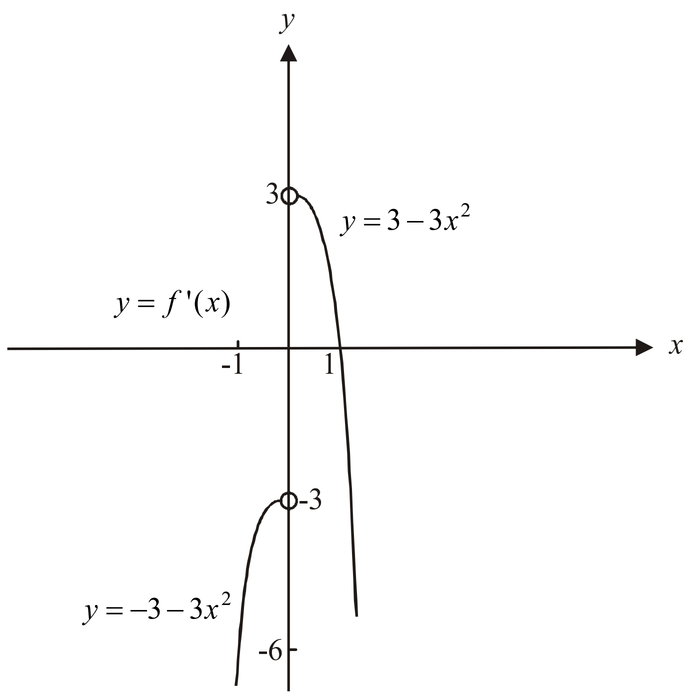
$$= \begin{cases} 3x - x^3 & \text{for } x \geq 0 \\ -3x - x^3 & \text{for } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 3 - 3x^2 & \text{for } x > 0 \\ -3 - 3x^2 & \text{for } x < 0 \end{cases}$$

Note from the graph of $y = f(x)$, at $x = 0$ there is a cusp, so $f'(x)$ is not defined at $x = 0$.

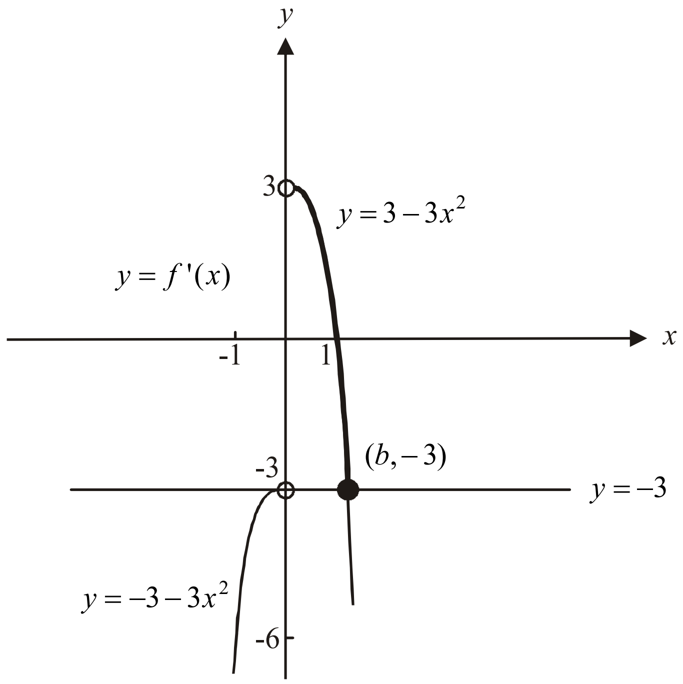
(1 mark) – first branch and domain
(1 mark) – second branch and domain

ii.



(1 mark) – first branch
(1 mark) – second branch
(1 mark) open circles at $x = 0$

g. Looking at the graph of $y = f'(x)$, we see that there is a unique value of $f'(x)$ (i.e. $f'(x)$ has only one value) for that section of the graph marked heavily.



To find *b*, substitute $y = -3$ into $y = 3 - 3x^2$

$$-3 = 3 - 3x^2$$
$$-6 = -3x^2$$
$$x^2 = 2$$
$$x = \sqrt{2}, \quad x > 0 \text{ for this branch}$$

So $b = \sqrt{2}$. **(1 mark)**

The x values required are $x \in \left(0, \sqrt{2}\right]$.

(1 mark)

(Note that for each point that has an x coordinate such that $x \in (-\infty, 0) \cup (\sqrt{2}, \infty)$ there is another point that shares the same value of $f'(x)$; that is, has the same gradient).

Total 15 marks

Question 2

a. $\mu = 45, \quad \sigma = 3$

$\Pr(X < 40) = \text{norm Cdf}(-\infty, 40, 45, 3)$

$= 0.04779...$

So 5% (correct to the nearest per cent) record less than 40 minutes.

(1 mark)

b. $\text{inv Norm}(0.3, 45, 3) = 43.4268...$

The slowest time needed to receive a medallion is 43.427 minutes (correct to 3 decimal places). **(1 mark)**

Check your answer: $\Pr(X < 43.4268...) = \text{normal Cdf}(-\infty, 43.4268, 45, 3) = 0.3$

c. This involves conditional probability

$\Pr(X < 40 \mid X < 43.4268)$

$= \dfrac{\Pr(X < 40)}{\Pr(X < 43.4268)}$

$= \dfrac{0.04779}{0.3}$ (from parts a. and b.)

$= 0.1593$ (correct to 4 decimal places)

(1 mark)

d. This is a binomial distribution with $n = 5, p = 0.3$. **(1 mark)**

Method 1 – using CAS

binomPdf(5, 0.3, 2)

$\Pr(X = 2) = 0.3087$

(1 mark)

Method 2 – by hand

$\Pr(X = 2) = {}^5C_2 (0.3)^2 (0.7)^3$

$= 0.3087$

(1 mark)

e. $\Pr(X < 2) = \displaystyle\int_1^2 \frac{5}{4x^2} dx$ **(1 mark)**

$= \dfrac{5}{8}$

(1 mark)

f. $\text{var}(X) = E(X^2) - (E(X))^2$

Now, $E(\text{X}) = \int_1^5 \left(x \times \frac{5}{4x^2} \right) dx$

$= \frac{5}{4} \log_e(5)$

$= 2.0118...$ **(1 mark)**

$E(X^2) = \int_1^5 \left(x^2 \times \frac{5}{4x^2} \right) dx$

$= 5$ **(1 mark)**

$\text{var}(X) = 5 - (2.0118...)^2$

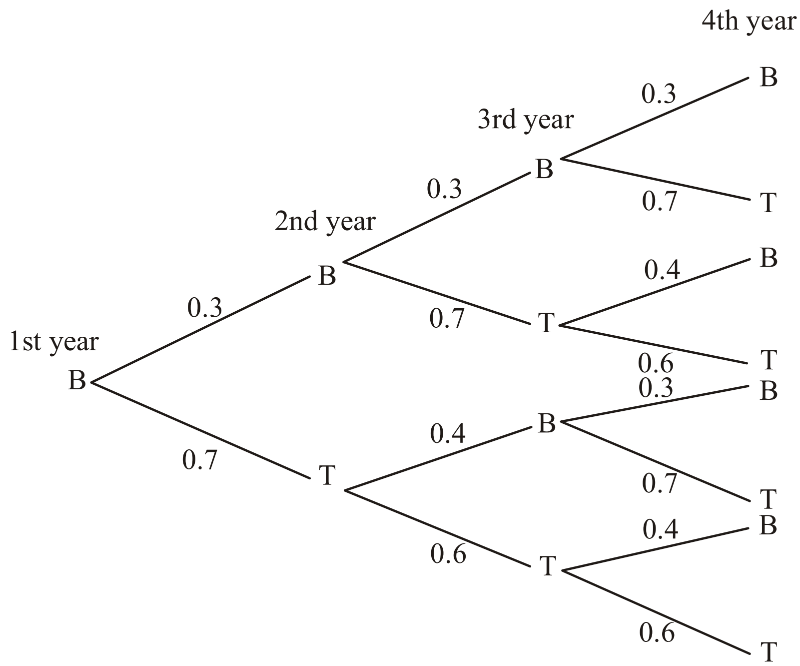
$= 0.9527$ (to 4 decimal places)

(1 mark)

g. **i.** $0.7 \times 0.6 \times 0.6 = 0.252$

(1 mark)

ii. Draw a tree diagram.



Pr(bib in at least 2 of next 3 years)

$= BBB + BBT + BTB + TBB$ **(1 mark)**

$= 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.7 + 0.3 \times 0.7 \times 0.4 + 0.7 \times 0.4 \times 0.3$

$= 0.258$

(1 mark)

iii. Use the transition matrix

$$\begin{array}{c} \text{one year} \\ \begin{array}{cc} B & T \end{array} \end{array}$$

$$\begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \begin{array}{c} B \\ T \end{array} \text{ next year}$$

and the state matrix for the first year $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{array}{c} B \\ T \end{array}$

$$\begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.36363... \\ 0.63636... \end{bmatrix}$$

The probability that Jane will wear a bib in the tenth year is 0.3636 (correct to 4 decimal places).

(1 mark) – correct transition matrix
(1 mark) correct power of 9
(1 mark) correct answer

iv. Try $n = 20$

$$\begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.36363... \\ 0.63636... \end{bmatrix}$$

From part **iii.**, we see that a steady state has been reached.
The probability that Jane will wear a tattoo in the long term is 0.6364 (correct to 4 decimal places).

(1 mark)
Total 17 marks

Question 3

a. **i.** $h(x) = \cos(e^{\frac{x}{10}}) + 1$
$h(x) = f(g(x))$

So $g(x) = e^{\frac{x}{10}}$ and $f(x) = \cos(x) + 1$

(1 mark)

ii. $g(f(x))$ exists iff $r_f \subseteq d_g$
Now, $d_g = R$, $r_g = (0,\infty)$
$d_f = R$, $r_f = [0,2]$

Since $[0,2] \subseteq R, g(f(x))$ exists.

(1 mark)

b. $$h'(x) = \frac{-e^{\frac{x}{10}} \times \sin(e^{\frac{x}{10}})}{10}$$

(1 mark)

c. <u>Method 1</u>
Solve $h(x) = 0$ for x.
Choosing the least value of x, we have $x = 10\ln(\pi)$.
The required point is $(10\ln(\pi), 0)$.

(1 mark)

<u>Method 2</u>
Solve $h'(x) = 0$ for x.
Choosing the least value of x, we have $x = 10\ln(\pi)$.
Since $h(10\ln(\pi)) = 0$ the required point is $(10\ln(\pi), 0)$.

(1 mark)

d. **i.** <u>Method 1</u>
$h(x) = \cos(e^{\frac{x}{10}}) + 1$

Max value of $\cos(e^{\frac{x}{10}}) = 1$
So max value of $h(x)$ is $1 + 1 = 2$

(1 mark)

<u>Method 2</u>
Using the result from part **c.**, <u>Method 2</u>
Solve $h'(x) = 0$ for x.
Local maxima occur at $x = 10\ln(2\pi), x = 10\ln(4\pi), \ldots$
Now $h(10\ln(2\pi)) = 2$
$h(10\ln(4\pi)) = 2$ and so on.
The maximum value of h is 2.

(1 mark)

ii. Method 1 – following on from Method 1 in part **i.**,
Solve $h(x)=2$ for x
$x=10\log_e(2\pi)$
and $x=10\log_e(4\pi)$
The distances from the fence where the maximum height occurs are
$10\log_e(2\pi)$ metres and $10\log_e(4\pi)$ metres.
(1 mark) **(1 mark)**

Method 2 – following on from Method 2 in part **i.**,
Local maxima occur at
$x=10\log_e(2\pi)$
and $x=10\log_e(4\pi)$
The distances from the fence where the maximum height occurs are
$10\log_e(2\pi)$ metres and $10\log_e(4\pi)$ metres.
(1 mark) **(1 mark)**

e. We are looking for the average value of the function h.

$$\text{average value} = \frac{1}{10\log_e(5\pi)-0}\int_0^{10\log_e(5\pi)} h(x)\,dx$$ **(1 mark)**

$$= 0.87893...$$
$$= 0.88\text{m (correct to 2 decimal places)}$$

The landfill will be 0.88m above the ground after it has been levelled.
(1 mark)

f. $h'(x)$ gives us the gradient of a tangent to the pile at x.
$\dfrac{-1}{h'(x)}$ gives us the gradient of a normal to the pile at x.
Solve $\dfrac{-1}{h'(x)}=6.5$ for x. **(1 mark)**
$x=4.3128....,\ 9.0374...,\ 18.7508...,\ 22.2547...$
but $x\in[5,\ 10]$ so $x=9.0374$

Now $h(9.0374)=0.217904$
So the equation of the normal is $y-0.217904=6.5(x-9.0374)$ **(1 mark)**

When $y=0,\quad x=9.00388$

The horizontal distance from the fence is 9.004m (correct to 3 decimal places).
(1 mark)
Total 12 marks

Question 4

a. Define $f(x)=\dfrac{2500}{x^2-20x+600}$

$f(0)=\dfrac{25}{6}$

A is the point $\left(0,\dfrac{25}{6}\right)$.

(1 mark)

b. Method 1

Graph $y=f(x)$.

T is a local maximum with coordinates (10,5).

The *y*-coordinate of *A* is $\dfrac{25}{6}$ from part **a.**

The difference in height between points *T* and *A* is $5-\dfrac{25}{6}=\dfrac{5}{6}$m.

(1 mark)

Method 2

Solve $f'(x)=0$ for x

$x=10$

Now $f(10)=5$

The *y*-coordinates of *A* is $\dfrac{25}{6}$ from part **a**.

The difference in height between points *T* and *A* is $5-\dfrac{25}{6}=\dfrac{5}{6}$m.

(1 mark)

c. At point *B*, $f'(x)=\dfrac{40}{441}$

Solve $f'(x)=\dfrac{40}{441}$ for *x*.

$x=5$

(1 mark)

Note that the other solution of $x=-16.70...$ is outside the domain of *f*.

$f(5)=\dfrac{100}{21}$

B is the point $\left(5,\dfrac{100}{21}\right)$.

(1 mark)

d. The gradient of the line BC is $\frac{4}{21}$.

The equation of the straight line is given by $y-y_1=m(x-x_1)$

Method 1

Using $C(22,8)$

$$y-8=\frac{4}{21}(x-22)$$

$$y=\frac{4x}{21}+\frac{80}{21}$$

(1 mark)

Method 2

Using point $B\left(5,\frac{100}{21}\right)$ from part **c.**,

$$y-\frac{100}{21}=\frac{4}{21}(x-5)$$

$$y=\frac{4x}{21}+\frac{80}{21}$$

(1 mark)

e. C is the point (22,8).

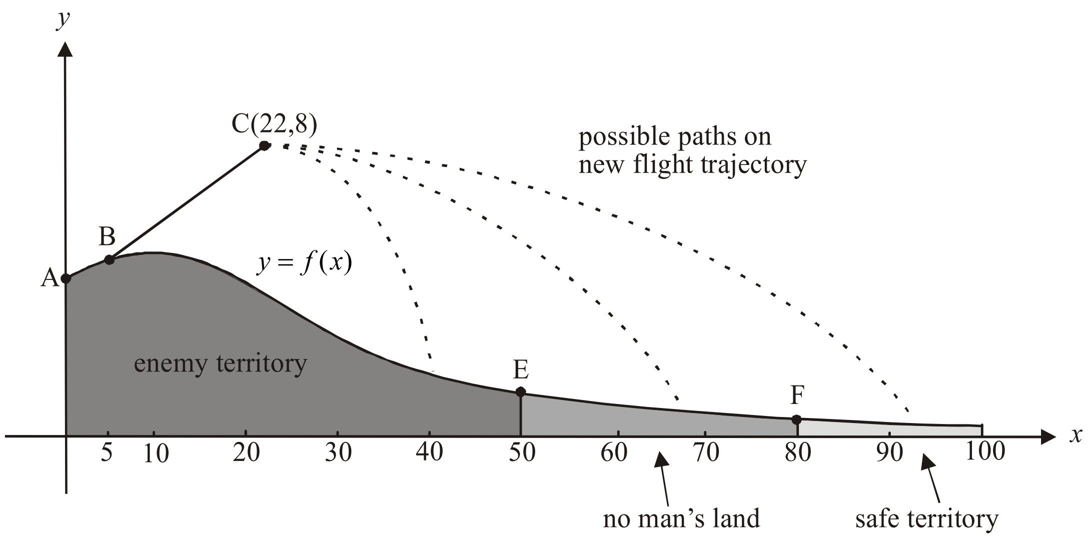
At the point where $x=22$, $f(22)=\frac{625}{161}$.

At point C, Victoria is $8-\frac{625}{161}=4.1\text{m (correct to 1 decimal place)}$ above the ground.

(1 mark)

f. The new flight trajectory is that of an inverted parabola with a turning point at (22,8). Therefore the maximum height above sea level that Victoria will have is 8m. (Note that this occurs at point C so Victoria loses altitude on this new flight trajectory).

(1 mark)



g. **i.** Point E is located at the point where $x = 50$.

$f(50) = \frac{25}{21}$

So E is the point $\left(50, \frac{25}{21}\right)$.

Solve $\frac{25}{21} = -\frac{1}{k}(50-22)^2 + 8$ for k.

$$\begin{aligned} k &= \frac{16\,464}{143} \\ &= 115.13 \text{ (correct to 2 decimal places)} \end{aligned}$$

(1 mark)

ii. Point F is located at the point where $x = 80$

$f(80) = \frac{25}{54}$

So F is the point $\left(80, \frac{25}{54}\right)$.

Solve $\frac{25}{54} = -\frac{1}{k}(80 - 22)^2 + 8$ for k

$$\begin{aligned} k &= \frac{181\,656}{407} \\ &= 446.33 \text{ (correct to 2 decimal places)} \end{aligned}$$

(1 mark)

The parameter k affects the 'width' of the parabola with equation
$y = -\frac{1}{k}(x-22)^2 + 8$.
Comparing the value of k in part **i.**, we see that as the parabola becomes 'wider' the value of k increases.
Safe territory extends to the right of F and beyond the point where $x = 100$.
So Victoria will land in safe territory for $k \geq 446.33$ (correct to 2 decimal places).

(1 mark)

h. Let (x', y') be an image point.

$$\begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} 0.8x \\ 0.4y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

So $\quad 0.8x = x'$

$$x = \frac{x'}{0.8}$$

and $\quad 0.4y = y'$

$$y = \frac{y'}{0.4}$$

(1 mark)

The path defined by $y = -\frac{1}{450}(x-22)^2 + 8$ becomes

$$\frac{y'}{0.4} = -\frac{1}{450}\left(\frac{x'}{0.8} - 22\right)^2 + 8$$

The image equation and hence the equation of the transformed flight path is

$$y = -\frac{0.4}{450}\left(\frac{x}{0.8} - 22\right)^2 + 8 \times 0.4$$

$$y = -\frac{1}{1125}\left(\frac{x}{0.8} - 22\right)^2 + 3.2$$

(1 mark)

i. Method 1

Find where the flight path hits the ground; that is,

solve $\quad f(x) = -\frac{1}{450}(x-22)^2 + 8 \quad$ for x

$x = -34.022....\quad$ or $\quad x = 80.249...$

Reject the first answer since $x > 0$. So $x = 80.249...$

(1 mark)

Since $x > 80$ and safe territory occurs for $x \geq 80$, Victoria lands in safe territory.

(1 mark)

Method 2

$f(80) = \frac{25}{54} = 0.46296$ so at point F the ground is 0.46296m above sea level.

On Victoria's flight path when $x = 80$, $y = \frac{118}{225} = 0.5244$ so at point F, Victoria is 0.5244 m above sea level. **(1 mark)**

Since $0.5244 > 0.46296$, Victoria is still above the ground at $x = 80$ so we know that she will land to the right of point F and therefore she will land in safe territory.

(1 mark)

Total 14 marks