INSIGHT
Trial Exam Paper

2010
MATHEMATICAL METHODS (CAS)
Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

a. Let $y = \dfrac{\sin 2x}{e^x}$. Find $\dfrac{dy}{dx}$.

2 marks

Solution

Using the quotient rule

$$u = \sin 2x, \quad \frac{du}{dx} = 2\cos 2x$$
$$v = e^x \quad , \quad \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{e^x \times 2\cos 2x - e^x \times \sin 2x}{e^{2x}} = \frac{2\cos 2x - \sin 2x}{e^x}$$

Mark allocation

- 1 method mark for knowing to use quotient rule
- 1 answer mark for correct derivative

b. If $f(x) = e^{\sqrt{x}}$, find $f'(16)$.

2 marks

Solution

Using chain rule gives $f'(x) = \dfrac{1}{2\sqrt{x}} e^{\sqrt{x}}$.

Substitute in $x = 16$, *gives* $\dfrac{1}{2\sqrt{16}} e^{\sqrt{16}} = \dfrac{1}{8} e^4$

Mark allocation

- 1 mark for correct chain rule
- 1 mark for correct answer

Question 2

a. Write $\frac{3x-1}{x+1}$ in the form $\frac{a}{x+1}+b$. State the values of a and b.

2 marks

Solution

Re-expression or otherwise gives

$$\frac{3x-1}{x+1}=\frac{3(x+1)-1-3}{x+1}=3-\frac{4}{x+1}=-\frac{4}{x+1}+3$$

so $a=-4,\ b=3$

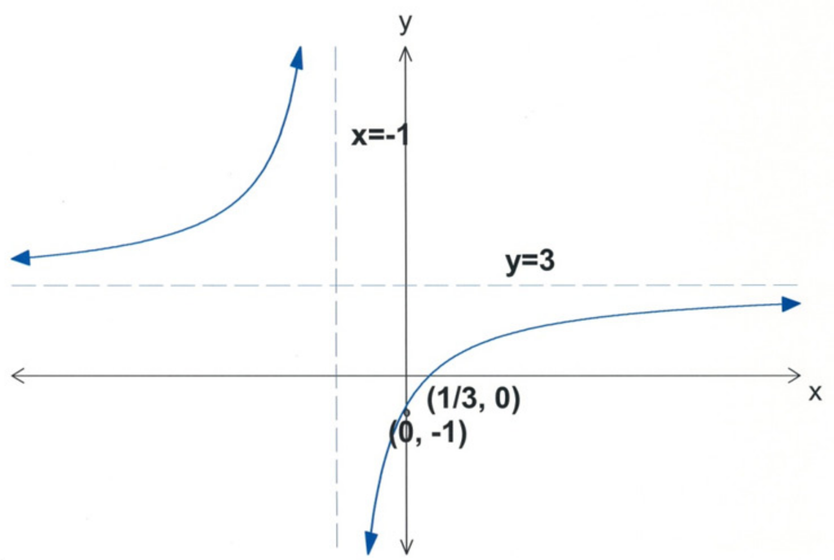
Mark allocation

- 1 mark for correct value of a
- 1 mark for correct value of b

b. Sketch the graph of $f:R\backslash\{-1\}\rightarrow R,\ f(x)=\frac{3x-1}{x+1}$. Label all axis intercepts as coordinates. Label each asymptote with its equation.

2 marks

Solution



Mark allocation

- 1 mark for shape
- 1 mark for all correct and labelled

Question 3

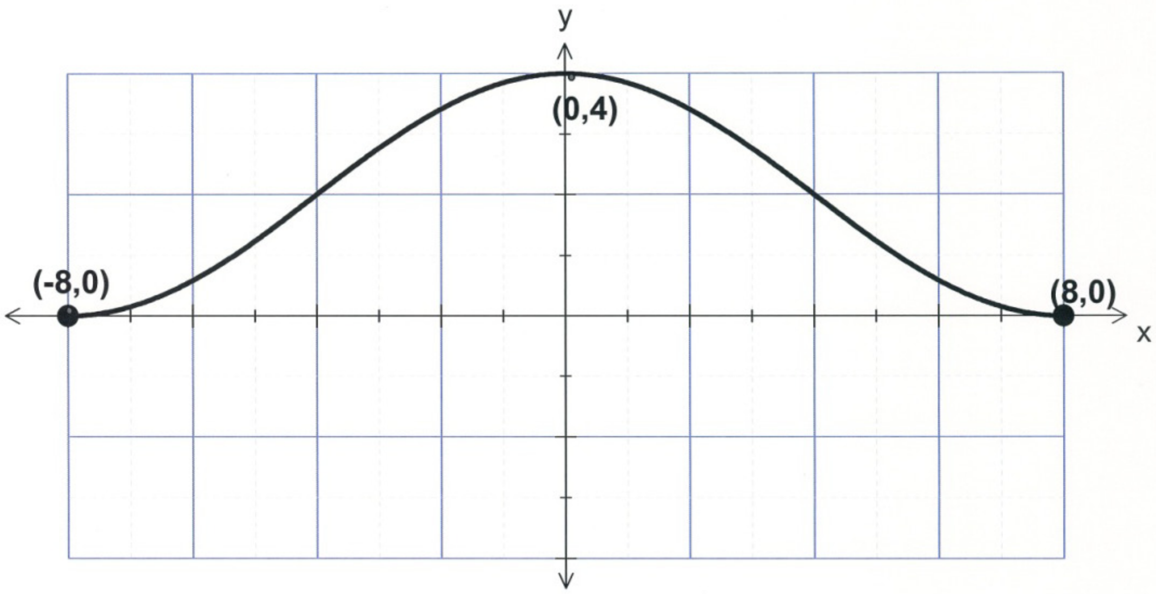
Sketch the graph of $y = 2\cos\left(\frac{\pi x}{8}\right) + 2$ for $x \in [-8, 8]$. Label axis intercepts and endpoints with their coordinates.

3 marks

Solution

amplitude is 2 and period $\frac{2\pi}{n} = \frac{2\pi}{\frac{\pi}{8}} = 16$

graph has been shifted up 2 units



Mark allocation

- 1 mark for correct shape and position of graph
- 1 mark for showing one cycle
- 1 mark for graph correct and points labelled

Question 4

Ciara is a very good netballer. She plays goal shooter in her team and knows that from past experience her probability of scoring a goal depends on the success of her previous attempt. She knows that if she has scored a goal previously then her probability of scoring a goal on the next attempt is 0.7. If she is unsuccessful on the previous attempt, her probability of being unsuccessful on the next attempt is 0.8.

The probabilities associated with each state are represented in the transition matrix:

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

a. If she has scored a goal, what is the chance of her not scoring on her next attempt?

1 mark

Solution

This corresponds to the element in the first column, bottom row of the matrix — 0.3

$$\begin{array}{cc} & \text{old} \\ & \begin{array}{cc} \text{G} & \text{M} \end{array} \\ \begin{array}{r} \text{new} \quad \text{G} \\ \text{M} \end{array} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

Mark allocation

- 1 mark for correct answer

b. She has five shots at goal. Her first attempt is a goal. What is the probability that it takes her until her final shot to score another goal?

2 marks

Solution

$$\Pr(GMMMG) = 1 \times 0.3 \times 0.8 \times 0.8 \times 0.2 = 0.0384$$

Mark allocation

- 1 mark for working
- 1 mark for answer

Question 4 – continued
TURN OVER

c. Find the probability of Ciara scoring a goal in the long term.

2 marks

Solution

This involves calculating the steady state probabilities

using the general matrix $\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$

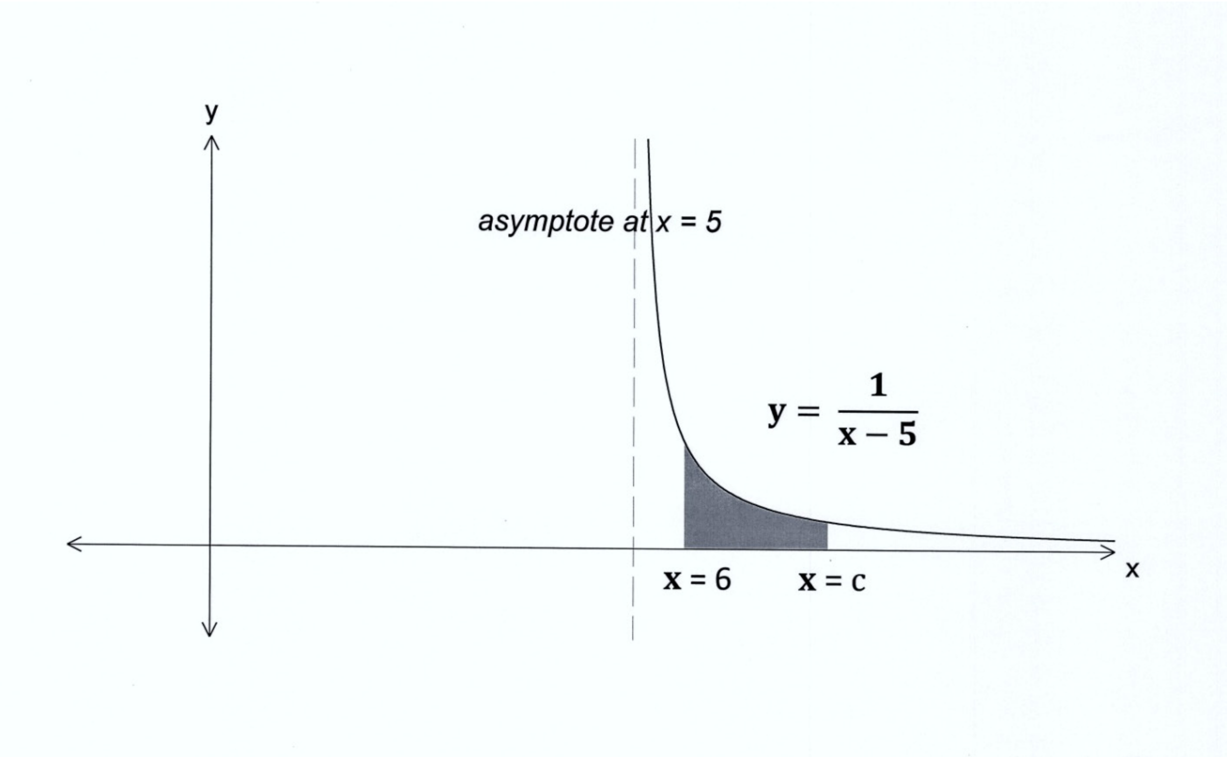
the $\Pr(G) = \dfrac{b}{a+b} = \dfrac{0.2}{0.2+0.3} = \dfrac{0.2}{0.5} = 0.4$

Mark allocation

- 1 mark for using steady state matrix
- 1 mark for answer

Question 5

The shaded area is equal to $\frac{1}{2}$. Find the value of c.



3 marks

Solution

Set up the integral gives

$$\int_6^c \frac{1}{x-5}\,dx = \frac{1}{2}$$

$$\left[\log_e|x-5|_6^c\right] = \frac{1}{2}$$

$$\log_e(c-5) - \log_e 1 = \frac{1}{2}$$

$$\log_e(c-5) = \frac{1}{2}$$

$$e^{\frac{1}{2}} = c-5$$

$$c = 5 + e^{\frac{1}{2}}$$

Mark allocation

- 1 mark for setting up correct integral
- 1 mark for correct antidifferentiation
- 1 mark for correct answer

Question 6

The duration of telephone calls to the home loan department of a mortgage broker is a random variable X minutes with probability density function

$$f(x)=\begin{cases}\frac{1}{4}e^{-\frac{x}{4}} & x>0\\ 0 & x\le 0\end{cases}$$

a. Find $\Pr(4<X<8)$

2 marks

Solution

$$\Pr(4<X<8)=\int_4^8 \frac{1}{4}e^{\frac{-x}{4}}\,dx=\left[-e^{\frac{-x}{4}}\right]_4^8=-e^{-2}+e^{-1}=\frac{1}{e}-\frac{1}{e^2}$$

Mark allocation

- 1 mark for setting up the correct integral
- 1 mark for correct answer

b. Find the probability that a telephone call will last more than 12 minutes.

2 marks

Solution

$$\begin{aligned}\Pr(X>12)&=1-\Pr(X<12)\\&=1-\int_0^{12}\frac{1}{4}e^{\frac{-x}{4}}\,dx\\&=1-\left[-e^{-\frac{x}{4}}\right]_0^{12}\\&=1-(-e^{-3}+e^{0})\\&=\frac{1}{e^3}\end{aligned}$$

Mark allocation

- 1 mark for setting up expression involving integral
- 1 mark for answer

Question 6 – continued

c. Find the value of a such that 90% of telephone calls last less than a minutes.

3 marks

Solution

$$\Pr(X < a) = 0.9$$

$$0.9 = \int_0^a \frac{1}{4} e^{\frac{-x}{4}} \, dx$$

$$= \left[-e^{\frac{-a}{4}} + e^0 \right]$$

$$-0.1 = -e^{\frac{-a}{4}}$$

$$0.1 = e^{\frac{-a}{4}}$$

$$\log_e 0.1 = \frac{-a}{4}$$

$$a = -4 \log_e 0.1$$

Mark allocation

- 1 mark for setting up an integral
- 1 mark for getting $e^{\frac{-a}{4}} = 0.1$
- 1 mark for answer

Question 7

Consider the function $f: R \to R,\ f(x) = 2x^3 e^{-4x}$

a. $f'(x)$ may be written in the form $f'(x) = e^{-4x}(ax^2 + bx^3)$ where *a* and *b* are real constants. Find the values of *a* and *b*.

2 marks

Solution

Using the product rule gives

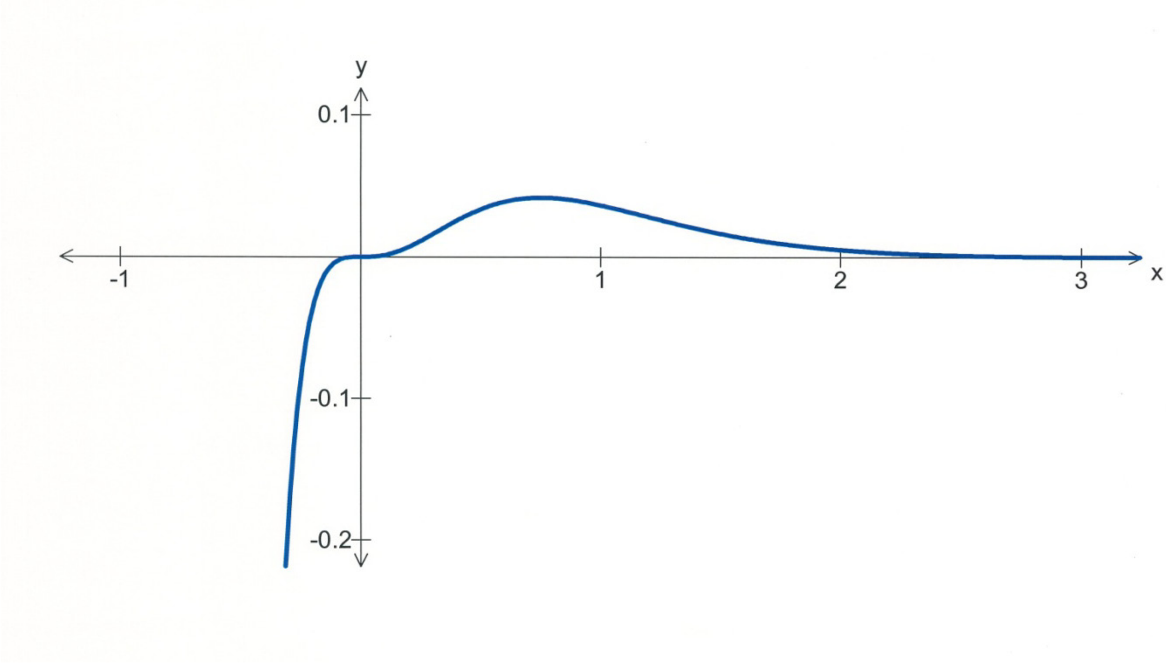
$$
\begin{aligned}
f'(x) &= 6x^2 e^{-4x} + -8x^3 e^{-4x} \\
&= e^{-4x}(6x^2 - 8x^3)
\end{aligned}
$$
$$\textit{so}\ \ a = 6 \text{ and } b = -8$$

Mark allocation

- 1 mark for using product rule correctly
- 1 mark for a and b both correct

Question 7 – continued

The graph of $y = f(x)$ is as shown.



b. Find the exact coordinates of the two stationary points and state their nature.

3 marks

Solution

let

$$f'(x) = 0$$
$$e^{-4x}(6x^2 - 8x^3) = 0$$
$$e^{-4x}x^2(6 - 8x) = 0$$
$$\text{gives } x = 0 \quad x = \frac{6}{8} = \frac{3}{4}$$

substitute back into equation for $f(x)$ to get y-coordinates.
stationary point of inflexion at $(0,0)$

maximum turning point at $(\frac{3}{4}, \frac{27}{32}e^{-3})$

Mark allocation

- 1 mark for both x values correct
- 1 mark for both y values correct
- 1 mark for nature of the points correct

c. Find the equation of the tangent to the curve $y = f(x)$ where $x = 1$.

2 marks

Solution

$$\begin{aligned}
&at\ x = 1,\ f'(1) = e^{-4}(6-8) = -2e^{-4} \\
&at\ x = 1,\ f(x) = 2e^{-4} \\
&y - y_1 = m(x - x_1) \\
&y - 2e^{-4} = -2e^{-4}(x-1) \\
&y = -2e^{-4}x + 4e^{-4} = \frac{-2x}{e^4} + \frac{4}{e^4}
\end{aligned}$$

Mark allocation

- 1 mark for finding gradient
- 1 mark for correct equation

Question 8

A transformation is defined by the matrix $\begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}$. Find the equation of the image of the graph of the line with the equation $y = 3x + 5$ under this transformation.

2 marks

Solution

Need to examine the images of the points (*x, y*) under the matrix transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

so multiplying out gives

$$\begin{aligned}
&x' = -2y \\
&y = \frac{-x'}{2} \\
&y' = 3x \\
&x = \frac{y'}{3}
\end{aligned}$$

so substituting into the equation

$$\begin{aligned}
&y = 3x + 5 \quad gives \quad \frac{-x'}{2} = 3\frac{y'}{3} + 5 \\
&so \quad \frac{-x'}{2} = y' + 5 \\
&or \quad y = \frac{-x}{2} - 5
\end{aligned}$$

Question 8 – continued

Mark allocation

- 1 mark for setting up matrices involving points and images
- 1 mark for answer

Question 9

For the simultaneous linear equations:

$$mx - 6y = 6$$
$$4x - my = m$$

find the values of m for which the equations have infinitely many solutions.

2 marks

Solution

In both equations re-write the equations in the form $y = mx + c$

This gives

$$y = \frac{m}{6}x - 1$$
$$y = \frac{4}{m}x - 1$$

For there to be an infinite number of solutions, the equations need to be identical.

So $\frac{m}{6} = \frac{4}{m}$

$$m^2 = 24$$
$$m = \pm\sqrt{24}$$
$$m = \pm 2\sqrt{6}$$

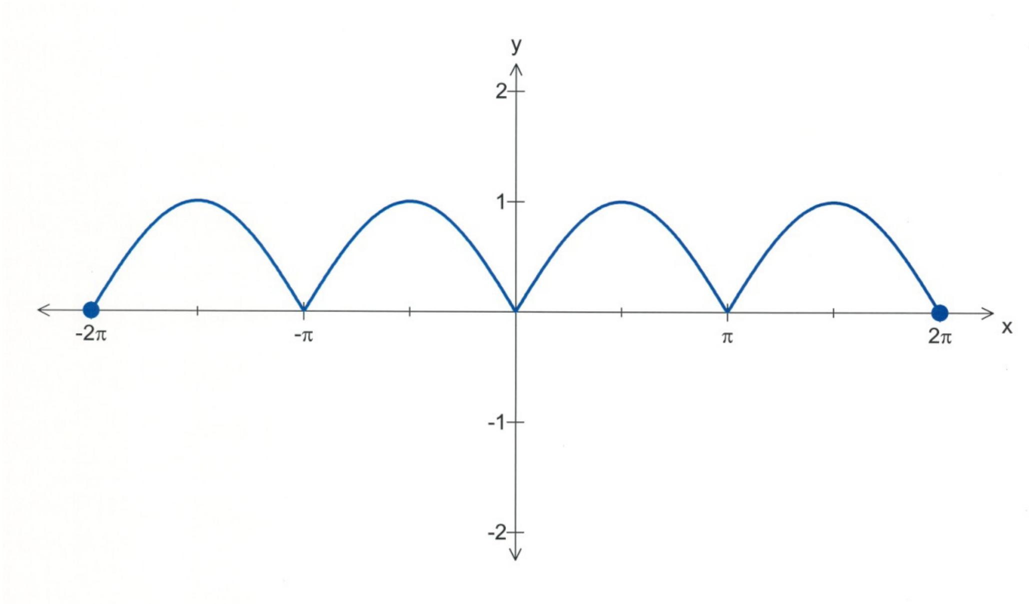
Mark allocation

- 1 mark for getting $\frac{m}{6} = \frac{4}{m}$
- 1 mark for answer

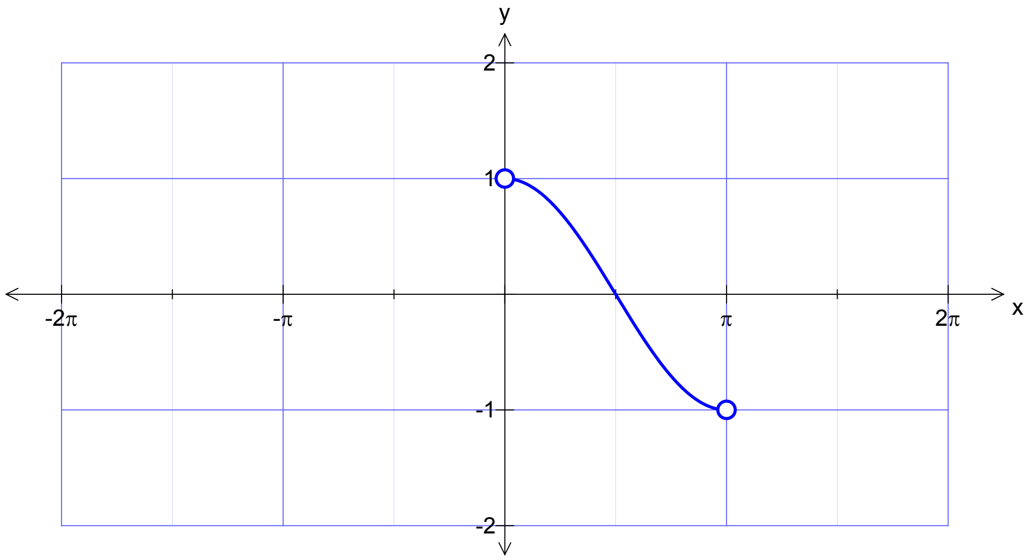
TURN OVER

Question 10

The graph of the function $f:[-2\pi, 2\pi] \to R,\ \ f(x) = |\sin(x)|$ is shown below



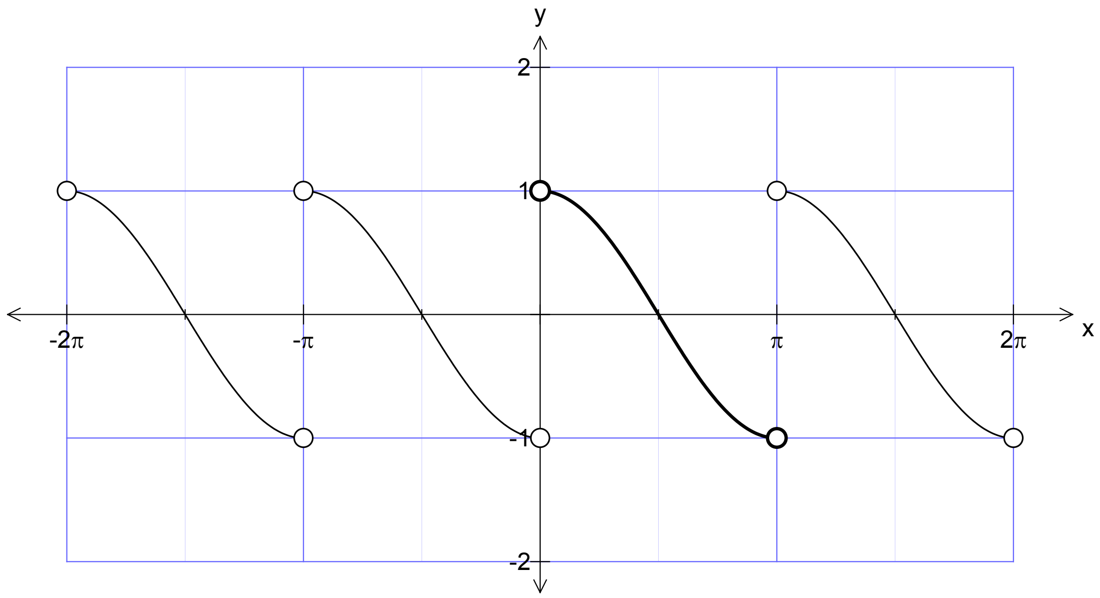
a. Part of the graph of the derivative function is drawn on the axes given. Complete the graph of the derivative function, f' on the axes given.



1 mark

Question 10 – continued

Solution



Mark allocation

- 1 mark for correct answer

b. State the rule for the derivative function.

2 marks

Solution

$$f'(x) = \begin{cases} \cos x, \ x \in (0, \pi) \cup (-2\pi, -\pi) \\ -\cos x, x \in (\pi, 2\pi) \cup (-\pi, 0) \end{cases}$$

Mark allocation

- 1 mark for indicating a cos function with domain $(0, \pi)$
- 1 mark for correct hybrid function

END OF SOLUTIONS PAPER