INSIGHT

Trial Exam Paper

2010

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

SECTION 1

Question 1

The smallest positive value of x for which $\tan 4x = 1$ is

A. π

B. $\frac{\pi}{4}$

C. $\frac{\pi}{8}$

D. $\boldsymbol{\frac{\pi}{16}}$

E. 2π

Answer is D

Solution

$\tan 4x = 1$

$4x = \frac{\pi}{4}$

$x = \frac{\pi}{16}$

Question 2

The graph with the equation $y = |x-2| + 4$ is reflected in the y-axis and then translated 1 unit down. The resulting graph has the equation

A. $y = -|x-2| + 3$

B. $y = |-x+2| + 3$

C. $y = -|x+2| - 1$

D. $\boldsymbol{y = |x+2| + 3}$

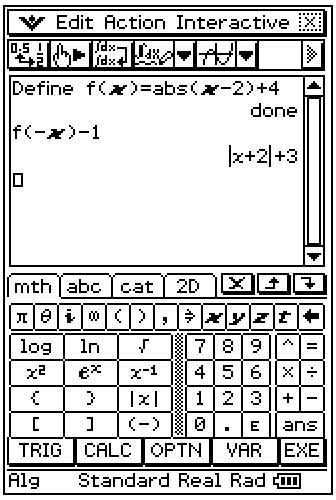
E. $y = |x+2| - 5$

Answer is D

Section 1 – Question 2 – continued

Solution

Reflecting in the y-axis produces $f(-x)$. Shifting down 1 unit gives $f(-x)-1$.
Using CAS to simplify gives



Question 3

The range of the function $f:[-5, 4) \to R,\ f(x) = (x-3)^{\frac{2}{3}}$ is

A. $[3,\infty)$

B. $[-5,4)$

C. $\mathbf{[0,4]}$

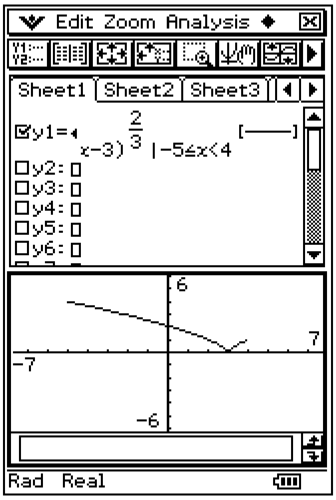
D. $(0,4)$

E. $(0,4]$

Answer is C

Solution

Looking at the graph the lowest point is given by the vertex at (3, 0) and the highest point at the left endpoint ie. When $x=-5,\ y=(-5-3)^{\frac{2}{3}}=(-8)^{\frac{2}{3}}=4$



so range is [0, 4].

Question 4

For the system of equations

$$z = 1$$
$$y - x = 2$$
$$x + y = 5$$

An equivalent matrix representation is

A.

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

B.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

C.

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

D.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

E.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Answer is A

Solution

Multiplying the rows and columns in the matrices in alternative A gives the stated equations.

Section 1 – continued

Question 5

A bag contains 4 white, 3 red and 3 black balls. Two balls are selected without replacement. The probability that they are both the same colour is

A. $\frac{2}{45}$

B. $\frac{6}{25}$

C $\frac{8}{15}$

D. $\frac{1}{15}$

E. $\mathbf{\frac{4}{15}}$

Answer is E

Solution

Both the same colour means they can be 2 white or 2 red or 2 black balls.

$$\begin{aligned}\Pr(\textit{same colour}) &= \Pr(WW) + \Pr(RR) + \Pr(BB) \\ &= \frac{4}{10}\times\frac{3}{9}+\frac{3}{10}\times\frac{2}{9}+\frac{3}{10}\times\frac{2}{9}=\frac{24}{90}=\frac{4}{15}\end{aligned}$$

Question 6

The function $f : B \to R,\ f(x) = \left|\log_e(2x^5)\right|$ will have an inverse function if

A. $B = [2^{\frac{1}{5}}, \infty)$

B. $\boldsymbol{B = [2^{-\frac{1}{5}}, \infty)}$

C. $B = [-2^{\frac{1}{5}}, \infty)$

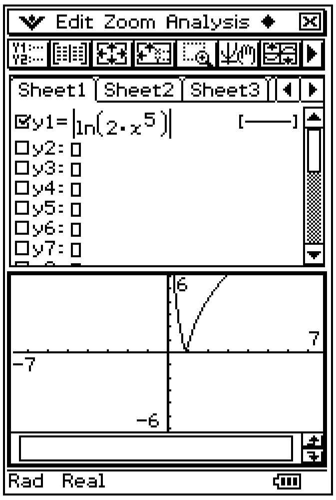
D. $B = [0, \infty)$

E. $B = (0, \infty)$

Answer is B

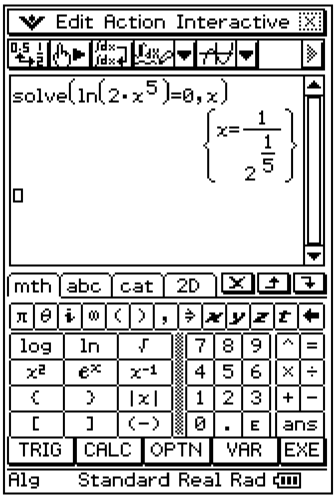
Solution

For an inverse function to exist the function must be one to one. Look at the graph and determine the domain so that the function passes the horizontal line test.



Graph is one to one from the x-intercept onwards. To find the x-intercept solve $\left|\log_e(2x^5)\right| = 0$ *or* $\log_e(2x^5) = 0.$

Using CAS this gives



so suitable domain is $[2^{-\frac{1}{5}}, \infty)$

Question 7

The function g has the rule $g(x)=\sqrt{b-ax}$ for $a,b>0$. The maximal domain of g is

A. $x \le \frac{a}{b}$

B. $x > \frac{b}{a}$

C. $x < \frac{b}{a}$

D. $x \ge \frac{b}{a}$

E. $\boldsymbol{x \le \frac{b}{a}}$

Answer is E

Solution

Maximal domain is for $b-ax \ge 0$

$$b \ge ax$$

$$x \le \frac{b}{a}$$

Question 8

The average value of the function $y = \sin(2x)$ over the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$ is

A. $\frac{3}{\pi}$

B. $\frac{6}{\pi}$

C. $\frac{3}{2\pi}$

D. $\frac{1}{\pi}$

E. $\frac{1}{2}$

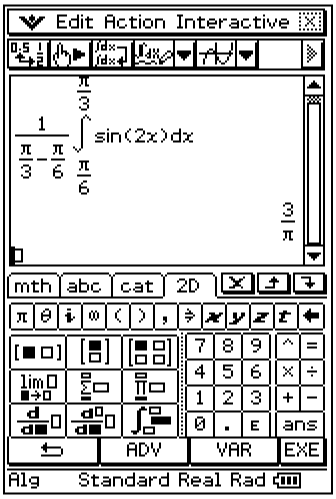
Answer is A

Solution

The average value of a function is defined as

$\frac{1}{b-a}\int_a^b f(x)\,dx$ for a function $f(x)$ $x \in [a,b]$

so the average value of this function is $\frac{1}{(\frac{\pi}{3}-\frac{\pi}{6})}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(2x)\,dx$. Using CAS this equals



Question 9

If $y = 5a^{3x} - 2b$, then x equals

A. $\frac{1}{3}\log_a(5(y+2b))$

B. $\frac{1}{3}\log_a(\frac{y}{5}) + 2b$

C. $\frac{y+2b}{15a}$

D. $\frac{1}{15}\log_a(y+2b)$

E. $\mathbf{\frac{1}{3}\log_a(\frac{y+2b}{5})}$

Answer is E

Solution

Making x the subject gives

$$5a^{3x} = y + 2b$$

$$a^{3x} = \frac{y+2b}{5}$$

$$3x = \log_a \frac{y+2b}{5}$$

$$x = \frac{1}{3}\log_a \frac{y+2b}{5}$$

Question 10

The radius of a sphere is increasing at a rate of 4 mm/min.
When the radius is 5 mm, the rate at which the surface area is increasing, in mm^2/min is

A. ***160π***

B. 32π

C. 400π

D. 160

E. 100π

Answer is A

Solution

To solve use the rate equation $\frac{dS}{dt} = \frac{dr}{dt} \times \frac{dS}{dr}$ with $S = 4\pi r^2$ *so* $\frac{dS}{dr} = 8\pi r$.
This gives

$$\frac{dS}{dt} = 4 \times 8\pi r$$

$$\textit{with } r = 5 \textit{ then}$$

$$\frac{dS}{dt} = 160\pi$$

Question 11

The graph of $y = kx - 5$ intersects the graph of $y = x^2 - 6x$ at two distinct points for

A. $\{k : -2\sqrt{5} - 6 < k < 2\sqrt{5} + 6\}$

B. $\{k : k < 2\sqrt{5} + 6\}$

C. $\boldsymbol{\{ k : -2\sqrt{5} - 6 > k \} \cup \{ k : k > 2\sqrt{5} - 6 \}}$

D. $\{k : k > -2\sqrt{5} - 6\}$

E. $\{k : k = -2\sqrt{5} - 6\} \cup \{k : k = 2\sqrt{5} + 6\}$

Answer is C

Solution

The graphs intersect when

$$x^2 - 6x = kx - 5$$

$$x^2 + x(-6 - k) + 5 = 0$$

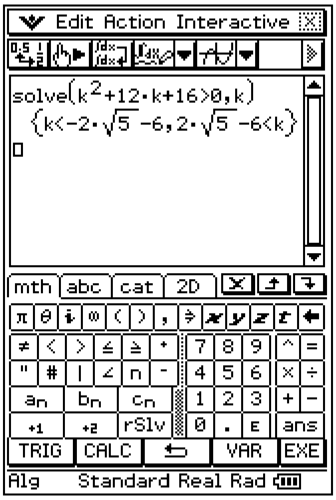
Section 1 – Question 11 – continued

for two solutions want $\Delta > 0$ so

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-6-k)^2 - 4.1.5 \\ &= 36 + 12k + k^2 - 20 \\ &= k^2 + 12k + 16\end{aligned}$$

so $k^2 + 12k + 16 > 0$

using CAS this gives



Question 12

A fair die is tossed 8 times.
The probability of getting a six at least once is

A. $\boldsymbol{1-(\frac{5}{6})^8}$

B. $1-\left(\frac{1}{6}\right)^8$

C. $\frac{5}{6}$

D. $\frac{1}{6}$

E. $1-(\frac{5}{6})^7$

Answer is A

Solution

$$\begin{aligned}\Pr(\textit{at least once}) &= 1 - \Pr(\textit{not at all}) \\ &= 1 - \Pr(\textit{no sixes}) \\ &= 1 - \left(\frac{5}{6}\right)^8\end{aligned}$$

Question 13

Which one of the following statements is **not** true about the function $f: R \to R,\ f(x) = |3x - 5|$.

A. The graph of f is continuous everywhere

B. $\boldsymbol{f'\left(\frac{5}{3}\right) = 0}$

C. $f(x) \geq 0$ for all x

D. $f'(x) > 0$ for all $x > \frac{5}{3}$

E. $f'(x) < 0$ for all $x < 0$

Answer is B

Solution

The graph is not differentiable at the vertex at $x = \frac{5}{3}$

Question 14

Let $f: R \to R$ be a differentiable function.
The derivative of $f(\sqrt{\sin x})$ is

A. $f'(\sqrt{\sin x})$

B. $\sqrt{\cos x}\, f'(\sqrt{\sin x})$

C. $\frac{\cos x}{2\sqrt{\sin x}} f'(\sqrt{\cos x})$

D. $\boldsymbol{\frac{\cos x}{2\sqrt{\sin x}} f'(\sqrt{\sin x})}$

E. $\frac{\sin x}{2\sqrt{\cos x}} f'(\sqrt{\sin x})$

Answer is D

Solution

Using the chain rule, $\frac{d}{dx}(f(g(x)) = g'(x) f'(g(x))$

$$\frac{d}{dx} f(\sqrt{(\sin x)}) = f'(\sqrt{\sin x}) \times \frac{d}{dx}\sqrt{\sin x}$$

$$= f'(\sqrt{\sin x}) \times \frac{1}{2\sqrt{\sin x}} \cos x$$

Question 15

The system of linear simultaneous equations

$$5x + ay = 0$$
$$(2a+1)x + 2y = 0$$

where a is a real constant, has infinitely many solutions for

A. $\boldsymbol{a \in \{\frac{-5}{2}, 2\}}$

B. $a \in R \setminus \{\frac{-5}{2}, 2\}$

C. $a \in \{\frac{5}{2}, -2\}$

D. $a \in R \setminus \{\frac{5}{2}, -2\}$

E. $a \in R$

Answer is A

Solution

Best to look at the determinant
$\det A = ad - bc = 10 - a(2a+1) = 10 - 2a^2 - a$.
When $\det A = 0$, system will have either infinitely many solutions or no solutions.

Let

$$10 - 2a^2 - a = 0$$
$$2a^2 + a - 10 = 0$$
$$a = \frac{-5}{2}, 2$$

Check—

$a = \frac{-5}{2}$, gives $2x - y = 0$ (one line) $\therefore$ infinite solutions
and $a = 2$ gives $5x + 2y = 0$ (one line) $\therefore$ infinite solutions

Question 16

The transformation $T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ maps the curve with the equation $y = x^2$ to the curve with the equation

A. $y = \dfrac{-5(x+2)^2}{9} - 7$

B. $y = \dfrac{-5(x-2)^2 + 7}{3}$

C. $y = \dfrac{-5(x+2)^2 + 7}{3}$

D. $\boldsymbol{y = \dfrac{-5(x-2)^2}{9} + 7}$

E. $y = \dfrac{-9(x-2)^2}{5} + 7$

Answer is D

Solution

Expanding the matrix gives

$$\left.\begin{aligned} x' &= 3x + 2 \\ y' &= -5y + 7 \end{aligned}\right\}$$

$$\Rightarrow \frac{x'-2}{3} = x \quad and \quad \frac{y'-7}{-5} = y$$

substitute into $y = x^2$ to get

$$\frac{y-7}{-5} = \left(\frac{x-2}{3}\right)^2$$

$$so \quad y = \frac{-5(x-2)^2}{9} + 7$$

Question 17

The graph of $y = x^3 - bx$ for $b \in R^+$ has turning points at $x = 3$ and $x = -3$.
The function $y = |x^3 - bx|$ is an increasing function for

A. $x \in R$

B. $x \in \{x : -3 \le x \le 3\}$

C. $\boldsymbol{x \in \{x : -\sqrt{b} \le x \le -3\} \cup \{x : 0 \le x \le 3\} \cup \{x : x \ge \sqrt{b}\}}$

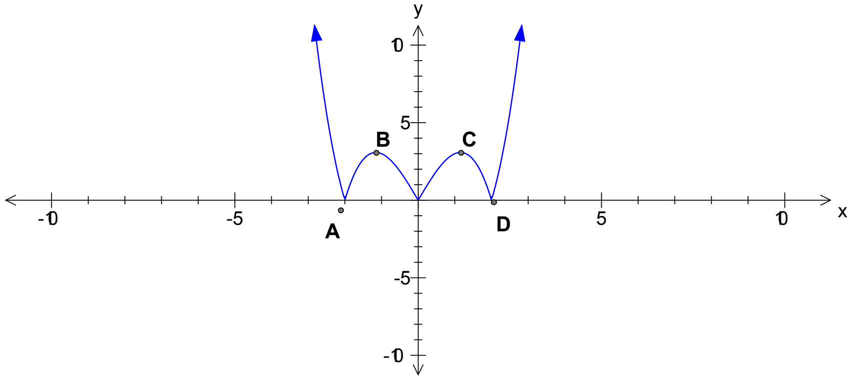
D. $x \in \{x : x \le -3\} \cup \{x : 0 \le x \le 3\} \cup \{x : x \ge \sqrt{b}\}$

E. $x \in \{x : -\sqrt{b} \le x \le -3\} \cup \{x : x \ge \sqrt{b}\}$

Answer is C

Solution

Given that the turning points are at x=3 and x=-3
I found dy/dx and given that dy/dx =0 when x=3 found that b=27 (similarly for x=-3) so I actually graphed the exact function required
Consider the cubic graph $y = |x^3 - bx|$ sketched as an example with $b = 4$



Factorising $|x^3 - bx|$ gives $|x^3 - bx| = |x(x^2 - b)| = |x(x - \sqrt{b})(x + \sqrt{b}|$. This means there are *x*-intercepts at $x = 0,\ x = \sqrt{b},\ x = -\sqrt{b}$

The graph is increasing (has a positive gradient) from the intercept at A ($(-\sqrt{b}, 0)$ to the turning point at B ($x = -3$), from the origin to the turning point at C ($x = 3$)and then from the intercept at D $(\sqrt{b}, 0)$ onwards.

Alternatively—

The value of b can be calculated. For $y = x^3 - bx$, $\dfrac{dy}{dx} = 3x^2 - b$ and $\dfrac{dy}{dx} = 0$, at $x = 3$ so

$3x^2 - b = 0$

$3(3)^2 - b = 0$

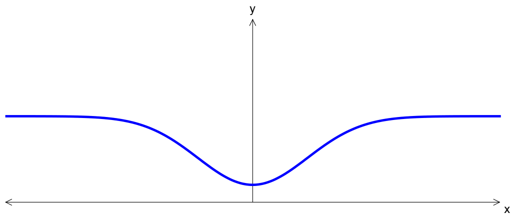
$27 - b = 0$

$b = 27$

With this knowledge the graph of $y = |x^3 - 27x|$ can be sketched on the calculator and the values for where the curve is increasing can be determined as above.

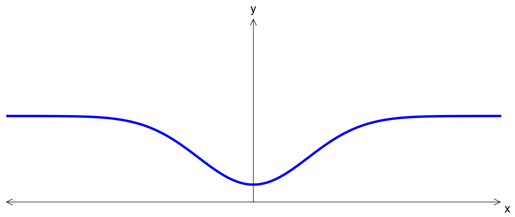
Question 18

The graph of $y = f(x)$ is shown

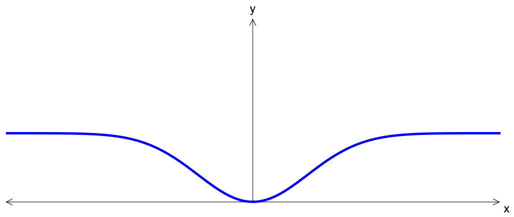


The graph of an antiderivative of *f* could be

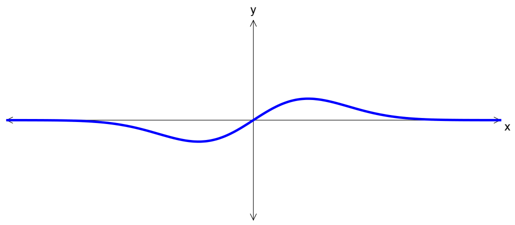
A.



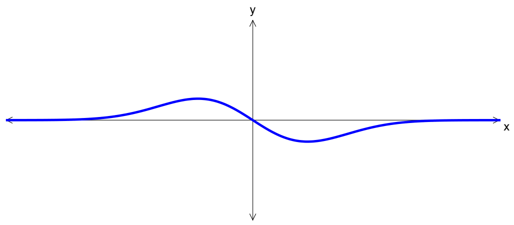
B.



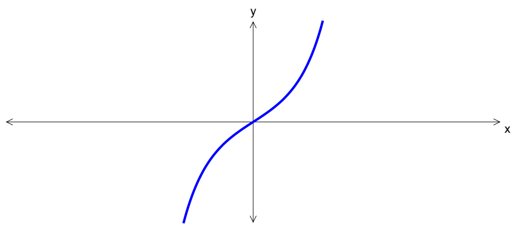
C.



D.



E.



Answer is E

Solution

The graph of the antiderivative would be a graph whose gradient, as you move left to right, is a positive constant, then loses some of its steepness to a smaller positive value, then increases steepness again to another positive value.

Question 19

Phil likes to drink both coffee and tea. He drinks one cup of either each day and likes to switch from day to day. If he drinks coffee one day he drinks coffee the next with a probability of 0.65. If he drinks tea one day he drinks coffee the next with a probability of 0.3. The transition matrix T representing the situation is

A. $\begin{bmatrix} 0.65 & 0.7 \\ 0.35 & 0.3 \end{bmatrix}$

B. $\begin{bmatrix} 0.35 & 0.3 \\ 0.65 & 0.7 \end{bmatrix}$

C. $\begin{bmatrix} \boldsymbol{0.65} & \boldsymbol{0.3} \\ \boldsymbol{0.35} & \boldsymbol{0.7} \end{bmatrix}$

D. $\begin{bmatrix} 0.65 & 0.35 \\ 0.3 & 0.7 \end{bmatrix}$

E. $\begin{bmatrix} 0.65 \\ 0.3 \end{bmatrix}$

Answer is C

Solution

Let C be the probability he drinks coffee, and T be the probability he drinks tea.
The information presented gives $\Pr(C \mid C) = 0.65$ *and* $\Pr(C \mid T) = 0.3$.

So
$\begin{bmatrix} C \mid C & C \mid T \\ T \mid C & T \mid T \end{bmatrix}$ becomes $\begin{bmatrix} 0.65 & 0.3 \\ T \mid C & T \mid T \end{bmatrix}$.

The columns need to add to one so the missing values become $\begin{bmatrix} 0.65 & 0.3 \\ 0.35 & 0.7 \end{bmatrix}$.

Question 20

Let $f : R \to R,\ f(x) = e^x + e^{-x}$.
Then $[f(u)]^2 =$

A. $f(2u)$

B. $f(u^2)$

C. $\boldsymbol{f(2u) + 2}$

D. $2f(u)$

E. $f(2u) - 2$

Answer is C

Solution

$$[f(u)]^2 = (e^u + e^{-u})^2 = e^{2u} + 2e^u e^{-u} + e^{-2u} = e^{2u} + e^{-2u} + 2$$
$$= f(2u) + 2$$

Question 21

The random variable X has a normal distribution with a mean of 20 and a standard deviation of 0.5.
If the random variable Z has the standard normal distribution, then the probability that X is greater than 21.5 is

A. $1-\Pr(Z>3)$

B. $1-\Pr(Z<-3)$

C. $\Pr(Z<3)$

D. ***$\Pr(Z<-3)$***

E. $\Pr(-3<Z<3)$

Answer is D

Solution

$X \sim N(\mu=20, \sigma=0.5)$ $\Pr(X>21.5)$ convert to standard normal gives

$\Pr(Z>\frac{21.5-20}{0.5})=\Pr(Z>3)$. Using symmetry this is equivalent to $\Pr(Z<-3)$.

Question 22

The probability density function for a continuous random variable X is given by

$$f(x)=\begin{cases}|1-x| & for\ 0<x<2\\ 0 & otherwise\end{cases}$$

The expected value $E(X)$ is

A. ***1***

B. $\frac{1}{2}$

C. $\frac{1}{6}$

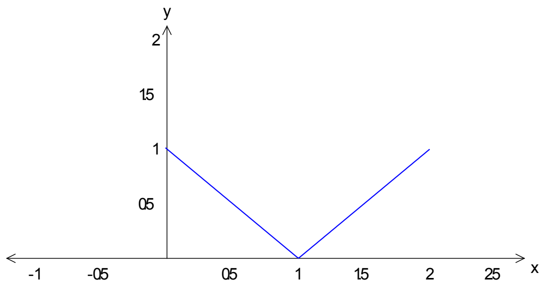
D. $\frac{1}{4}$

E. 0

Answer is A

Solution

Graph of the distribution gives



The distribution is symmetrical about $x=1$, therefore E(x)=1

END OF SECTION 1

SECTION 2

Question 1

Let $f:[0,\infty)\to R,\ f(x)=e^{\frac{x}{3}}-2$.

a. **i.** State the range of *f.*

1 mark

Solution
range is $[-1,\infty)$

note—need to look at the domain of function.

Mark Allocation

- 1 mark for answer

ii. Explain why f^{-1}, the inverse function of *f,* exists.

1 mark

Solution

The inverse function exists because the original function is one-to-one

Mark allocation

- 1 mark for answer

SECTION 2 – Question 1 – continued

iii. Find the rule for f^{-1} and state the domain.

2 marks

Solution

domain of f^{-1} = range of f = $[-1, \infty)$

to find equation swap x and y.

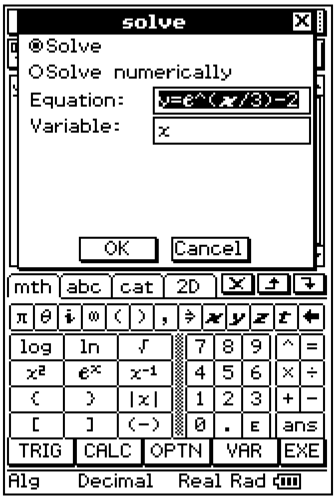
$$x = e^{\frac{y}{3}} - 2$$

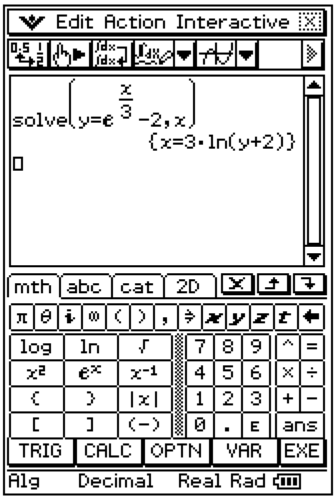
$$x + 2 = e^{\frac{y}{3}}$$

$$\log_e(x+2) = \frac{y}{3}$$

$$y = 3\log_e(x+2)$$

or using CAS



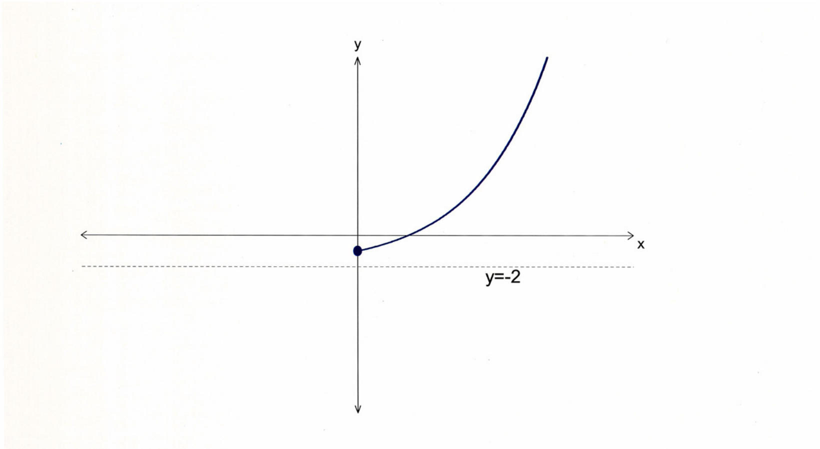


Mark allocation

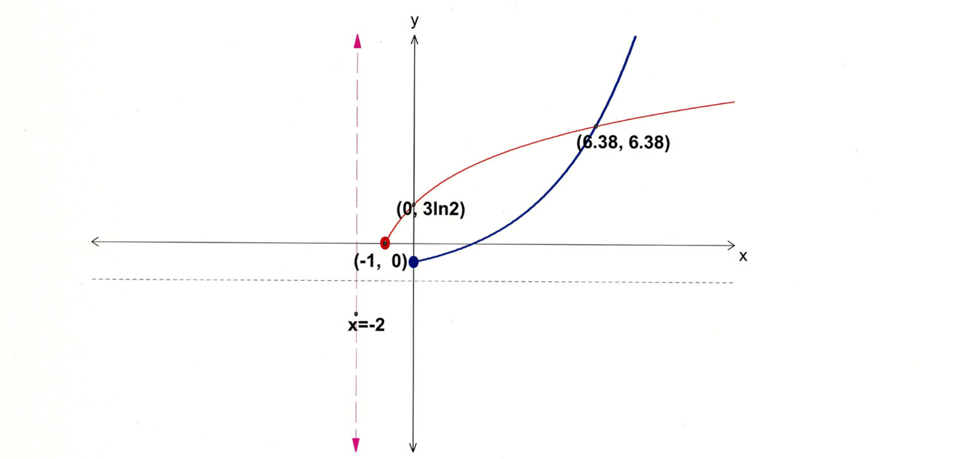
- 1 mark for equation
- 1 mark for domain

iv. The graph of $y = f(x)$ is shown below. On the same axes sketch the graph of $y = f^{-1}(x)$. Label any asymptote with its equation and any endpoint or intercept with its exact coordinates. Label the point of intersection between the two graphs as a coordinate correct to 2 decimal places.

2 marks



Solution

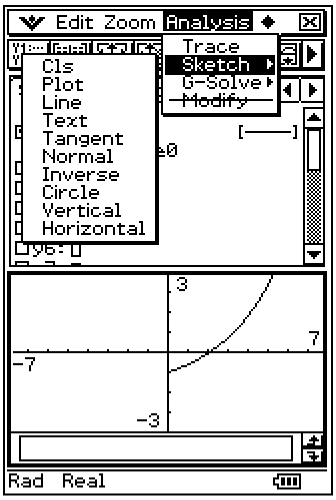


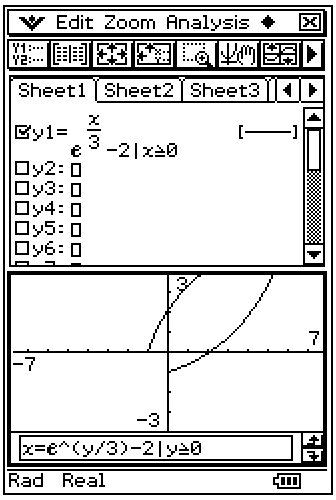
Mark allocation

- 1 mark for shape
- 1 mark for all labels correct

Tip

- *CAS can help here—use the graph screen, enter the original equation of f(x) and select draw inverse*





b. Let $g:[0, \infty) \to R,\ g(x) = x^2(e^{\frac{x}{a}} - b)$.

i. It is found that when $x = 1,\ g(x) = -3$ and when $x = 2,\ g(x) = -9$. Find the exact values of a and b.

3 marks

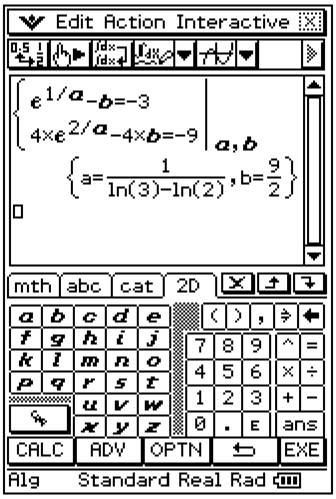
Solution

for $x = 1,\ y = -3$ $\quad -3 = e^{\frac{1}{a}} - b$

when $x = 2,\ y = -9$ $\quad -9 = 4(e^{\frac{2}{a}} - b)$

then solve simultaneously

using CAS this gives



so $x = \dfrac{1}{\log_e(\frac{3}{2})},\ y = \dfrac{9}{2}$

Mark allocation

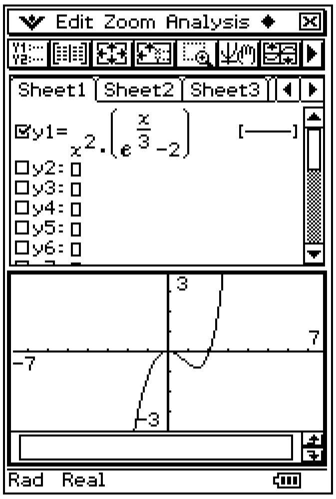
- 1 method mark for writing equations with values of x and y
- 1 method mark for using simultaneous equations
- 1 answer mark for correct answers

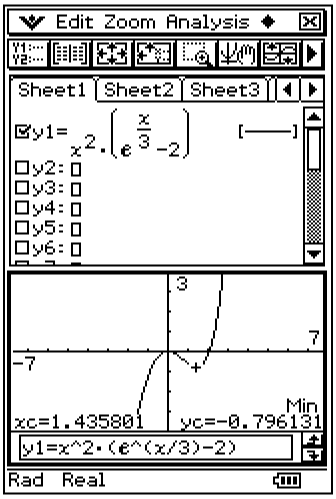
ii. For $a = 3$ *and* $b = 2$, $g(x) = x^2(e^{\frac{x}{3}} - 2)$. Find the least value of k, correct to 4 decimal places, such that $g : [k, \infty) \to R$, $g(x) = x^2(e^{\frac{x}{3}} - 2)$ is a one to one function.

2 marks

Solution

Need to look at the graph to determine where the graph is one to one.





The graph has a minimum turning point at approximately $x = 1.4358$ so it will be one-to-one from this point on.

so $k = 1.4358$

Mark allocation

- 1 mark for finding turning point
- 1 mark for answer

c. For the function $q : R \to R$, $q(x) = (x^2 - bx + 6)e^{\frac{x}{3}}$ the derivative is

$q'(x) = \dfrac{(x^2 + x(6 - b) + 6 - 3b)e^{\frac{x}{3}}}{3}$. Show that the graph of $y = q(x)$ will have 2 stationary points for all values of b.

3 marks

Solution

For stationary points $q'(x)=0$ so $x^2+x(6-b)+(6-3b)=0$.
For 2 solutions $\Delta>0$ so find Δ.

$$a=1,\ b=6-b,\ c=6-3b$$
$$\Delta=b^2-4ac$$
$$=(6-b)^2-4(6-3b)$$
$$=36-12b+b^2-24+12b$$
$$=b^2+12$$

$b^2+12>0$ for all values of *b,* so $\Delta>0$ for all values of *b,* so 2 solutions (2 stationary points) for all values of *b.*

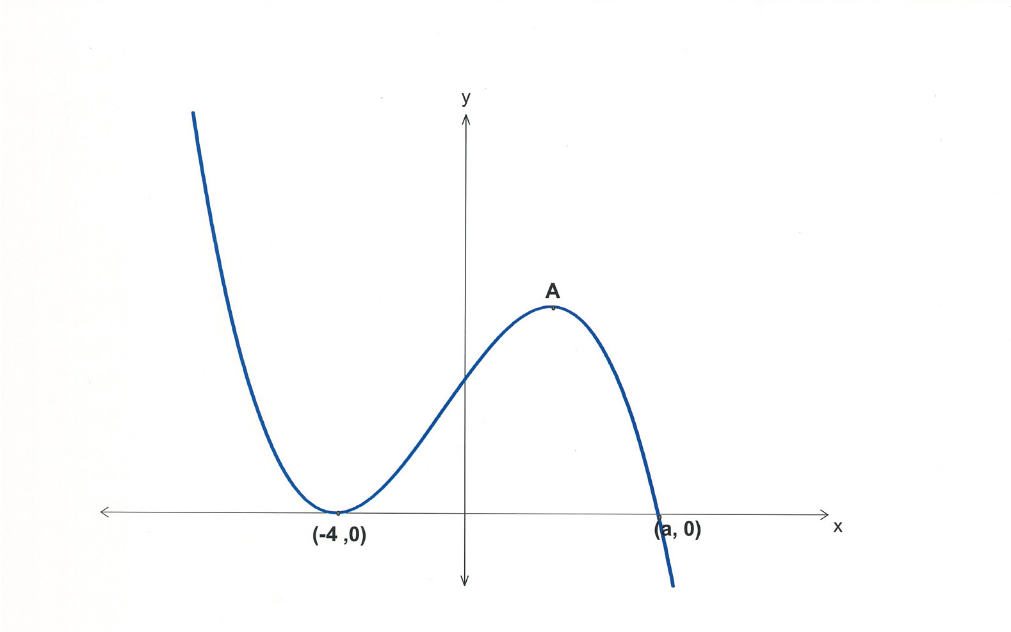
Mark allocation

- 1 method mark for knowing to find Δ
- 1 mark for finding Δ
- 1 mark for conclusion $b^2+12>0$

Total 14 marks

Question 2

The graph of $y = (x+4)^2(a-x)$ for $a \in (-4, 10]$ is shown below.

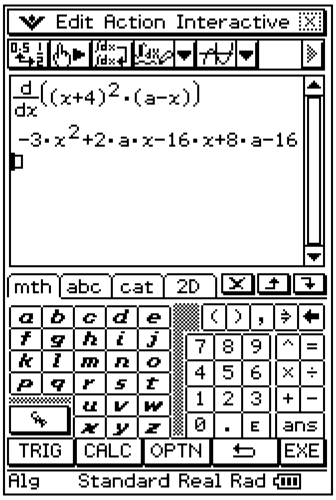


a. Find $\dfrac{dy}{dx}$.

1 mark

Solution

use CAS or otherwise



so $\dfrac{dy}{dx} = -3x^2 + 2ax - 16x + 8a - 16$

Mark allocation

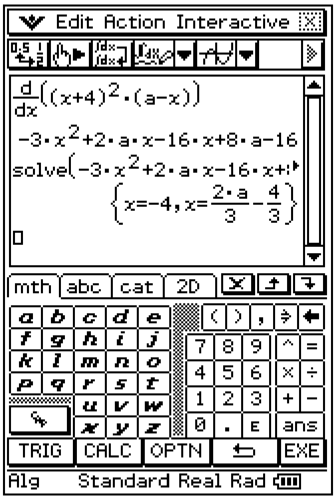
- 1 mark for answer

b. Find the coordinates of the turning points.

2 marks

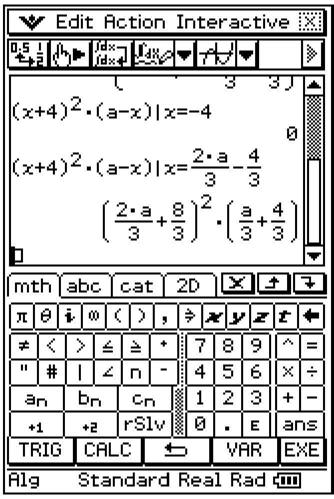
Solution

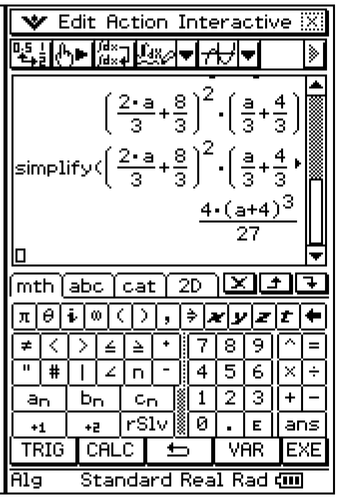
let $\frac{dy}{dx} = 0$ gives



so $x = -4$ and $x = \frac{2a-4}{3}$

substitute in to find y values





so the coordinates $(-4, 0)$ *and* $(\frac{2a-4}{3}, \frac{4(a+4)^3}{27})$

Mark allocation

- 1 mark for finding x values
- 1 mark for finding y values

c. Find the value of a such that the maximum turning point at A lies on the y-axis. State the coordinates of the maximum turning point.

2 marks

Solution

need $x = 0$ so $2a - 4 = 0$ therefore $a = 2$

coordinates of point are (0, 32)

Mark allocation

- 1 mark for $a = 2$
- 1 mark for coordinates

d. For $y = f(x)$, find the exact value of a such that the equation $f(x) = 5$ has exactly two solutions.

2 marks

Solution

Want the turning point to lie on line $y = 5$ so the y-coordinate of the turning point needs to equal 5.

$$\frac{4(a+4)^3}{27} = 5$$

therefore

$$4(a+4)^3 = 135$$

$$(a+4)^3 = \frac{135}{4}$$

$$a = \sqrt[3]{\frac{135}{4}} - 4$$

Mark allocation

- 1 mark for equating $\frac{4(a+4)^3}{27} = 5$
- 1 mark for $a = \sqrt[3]{\frac{135}{4}} - 4$

e. **i.** Write a definite integral that gives the area bounded by the graph and the x-axis.

1 mark

Solution

$$\int_{-4}^{a}(x+4)^2(a-x)\,dx$$

Mark allocation

- 1 mark for answer

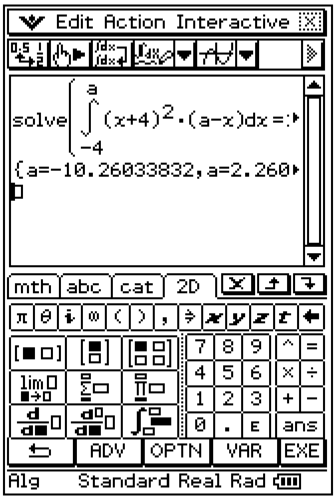
ii. Find the value of a, correct to 3 decimal places, such that the area bounded by the graph and the x-axis equals 128 sq units.

2 marks

Solution

$$\int_{-4}^{a}(x+4)^2(a-x)\,dx=128$$

use CAS to solve



so $a=2.260$

Mark allocation

- 1 mark for setting integral equal to 128
- 1 mark for answer—note must be 3 decimal places

f. Let T be a positive real number. Find the value of a such that the equation $f(x) = T$ has exactly two solutions for the largest value of T.

2 marks

Solution

An investigation of graphs on the graphing screen for different values of *a* reveals that as *a* increases the position of the turning point moves to the right and up so 2 solutions for maximum T will occur when *a* is greatest, ie *a* =10.

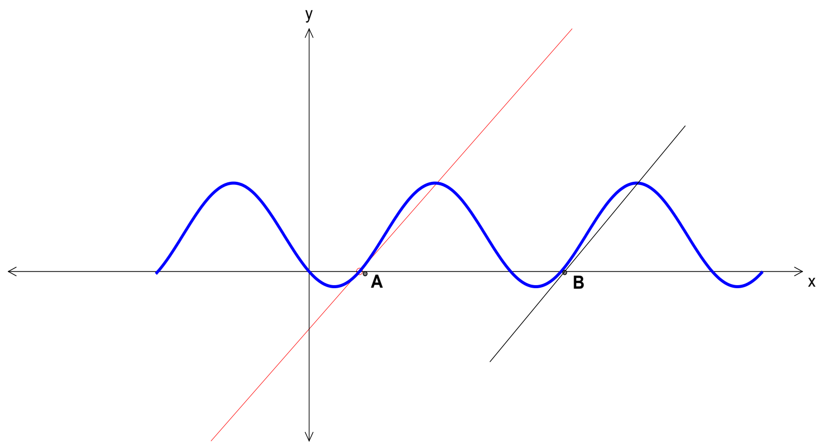
Mark allocation

- 2 marks for answer supported by valid reasoning.

Total 12 marks

Question 3

Part of the graph of the function $y = \sin x(\sin x - \cos x)$, with tangent lines drawn at A and B, is shown below.



a. State the period of the function.

1 mark

Solution

period is π

Mark allocation

- 1 mark for correct answer

b. State the coordinates of the points marked as *A* and *B*. Give exact values.

2 marks

Solution

$A\ (\frac{\pi}{4}, 0)\quad B(\frac{5\pi}{4}, 0)$

Mark allocation

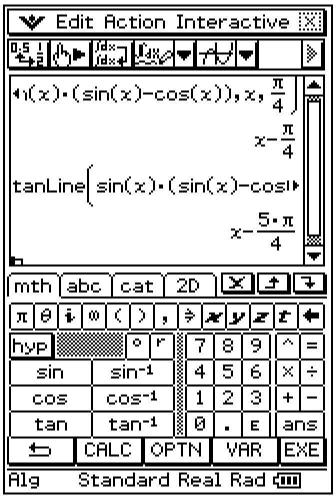
- 1 mark for A
- 1 mark for B

c. Find the exact minimum distance between the tangent lines.

3 marks

Solution

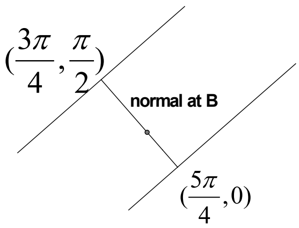
using CAS the tangent lines at A and B are



$$At\ A\ \ y = x - \frac{\pi}{4}$$

$$At\ B\ \ y = x - \frac{5\pi}{4}$$

minimum distance is the perpendicular distance between the two tangent lines, ie length of normal line at B to a point on the tangent at A.



the normal curve at B is $y = -x + \frac{5\pi}{4}$ (can be found again using CAS)

and the tangent at A intersects the normal at A at the point $(\frac{3\pi}{4}, \frac{\pi}{2})$ (found using intersect on CAS) so the minimum distance is the distance between $(\frac{3\pi}{4}, \frac{\pi}{2})$ and $B(\frac{5\pi}{4}, 0)$

distance =

$$\sqrt{(\frac{5\pi}{4} - \frac{3\pi}{4})^2 + (\frac{\pi}{2} - 0)^2}$$

$$\sqrt{(\frac{\pi}{2})^2 + (\frac{\pi}{2})^2}$$

$$\frac{\pi}{\sqrt{2}}$$

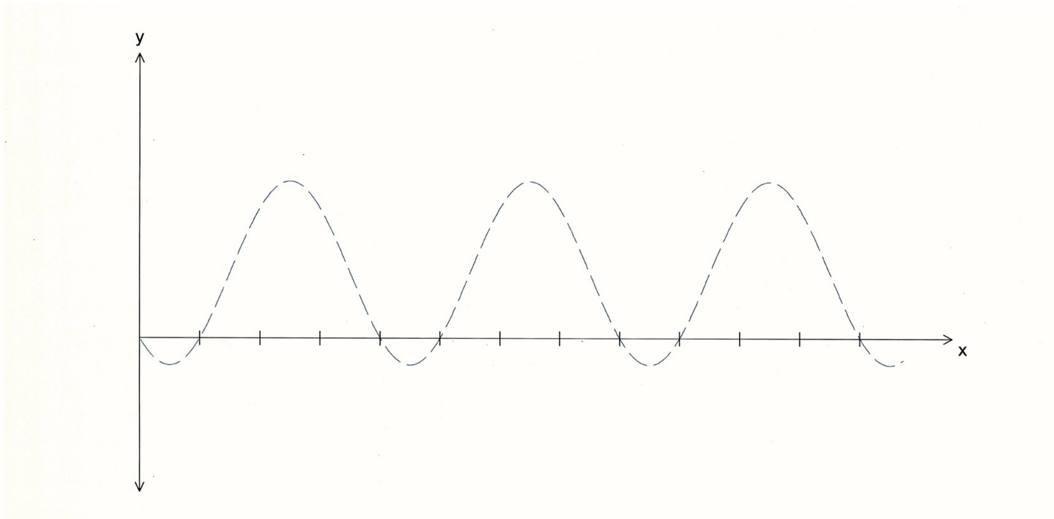
Mark allocation

- 1 mark for finding both tangent lines
- 1 mark for normal line
- 1 mark for answer

Renovations at the State Library include the positioning of a wallpaper frieze, or border pattern, at the top of the walls in the great hall.

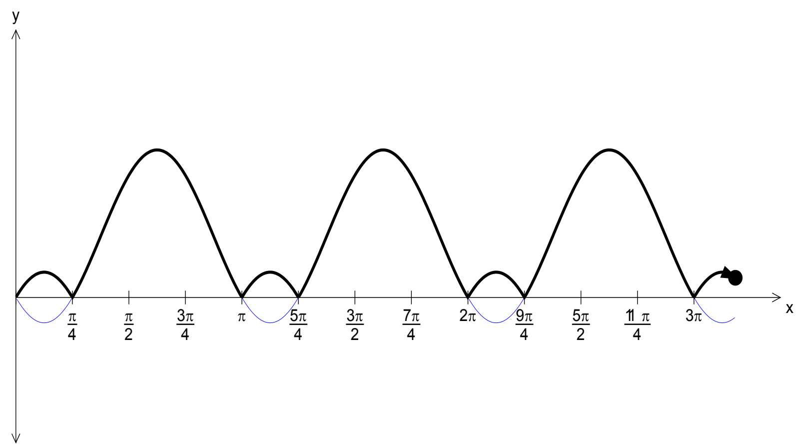
The pattern for the frieze is described by the rule $f(x) = |\sin x(\sin x - \cos x)|$ with all measurements in metres.

d. Sketch the graph of $y = f(x)$ *for* $x \in [0, 10]$. Label x-intercepts as exact values.



2 marks

Solution



Mark allocation

- 1 mark for shape
- 1 mark for x-intercepts labelled

e. State the general equation that gives the exact values of the x-coordinates of the turning points.

2 marks

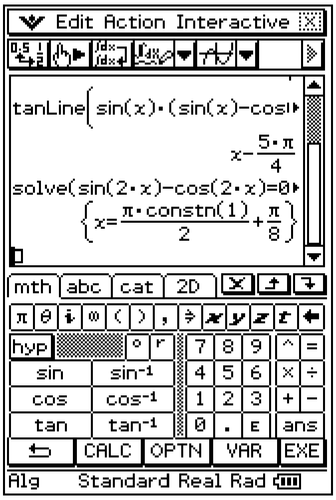
Solution

$$\frac{d}{dx}(\sin x(\sin x - \cos x))$$

$$= \sin(2x) - \cos(2x)$$

Absolute value is not required to find x coordinates.

Let $\frac{dy}{dx} = 0$ and solve using CAS



so the general solution is $x = \frac{\pi}{8} + \frac{n\pi}{2}$ $, n \in J$ or equivalently $x = \left(\frac{4n-3}{8}\right)\pi, \;\; n \in J$

Mark allocation

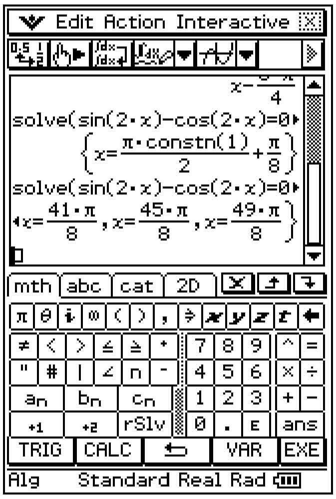
- 1 mark for setting $\frac{dy}{dx} = 0$
- 1 mark for answer

f. Find the exact values of the *x*-coordinates of the turning points in the section of wallpaper from 15m to 20m.

2 marks

Solution

solve the derivative equal to zero over the domain $15 \le x \le 20$



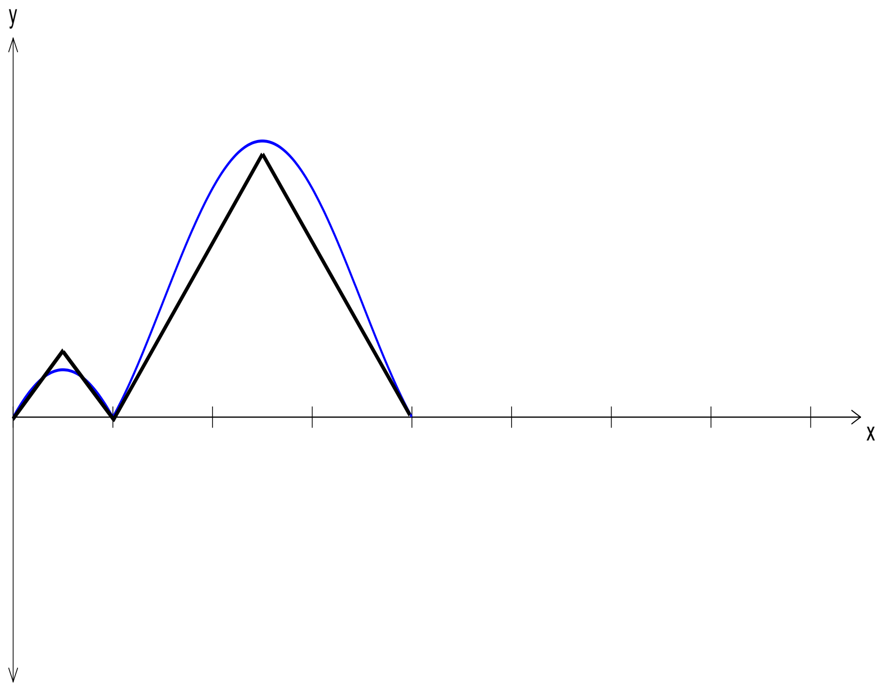
so $x = \frac{41\pi}{8}, \frac{45\pi}{8}, \frac{49\pi}{8}$

Mark allocation

- 1 mark for setting derivative equal to zero over the domain
- 1 mark for answers

For decorative purposes it is desirable to paint under the arches of the wallpaper frieze. Mathematically this is described as the area between the graph and the x-axis. One section of wallpaper measures nearly 20m and is cut at the x-intercept at $x = \frac{25\pi}{4}$.

It is decided to approximate the area under the curve using triangles formed by drawing tangents to the curve at the x-intercepts.



g. Find the intersection points of the tangent lines forming the triangles for the first two arches. State coordinates as exact values.

2 marks

Solution

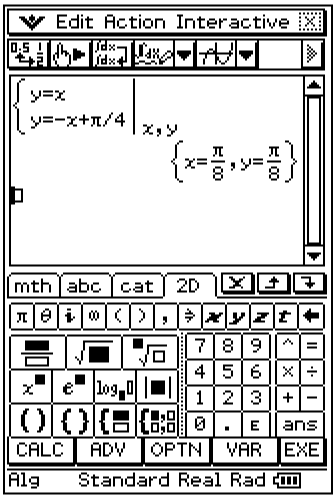
Use CAS to find the equations of the tangents to the curve $y = \sin x(\sin x - \cos x)$ at $x = 0,\ x = \frac{\pi}{4},\ x = \pi$ and reflect across x-axis as necessary.

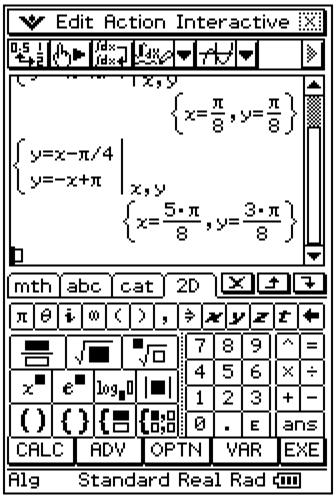
$$at\ x = 0,\ y = x$$

$$at\ x = \frac{\pi}{4},\ y = -x + \frac{\pi}{4}\ and\ y = x - \frac{\pi}{4}$$

$$at\ x = \pi,\ y = -x + \pi$$

First arch: the tangent lines intersect when $y = x$ intersects with $y = -x + \frac{\pi}{4}$





Second arch: the tangent lines intersect when $y = x - \dfrac{\pi}{4}$ meets $y = -x + \pi$

so the points of intersection are at $\left(\dfrac{\pi}{8}, \dfrac{\pi}{8}\right)$ *and* $\left(\dfrac{5\pi}{8}, \dfrac{3\pi}{8}\right)$

Mark allocation

- 1 mark for finding tangent lines
- 1 mark for both coordinates

h. Find, by how much the triangle method underestimates the true area under the curve for $\dfrac{25\pi}{4}m$ of wallpaper frieze. Answer correct to 2 decimal places.

3 marks

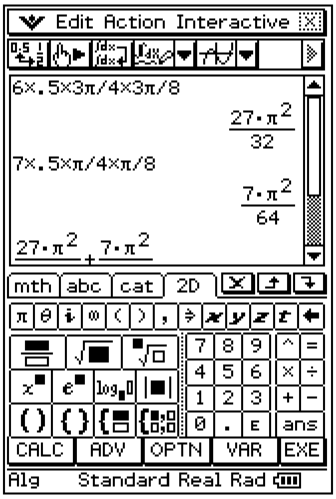
Solution

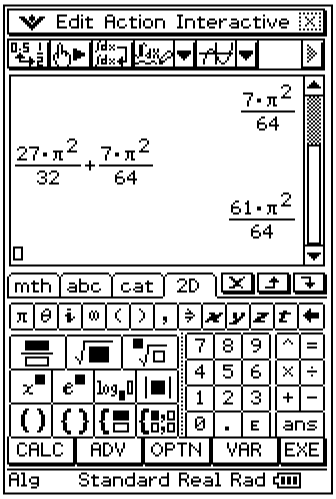
In $\dfrac{25\pi}{4}m$ of the wallpaper frieze there are 6 large and 7 small triangles

Approximate area using triangles

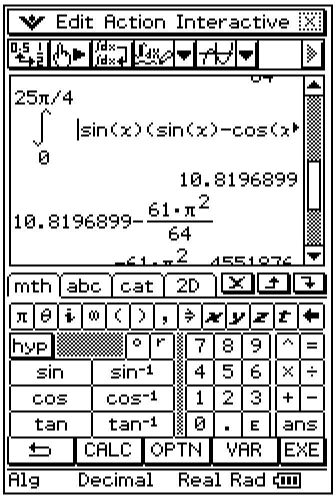
$= 6(0.5 \times \dfrac{3\pi}{4} \times \dfrac{3\pi}{8}) + 7(0.5 \times \dfrac{\pi}{4} \times \dfrac{\pi}{8})$

$= \dfrac{61\pi^2}{64}$

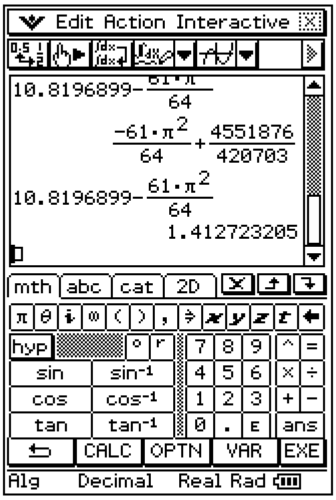




The true area using calculus is found by evaluating $\int_0^{\frac{25\pi}{4}} \left|\sin x(\sin x - \cos x)\right| dx$. This can be found using CAS.



So the difference in the areas is 1.41 m^2



Mark allocation

- 1 mark for finding exact areas of the triangles
- 1 mark for finding true area
- 1 mark for answer

Total 17 marks

Question 4

A spinner is made up of four coloured sections and labelled with the numbers 1,2,3,4. The probability of the marker landing on any one section is described by the probability distribution table below-

n	1	2	3	4
Pr(N=n)	$\frac{2k^2-1}{14}$	$\frac{2k}{14}$	$\frac{k}{14}$	$\frac{k-1}{14}$

a. Show that $k = 2$.

1 mark

Solution

The sum of the probabilities needs to equal 1

so

$$2k^2 - 1 + 2k + k + k - 1 = 14$$
$$2k^2 + 4k - 16 = 0$$
$$k = -4,\ k = 2$$

so $k = 2,\ as\ probabilities > 0$

or alternatively could substitute *k=2* and show the probs sum to 1.

Mark allocation

- 1 mark for summing the probabilities to 1 leading to correct solution

The spinner is spun twice.

b. Find the probability that the product of the numbers is an even number.

2 marks

Solution

easier to look at the $\Pr(even) = 1 - \Pr(odd)$

$$\Pr(even) = 1 - \Pr(odd)$$
$$= 1 - [(1,1) + (1,3) + (3,1) + (3,3)]$$
$$= \frac{115}{196}$$

Mark allocation

- 1 mark for $\Pr(even) = 1 - \Pr(odd)$ or collecting the even cases
- 1 mark for answer

The spinner is spun 6 times and the number the marker lands on is noted.

c. Find the probability that the marker lands on number 3 more than 2 times, correct to 3 decimal places.

2 marks

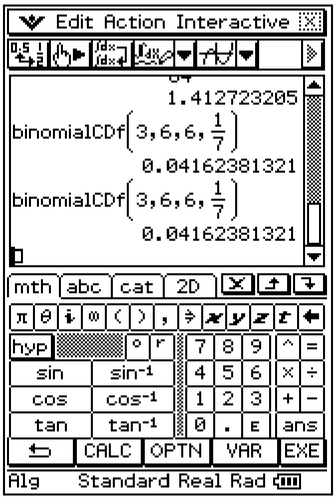
Solution

Let Y be the number of times the number 3 turns up

$$Y \sim Bi(n = 6, p = \frac{1}{7})$$

Using CAS

$$\Pr(Y > 2) = 0.042$$



Mark allocation

- 1 mark for defining the binomial distribution
- 1 mark for answer

d. What is the fewest number of spins required in order to be 80% sure of getting a 3 at least twice?

3 marks

Solution

$$Y \sim Bi(n = unknown,\ p = \frac{1}{7})$$

$$\Pr(Y \geq 2) = 1 - [\Pr(y = 0) + \Pr(y = 1)]$$

$$0.8 = 1 - binomialCDf(1, n, 1/7)$$

$$n = 20$$

spinner needs to be spun 20 times

Mark allocation

- 1 mark for setting up equation involving binomial with n trials and 0.8
- 1 mark for constructing $\Pr(Y \geq 2) = 1 - [\Pr(y = 0) + \Pr(y = 1)]$
- 1 mark for answer

The spinner is used for a game.
The spinner is spun twice and if the two numbers spun are identical then the player wins.
The game costs $2 to play and the player receives $5 back if they win and nothing if they lose.

e. Determine the player's expected profit.

2 marks

Solution

need to construct a pdf table

$\Pr(\textit{same number}) = (1,1) + (2,2) + (3,3) + (4,4) = \frac{5}{14}$

Let $P = \textit{profit made}$
so the table is

p	-2	3
Pr(P=p)	$\frac{9}{14}$	$\frac{5}{14}$

so $E(P) = \frac{-3}{14} = -0.21$ so expect to lose 21 cents.

Mark allocation

- 1 mark for finding probability of getting 2 numbers the same
- 1 mark for correct answer

The time (in minutes) spent playing the game by a particular player can be described by the random variable X, with a probability density function given by,

$$f(x)=\begin{cases} kx(b-x^2) & 0\le x\le 10 \\ 0, & otherwise \end{cases}$$

f. If the mean is $\frac{16}{3}$, find the value of k and b.

3 marks

Solution

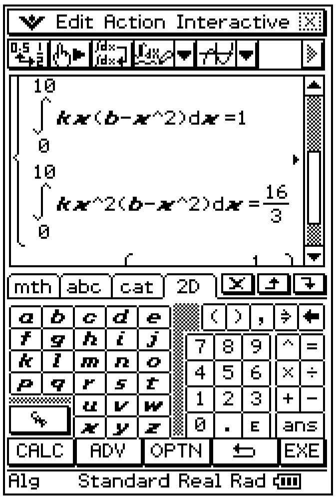
Using the information given

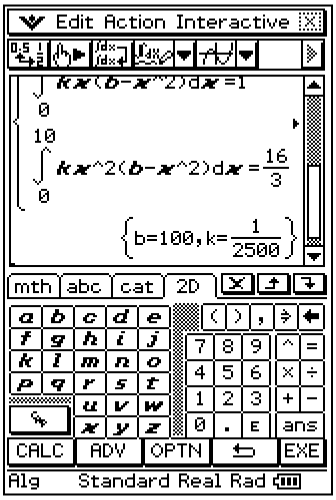
$$\int_0^{10} kx(b-x^2)dx = 1$$

$$\int_0^{10} x\times kx(b-x^2)\ dx = \frac{16}{3}$$

solving simultaneously using CAS gives

$$b=100,\ k=\frac{1}{2500}$$





Mark allocation

- 1 mark for setting up the first equation
- 1 mark for setting up the second equation
- 1 mark for both b and k correct

g. Find the exact value of the mode.

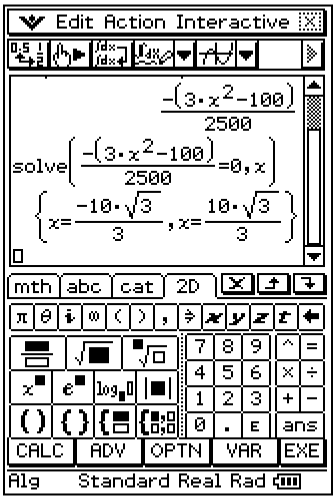
2 marks

Solution

the mode is the x-value of the turning point for the function.

so when $\frac{d}{dx}[\frac{x}{2500}(100-x^2)]=0 \quad for\ 0 \le x \le 10$

using CAS this gives



$x=\frac{10\sqrt{3}}{3}$

Mark allocation

- 1 mark for setting derivative equal to zero
- 1 mark for correct answer.

Total 15 marks

END OF SOLUTIONS BOOK