INSIGHT

Trial Exam Paper

2010

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

QUESTION AND ANSWER BOOK

Reading time: 15 minutes

Writing time: 2 hours

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white out liquid/tape.

Materials provided

- The question and answer book of 25 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice questions inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected no marks will be awarded.

Question 1

The smallest positive value of x for which $\tan 4x = 1$ is

A. π

B. $\frac{\pi}{4}$

C. $\frac{\pi}{8}$

D. $\frac{\pi}{16}$

E. 2π

Question 2

The graph with the equation $y = |x-2| + 4$ is reflected in the y-axis and then translated 1 unit down. The resulting graph has the equation

A. $y = -|x-2| + 3$

B. $y = |-x+2| + 3$

C. $y = -|x+2| - 1$

D. $y = |x+2| + 3$

E. $y = |x+2| - 5$

Question 3

The range of the function $f:[-5, 4) \to R,\ f(x) = (x-3)^{\frac{2}{3}}$ is

A. $[3, \infty)$

B. $[-5, 4)$

C. $[0, 4]$

D. $(0, 4)$

E. $(0, 4]$

Question 4

For the system of equations

$\boldsymbol{z = 1}$

$\boldsymbol{y - x = 2}$

$\boldsymbol{x + y = 5}$

An equivalent matrix representation is

A.

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

B.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

C.

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

D.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

E.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Question 5

A bag contains 4 white, 3 red and 3 black balls. Two balls are selected without replacement. The probability that they are both the same colour is

A. $\frac{2}{45}$

B. $\frac{6}{25}$

C $\frac{8}{15}$

D. $\frac{1}{15}$

E. $\frac{4}{15}$

Question 6

The function $f : B \to R,\ f(x) = \left|\log_e(2x^5)\right|$ will have an inverse function if

A. $B = [2^{\frac{1}{5}}, \infty)$

B. $B = [2^{-\frac{1}{5}}, \infty)$

C. $B = [-2^{\frac{1}{5}}, \infty)$

D. $B = [0, \infty)$

E. $B = (0, \infty)$

Question 7

The function g has the rule $g(x) = \sqrt{b - ax}$ for $a, b > 0$. The maximal domain of g is

A. $x \le \frac{a}{b}$

B. $x > \frac{b}{a}$

C. $x < \frac{b}{a}$

D. $x \ge \frac{b}{a}$

E. $x \le \frac{b}{a}$

Question 8

The average value of the function $y = \sin(2x)$ over the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$ is

A. $\frac{3}{\pi}$

B. $\frac{6}{\pi}$

C. $\frac{3}{2\pi}$

D. $\frac{1}{\pi}$

E. $\frac{1}{2}$

Question 9

If $y = 5a^{3x} - 2b$, then x equals

A. $\frac{1}{3}\log_a(5(y+2b))$

B. $\frac{1}{3}\log_a(\frac{y}{5}) + 2b$

C. $\frac{y+2b}{15a}$

D. $\frac{1}{15}\log_a(y+2b)$

E. $\frac{1}{3}\log_a(\frac{y+2b}{5})$

Question 10

The radius of a sphere is increasing at a rate of 4 mm/min.
When the radius is 5 mm, the rate at which the surface area is increasing, in mm^2/min is

A. 160π

B. 32π

C. 400π

D. 160

E. 100π

Question 11

The graph of $y = kx - 5$ intersects the graph of $y = x^2 - 6x$ at two distinct points for

A. $\{k : -2\sqrt{5} - 6 < k < 2\sqrt{5} + 6\}$

B. $\{k : k < 2\sqrt{5} + 6\}$

C. $\{k : -2\sqrt{5} - 6 > k\} \cup \{k : k > 2\sqrt{5} - 6\}$

D. $\{k : k > -2\sqrt{5} - 6\}$

E. $\{k : k = -2\sqrt{5} - 6\} \cup \{k : k = 2\sqrt{5} + 6\}$

Question 12

A fair die is tossed 8 times.
The probability of getting a six at least once is

A. $1 - (\frac{5}{6})^8$

B. $1 - \left(\frac{1}{6}\right)^8$

C. $\frac{5}{6}$

D. $\frac{1}{6}$

E. $1 - (\frac{5}{6})^7$

Question 13

Which one of the following statements is **not** true about the function
$f : R \to R,\ f(x) = |3x - 5|$.

A. The graph of f is continuous everywhere

B. $f'\left(\frac{5}{3}\right) = 0$

C. $f(x) \geq 0$ for all x

D. $f'(x) > 0$ for all $x > \frac{5}{3}$

E. $f'(x) < 0$ for all $x < 0$

Question 14

Let $f: R \to R$ be a differentiable function.
The derivative of $f(\sqrt{\sin x})$ is

A. $f'(\sqrt{\sin x})$

B. $\sqrt{\cos x}\, f'(\sqrt{\sin x})$

C. $\dfrac{\cos x}{2\sqrt{\sin x}} f'(\sqrt{\cos x})$

D. $\dfrac{\cos x}{2\sqrt{\sin x}} f'(\sqrt{\sin x})$

E. $\dfrac{\sin x}{2\sqrt{\cos x}} f'(\sqrt{\sin x})$

Question 15

The system of linear simultaneous equations

$$\begin{aligned} 5x + ay &= 0 \\ (2a+1)x + 2y &= 0 \end{aligned}$$

where a is a real constant, has infinitely many solutions for

A. $a \in \{\frac{-5}{2}, 2\}$

B. $a \in R \setminus \{\frac{-5}{2}, 2\}$

C. $a \in \{\frac{5}{2}, -2\}$

D. $a \in R \setminus \{\frac{5}{2}, -2\}$

E. $a \in R$

Question 16

The transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ maps the curve with the equation $y = x^2$ to the curve with the equation

A. $y = \dfrac{-5(x+2)^2}{9} - 7$

B. $y = \dfrac{-5(x-2)^2 + 7}{3}$

C. $y = \dfrac{-5(x+2)^2 + 7}{3}$

D. $y = \dfrac{-5(x-2)^2}{9} + 7$

E. $y = \dfrac{-9(x-2)^2}{5} + 7$

Question 17

The graph of $y = x^3 - bx$ for $b \in R^+$ has turning points at $x = 3$ and $x = -3$.
The function $y = |x^3 - bx|$ is an increasing function for

A. $x \in R$

B. $x \in \{x : -3 \le x \le 3\}$

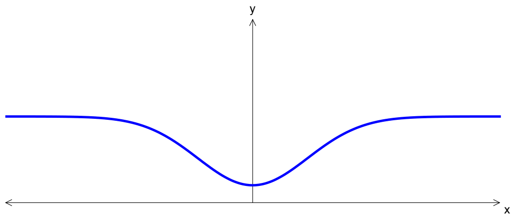
C. $x \in \{x : -\sqrt{b} \le x \le -3\} \cup \{x : 0 \le x \le 3\} \cup \{x : x \ge \sqrt{b}\}$

D. $x \in \{x : x \le -3\} \cup \{x : 0 \le x \le 3\} \cup \{x : x \ge \sqrt{b}\}$

E. $x \in \{x : -\sqrt{b} \le x \le -3\} \cup \{x : x \ge \sqrt{b}\}$

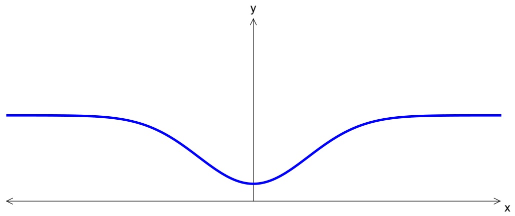
Question 18

The graph of $y = f(x)$ is shown



The graph of an antiderivative of f could be

A.



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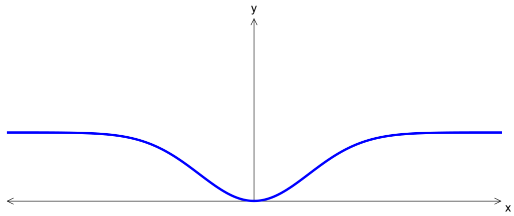

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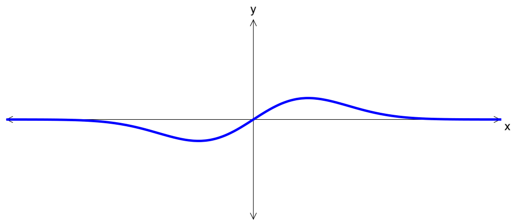
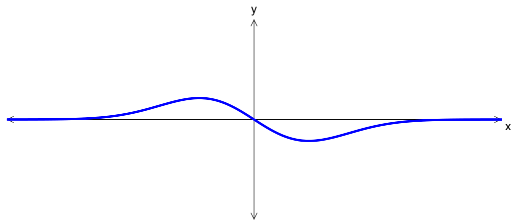
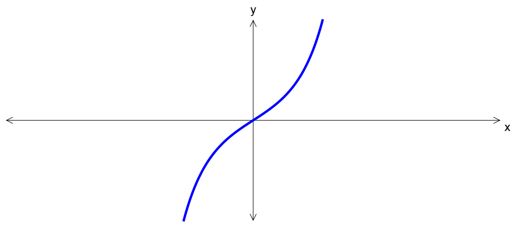


D.



E.



Question 19

Phil likes to drink both coffee and tea. He drinks one cup of either each day and likes to switch from day to day. If he drinks coffee one day he drinks coffee the next with a probability of 0.65. If he drinks tea one day he drinks coffee the next with a probability of 0.3. The transition matrix T representing the situation is

A. $\begin{bmatrix} 0.65 & 0.7 \\ 0.35 & 0.3 \end{bmatrix}$

B. $\begin{bmatrix} 0.35 & 0.3 \\ 0.65 & 0.7 \end{bmatrix}$

C. $\begin{bmatrix} 0.65 & 0.3 \\ 0.35 & 0.7 \end{bmatrix}$

D. $\begin{bmatrix} 0.65 & 0.35 \\ 0.3 & 0.7 \end{bmatrix}$

E. $\begin{bmatrix} 0.65 \\ 0.3 \end{bmatrix}$

Question 20

Let $f : R \to R,\ f(x) = e^{x} + e^{-x}$

Then $[f(u)]^{2} =$

A. $f(2u)$

B. $f(u^{2})$

C. $f(2u) + 2$

D. $2f(u)$

E. $f(2u) - 2$

Question 21

The random variable X has a normal distribution with a mean of 20 and a standard deviation of 0.5.
If the random variable Z has the standard normal distribution, then the probability that X is greater than 21.5 is

A. $1 - \Pr(Z > 3)$

B. $1 - \Pr(Z < -3)$

C. $\Pr(Z < 3)$

D. $\Pr(Z < -3)$

E. $\Pr(-3 < Z < 3)$

Question 22

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} |1-x| & \textit{for } 0 < x < 2 \\ 0 & \textit{otherwise} \end{cases}$$

The expected value $E(X)$ is

A. 1

B. $\frac{1}{2}$

C. $\frac{1}{6}$

D. $\frac{1}{4}$

E. 0

End of Section 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Exact value answers are required unless specified otherwise.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated diagrams are not drawn to scale.

Question 1

Let $f:[0, \infty) \rightarrow R,\ f(x) = e^{\frac{x}{3}} - 2$.

a. **i.** State the range of *f*.

1 mark

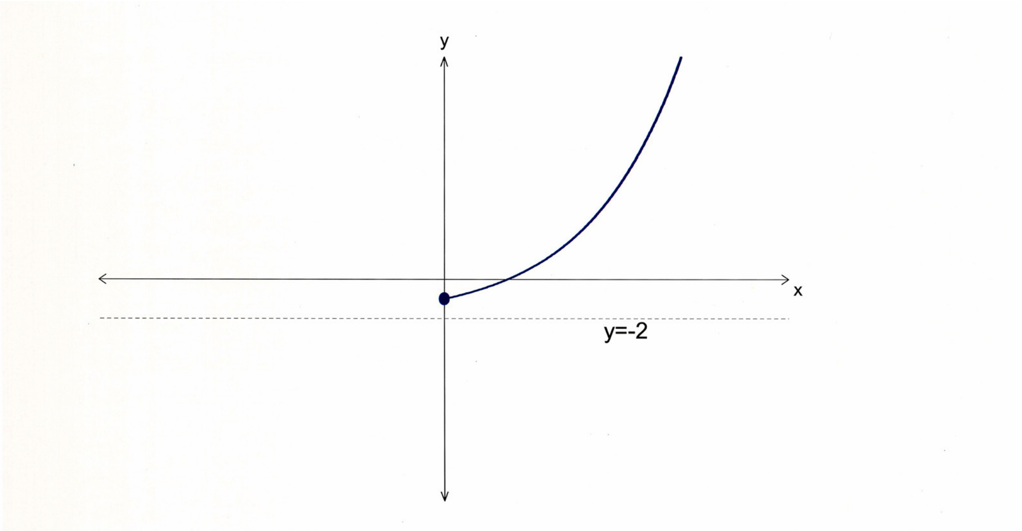
ii. Explain why f^{-1}, the inverse function of *f*, exists.

1 mark

iii. Find the rule for f^{-1} and state the domain.

2 marks

iv. The graph of $y = f(x)$ is shown below. On the same axes sketch the graph of $y = f^{-1}(x)$. Label any asymptote with its equation and any endpoint or intercept with its exact coordinates. Label the point of intersection between the two graphs as a coordinate correct to 2 decimal places.



2 marks

b. Let $g:[0, \infty) \to R,\ g(x) = x^2(e^{\frac{x}{a}} - b)$.

i. It is found that when $x = 1,\ g(x) = -3$ and when $x = 2,\ g(x) = -9$. Find the exact values of a and b.

3 marks

ii. For $a = 3$ *and* $b = 2,\ g(x) = x^2(e^{\frac{x}{3}} - 2)$. Find the least value of k, correct to 4 decimal places, such that $g:[k, \infty) \to R,\ \ g(x) = x^2(e^{\frac{x}{3}} - 2)$ is a one to one function.

2 marks

c. For the function $q : R \to R,\ q(x) = (x^2 - bx + 6)e^{\frac{x}{3}}$ the derivative is

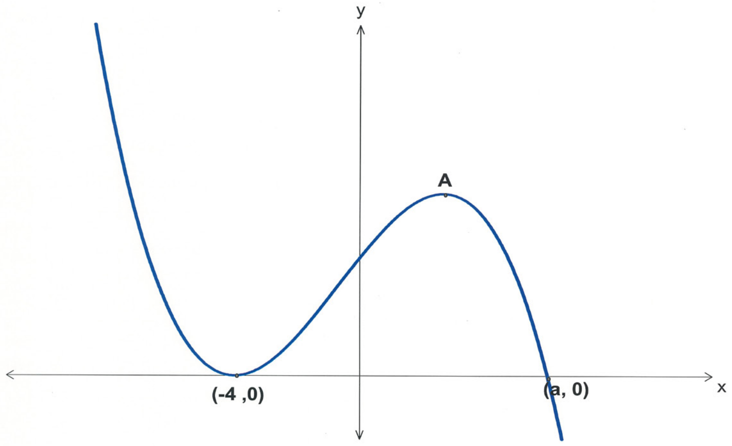
$q'(x) = \dfrac{(x^2 + x(6-b) + 6 - 3b)e^{\frac{x}{3}}}{3}$. Show that the graph of $y = q(x)$ will have 2 stationary points for all values of b.

3 marks

Total 14 marks

Question 2

The graph of $y = (x+4)^2(a-x)$ for $a \in (-4, 10]$ is shown below.



a. Find $\dfrac{dy}{dx}$.

1 mark

b. Find the coordinates of the turning points.

2 marks

c. Find the value of a such that the maximum turning point at A lies on the y-axis. State the coordinates of the maximum turning point.

2 marks

d. For $y = f(x)$, find the exact value of a such that the equation $f(x) = 5$ has exactly two solutions.

2 marks

e. **i.** Write a definite integral that gives the area bounded by the graph and the x-axis.

1 mark

ii. Find the value of a, correct to 3 decimal places, such that the area bounded by the graph and the x-axis equals 128 sq units.

2 marks

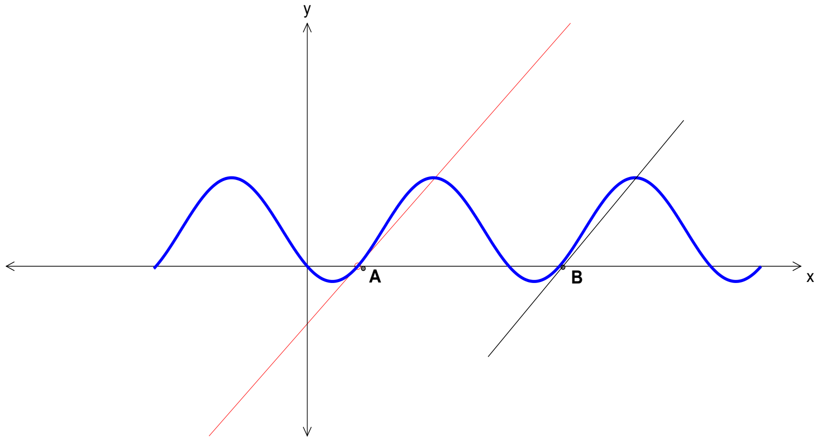
f. Let T be a positive real number. Find the value of a such that the equation $f(x) = T$ has exactly two solutions for the largest value of T.

2 marks

Total 12 marks

Question 3

Part of the graph of the function $y = \sin x(\sin x - \cos x)$, with tangent lines drawn at A and B, is shown below.



a. State the period of the function.

1 mark

b. State the coordinates of the points marked as A and B. Give exact values.

2 marks

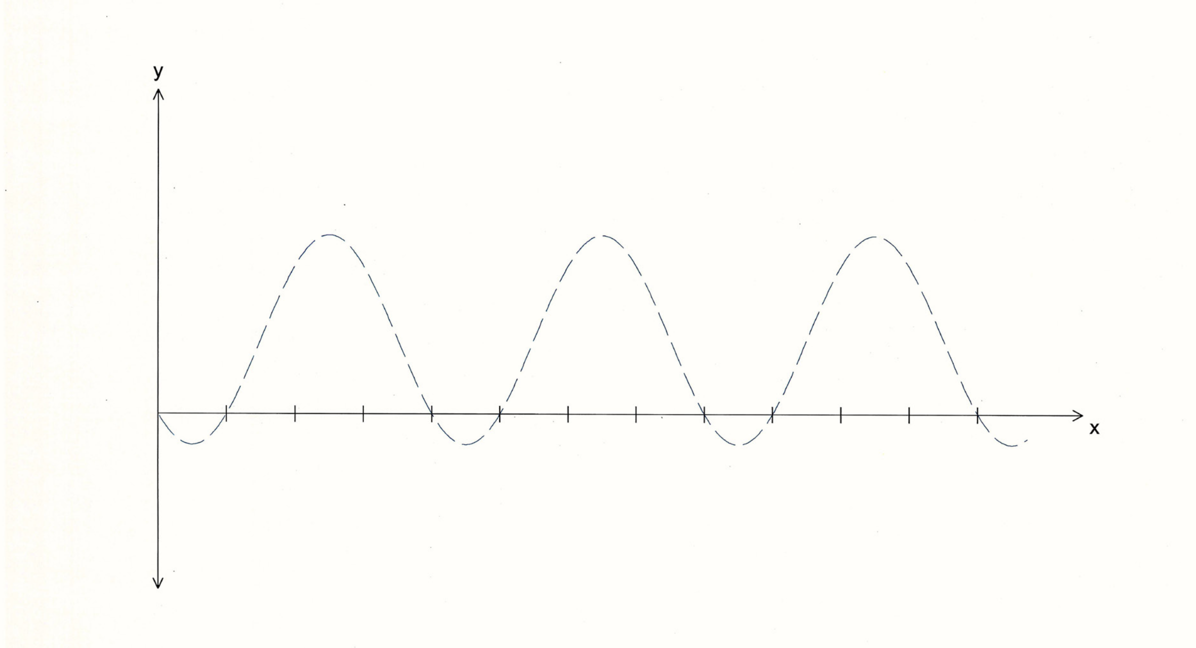
c. Find the exact minimum distance between the tangent lines.

3 marks

Renovations at the State Library include the positioning of a wallpaper frieze, or border pattern, at the top of the walls in the great hall.

The pattern for the frieze is described by the rule $f(x) = |\sin x(\sin x - \cos x)|$ with all measurements in metres.

d. Sketch the graph of $y = f(x)$ *for* $x \in [0, 10]$. Label x-intercepts as exact values.



2 marks

e. State the general equation that gives the exact values of the x-coordinates of the turning points.

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2 marks

f. Find the exact values of the x-coordinates of the turning points in the section of wallpaper from 15m to 20m.

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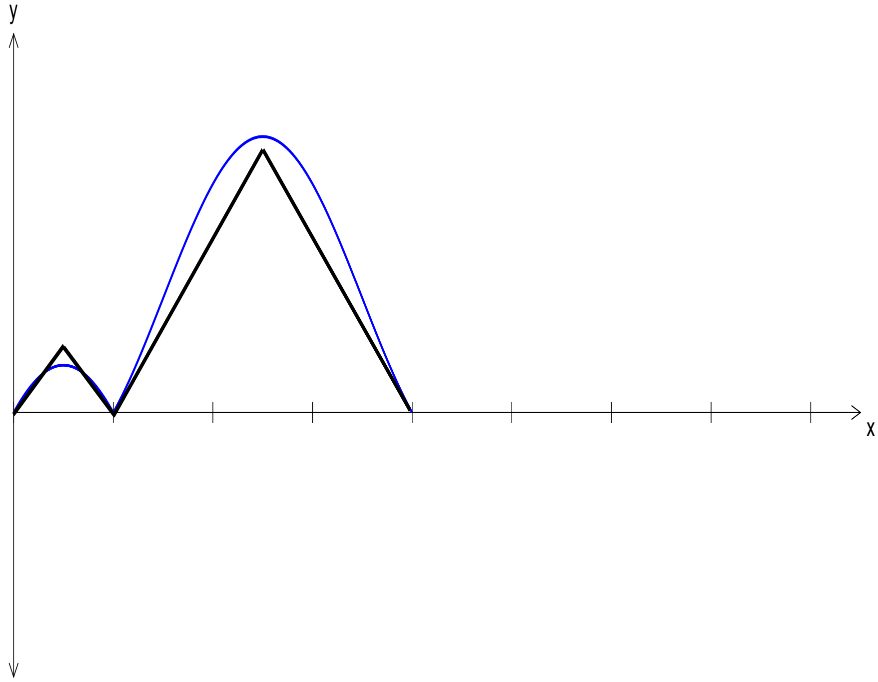
2 marks

For decorative purposes it is desirable to paint under the arches of the wallpaper frieze. Mathematically this is described as the area between the graph and the x-axis. One section of wallpaper measures nearly 20m and is cut at the x-intercept at $x = \dfrac{25\pi}{4}$.

It is decided to approximate the area under the curve using triangles formed by drawing tangents to the curve at the x-intercepts.



g. Find the intersection points of the tangent lines forming the triangles for the first two arches. State coordinates as exact values.

2 marks

h. Find, by how much the triangle method underestimates the true area under the curve for $\frac{25\pi}{4}m$ of wallpaper frieze. Answer correct to 2 decimal places.

3 marks

Total 17 marks

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Section 2 – continued

Question 4

A spinner is made up of four coloured sections and labelled with the numbers 1,2,3,4. The probability of the marker landing on any one section is described by the probability distribution table below-

n	1	2	3	4
Pr(N=n)	$\frac{2k^2-1}{14}$	$\frac{2k}{14}$	$\frac{k}{14}$	$\frac{k-1}{14}$

a. Show that $k = 2$.

1 mark

The spinner is spun twice.

b. Find the probability that the product of the numbers is an even number.

2 marks

The spinner is spun 6 times and the number the marker lands on is noted.

c. Find the probability that the marker lands on number 3 more than 2 times, correct to 3 decimal places.

2 marks

d. What is the fewest number of spins required in order to be 80% sure of getting a 3 at least twice?

3 marks

The spinner is used for a game.
The spinner is spun twice and if the two numbers spun are identical then the player wins.
The game costs \$2 to play and the player receives \$5 back if they win and nothing if they lose.

e. Determine the player's expected profit.

2 marks

The time (in minutes) spent playing the game by a particular player can be described by the random variable X, with a probability density function given by,

$$f(x)=\begin{cases}kx(b-x^2) & 0\le x\le 10\\ 0, & otherwise\end{cases}$$

f. If the mean is $\frac{16}{3}$, find the value of k and b.

3 marks

g. Find the exact value of the mode.

2 marks

Total 15 marks

END OF QUESTION AND ANSWER BOOK