MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2

2010 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1
Answer: B

Explanation:
Use CAS:
solve($2x + 8z = 26$ and $4x - 4y - 14z = -38$ and $8x - 4y + 2z = 14, x, y, z$).
Change the parameter of calculator to *p*.
$x = 13 - 4p,\ y = \frac{45-15p}{2},\ z = p$
or rref on calculator

Question 2
Answer: C

Explanation:
The equation of the horizontal asymptote is $y = 3$.
As $x \to \infty,\ y \to 3$.
The $y-$intercept has coordinates $(0, 1)$
The equation of the vertical asymptote is $x = 1$
The $x-$intercepts have coordinates $(0.18, 0)$ and $(1.82, 0)$

Question 3
Answer: E

Explanation:
$-1 - k^2$ is the $y-$intercept and there are values greater than $-1 - k^2$.

Question 4
Answer: A

Explanation:
$f(x) = |2e^x - 2|$ can be written as a hybrid function

$$f(x) = \begin{cases} 2e^x - 2 & x \geq 0 \\ -(2e^x - 2) & x < 0 \end{cases}$$

Find the derivative of the two separate functions

$$f'(x) = \begin{cases} 2e^x & x > 0 \\ -2e^x & x < 0 \end{cases}$$

Question 5
Answer: D

Explanation:
Average rate of change $= \frac{V(.25)-V(0)}{0.25} = \frac{0.213-0.211429}{0.25} = 0.0063$ m^3/h

Question 6
Answer: A

Explanation:
Use CAS Probability menu: inverse normal. Change the area to 0.75 because the calculators use $Z < c$ and not $Z > c$.
$c = 0.6745$

Question 7
Answer: A

Explanation:
$f'(x) = 0$ for $x = -1$ and $x = 1.25$ means that there are stationary points at $x = -1$ and $x = 1.25$. The gradient change from negative to positive at $x = 1.25$. $f'(x) < 0$ for all other real values of x means the gradient is the same before and after the stationary point, therefore $x = -1$ is a stationary point of inflection.

Question 8
Answer: E

Explanation:
Use CAS: $f(x) = \int(-2x^2 + 9x - 4)dx = \frac{-2x^3}{3} + \frac{9x^2}{2} - 4x$

Question 9
Answer: C

Explanation:
$2m + m + 4m + 3m = 1,\ m = \frac{1}{10}$.
$E(X) = \sum x \times \Pr(X = x) = \frac{2}{10} + \frac{2}{10} + \frac{12}{10} + \frac{12}{10} = \frac{28}{10} = \frac{14}{5}$

$\boldsymbol{x}$	1	2	3	4
$\Pr(\boldsymbol{X} = \boldsymbol{x})$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

Question 10
Answer: E

Explanation:
Use CAS: 1 $-$Normal Cdf$(-\infty, 2.416, 0, 1)$, $\Pr\big((Z > 2.416)\big) = 0.0078$

Question 11
Answer: C

Explanation:
$\int_0^4 2f(x) - 4dx = 2\int_0^4 f(x)dx - \int_0^4 4\,dx = 18 - [4x]_0^4 = 18 - 16 = 2$

Question 12
Answer: B

Explanation:
$\text{m} = \sin\left(\frac{\pi x}{6}\right) + d = 0$ at $x = 3$.
$\text{d} = -\sin\left(\frac{3\pi}{6}\right) = -1$
$\text{m} = \sin\left(\frac{\pi x}{6}\right) - 1$
$f(x) = \int \sin\left(\frac{\pi x}{6}\right) - 1dx = \frac{-6}{\pi}\cos\left(\frac{\pi x}{6}\right) - x + c$
$2 = \frac{-6}{\pi}\cos\left(\frac{3\pi}{6}\right) - 3 + c$
$c = 5$
$f(x) = \frac{-6}{\pi}\cos\left(\frac{\pi x}{6}\right) - x + 5$

Question 13
Answer: D

Explanation:
Use CAS: Binomial pdf $\left(30, \frac{1}{5}, 2, 2\right)$, $\Pr(X = 2) = 0.0337$.

Question 14
Answer: C

Explanation:
Use CAS: Inverse normal$(0.53, 0, 1)$, $z = 0.07527$
$z = \frac{x-\mu}{\sigma}$
$0.07527 = \frac{x-10}{2}$
$x = 10.1505$
or invNorm(0.53, 10, 2)

Question 15
Answer: B

Explanation:
Use CAS: norm Cdf$(-\infty, 27, 36, 3)$, $\Pr(X < 27) = 0.00135 = 0.0014$

Question 16
Answer: A

Explanation:
$f o g$ exists if ran $g \subseteq$ dom f, therefore $D = R \backslash (-2, 1)$.

Question 17
Answer: B

Explanation:
Use CAS: $\int 3x \times x^{\frac{1}{2}} dx = \int 3x^{\frac{3}{2}} dx = \frac{6\sqrt{x^5}}{5}$.

Question 18
Answer: D

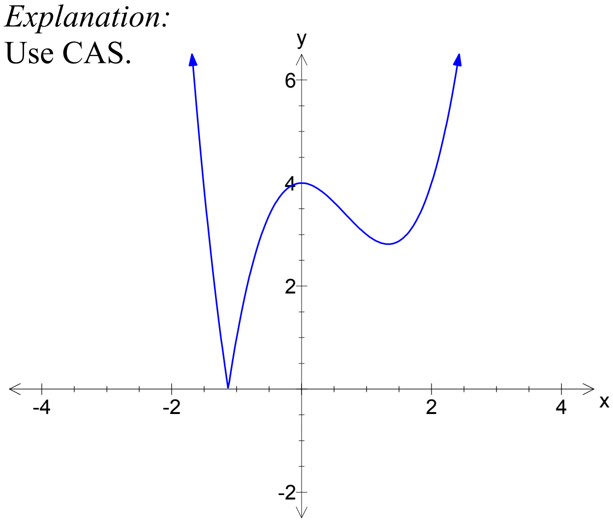
Explanation
: $\frac{1}{k}\int_2^5 (x^3 - 4)\, dx = 1$
$\frac{1}{k} \times \frac{561}{4} = 1$
$k = \frac{561}{4}$

Question 19
Answer: E

Explanation:
$l = \log_e(-x+1) = \log_e(-(-4)+1) = \log_e(5)$
$A = \log_e 5 + \log_e 4 + \log_e 3 + \log_e 2 = \log_e(5 \times 4 \times 3 \times 2) = \log_e 120$

Question 20
Answer: A

Explanation:
Use CAS.



$|x^3 - 2x^2 + 4|$

Question 21
Answer: C

Explanation:
$x = \frac{1}{3}\log_e(5y+2) - 1$
$3(x+1) = \log_e(5y+2)$
$e^{3x+3} - 2 = 5y$
Equation of the inverse function: $\frac{1}{5}(e^{3x+3} - 2)$

Question 22
Answer: E

Explanation:
The function $f: R \to R, f(x) = -5\sin\left(\frac{\pi x}{4}\right) + 1$, has period and range respectively:
period $= \frac{2\pi}{b} = \frac{2\pi}{1} \times \frac{4}{\pi} = 8$
maximum value $= -5(-1) = 6$ and minimum value $= -5(1) = -4$
range: $[-4, 6]$

SECTION 2: Analysis Questions
Question 1

a. $f(x) = \sqrt{2-3x} = (2-3x)^{\frac{1}{2}}$

Use chain rule: $m = f'(x) = \frac{1}{2}(2-3x)^{\frac{-1}{2}} \times -3 = \frac{-3}{2\sqrt{2-3x}}$

M1+A1

2 marks

b. Use CAS: $m = \frac{-3}{2\sqrt{2-3x}} = -4, \quad x = \frac{119}{192}$ and $\left(\frac{119}{192}, \frac{3}{8}\right)$

$y - y_1 = m(x - x_1)$

$y - \frac{3}{8} = -4\left(x - \frac{119}{192}\right)$

$y = -4x + \frac{119}{48} + \frac{3}{8}$

$g(x) = -4x + \frac{137}{48}$

M1+A1

2 marks

c. Domain of f is $\left(-\infty, \frac{2}{3}\right]$ and the range of g is R. The range of g is R. The range of g is not a subset of the domain of f (ran $g \nsubseteq$ dom f). So fog does not exist.

M1+A1

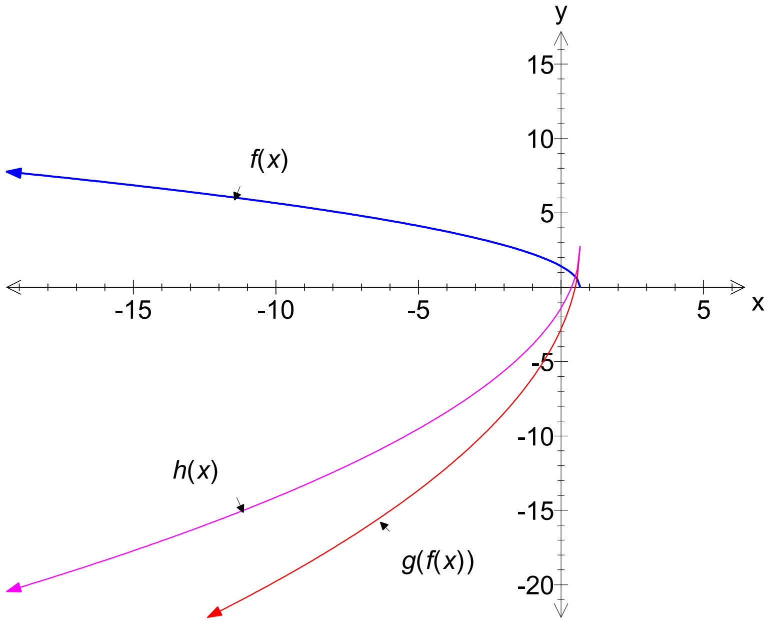
2 marks

d. $g(f(x)) = g(\sqrt{2-3x}) = -4\sqrt{2-3x} + \frac{137}{48}$, the domain is the set of all values in the domain of f (the domain of the inner function). Domain: $x \in \left(-\infty, \frac{2}{3}\right]$.

M2+A1

3 marks

e. and **f.**



A2
A2
2 +2 marks

Question 2

a. $\Pr(X \geq 49.62) = 0.4, \quad z = 0.253347 \; (\text{inv norm}(0.6, 0, 1))$

$\Pr(X < 51.37) = 0.75, \quad z = 0.67449 \; (\text{inv norm}(0.75, 0, 1))$

$z = \frac{X-\mu}{\sigma}, \quad 0.253347 = \frac{49.62-\mu}{\sigma}, \quad 0.67449 = \frac{51.37-\mu}{\sigma}$ use CAS to solve simultaneously.

$\mu = 48.57$ mm and $\sigma = 4.16$ mm

M2+A1
3 marks

b. Use CAS: norm Cdf(47.15, 49.87, 48.57, 4.16) $\quad \Pr(47.15 < X < 49.87) = 0.2562$

M1+A1
2 marks

c. Use CAS: norm Cdf($-\infty$, 51.42 , 48.57, 4.16), $\Pr(X < 51.42) = 0.7534$

M1+A1
2 marks

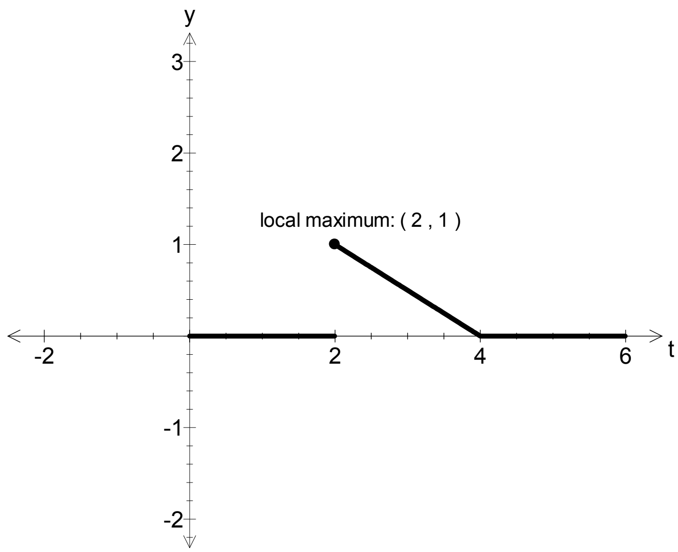
d. $\Pr(47.15 < X < 49.87 | X < 51.42) = \frac{\Pr(47.15 < X < 49.87)}{\Pr(X < 51.42)} = \frac{0.256248}{0.753358} = 0.3401$

M2+A1
3 marks

e. $\int_2^a (-0.5t+2)\,dt = 1, \quad a = 4$

A1
1 mark

f.



A2
2 marks

g. $\Pr\left(T > \frac{7}{3}\right) = \int_{\frac{7}{3}}^{4} (-0.5t+2)\,dt = 0.6944$

M1+A1
2 marks

h. $\mu = E(T) = \int_2^4 \left(t \times f(t)\right) dt = \int_2^4 (-0.5t + 2t)\,dt = 2.6667$

The average time taken to manufacture the large amount of candles is 2.6667 hours.

M1+A1
2 marks

i. $\text{sd}(T) = \sqrt{\text{var}(T)} = \sqrt{\int_2^4 t^2 \times (-0.5t+2)dt - (2.6667)^2} = 0.4714$

$\Pr\left(2.6667 - 2(0.4714) \le T \le 2.6667 + 2(0.4714)\right)$

$= \int_2^{3.6095} (-0.5t+2)\,dt = 0.9619$ because $f(t) = 0, t < 2.$

M2+A1
3 marks

Question 3

a. $h = \int \sin\left(\frac{\pi t}{6}\right) dx = \frac{-6}{\pi}\cos\left(\frac{\pi t}{6}\right) + c$

$0 = \frac{-6}{\pi}\cos(0) + c, \quad c = \frac{6}{\pi}$

$h = \frac{-6}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi}$

M2+A1

3 marks

b. Sketch the graphs and find points of intersection or solve $\frac{-6}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi} \leq 2$ on CAS.

between 6a.m. – 9:05 a.m., between 2:55 p.m. – 9:05 p.m, between 2 a.m. – 6 a.m

M1+A1

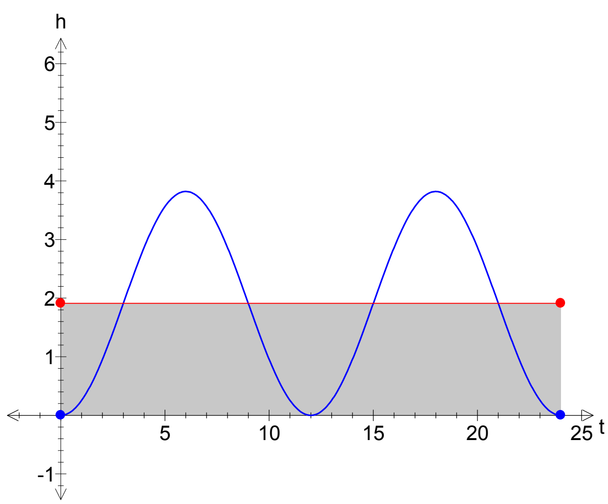
2 marks

c. average depth of $h = \frac{1}{24-0}\int_0^{24}\left(\frac{-6}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi}\right) dt = \frac{6}{\pi}$ m.

M1+A1

2 marks

d.



A3

3 marks

e. Reflection in the $x-$axis, dilation by a factor of $\frac{6}{\pi}$ from the $x-$axis, dilation by a factor of $\frac{6}{\pi}$ from the $y-$axis, translation of $\frac{6}{\pi}$ units up.

M1+A1

2 marks

f. Use CAS and solve $0 = \frac{-6}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi}$. Substitute the parameter of the calculator with an appropriate variable.

$t = 12n,\ n \in N.$

A1

1 mark

g. $k(t+\pi) = \cos(2t + 2\pi) = \cos(2t) = k(2t)$. The two expressions are identical and so $k(t) = \cos(2t)$ is a solution of the functional equation.

M1+A1

2 marks

Question 4

a. $A = 4hx + x^2 = 675$

$h = \frac{675 - x^2}{4x}$

M1+A1
2 marks

b.

i. $V = x^2 h = x^2 \left(\frac{675 - x^2}{4x}\right) = \frac{675x - x^3}{4}$

To be a maximum: $\frac{dV}{dx} = 0$

$\frac{dV}{dx} = \frac{675}{4} - \frac{3x^2}{4} = 0$

$3x^2 - 675 = 0$

$x^2 = 225$

$x = 15cm$ or use CAS – graph to find the maximum.

$h = \frac{675 - 15^2}{4 \times 15} = \frac{15}{2} cm$

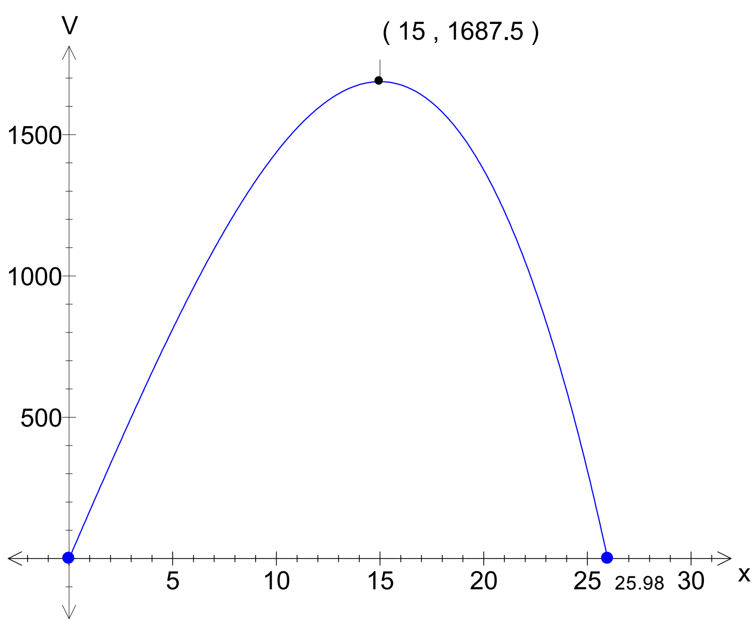
$V = 225 \times \frac{15}{2} = 1687.5 \text{ cm}^3$

M3+A1
4 marks

ii Domain: $(0, 25.98)$

M1+A1
2 marks

c.



A2
2 marks

