THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2011

Question 1

a.

$$y = \sqrt{2x^2 - 1}$$
$$= (2x^2 - 1)^{\frac{1}{2}}$$

Method 1

$$\frac{dy}{dx} = \frac{1}{2}(2x^2 - 1)^{-\frac{1}{2}} \times 4x$$ **(1 mark)**

$$= \frac{2x}{(2x^2 - 1)^{\frac{1}{2}}}$$

$$= \frac{2x}{\sqrt{2x^2 - 1}}$$

(1 mark)

Method 2

$$y = (2x^2 - 1)^{\frac{1}{2}} \qquad \text{let } u = 2x^2 - 1$$

$$y = u^{\frac{1}{2}} \qquad \frac{du}{dx} = 4x$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$ **(1 mark)**

$$= \frac{1}{2u^{\frac{1}{2}}}$$

$$= \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{(chain rule)}$$

$$= \frac{1}{2\sqrt{u}} \times 4x$$

$$= \frac{4x}{2\sqrt{2x^2 - 1}}$$

$$= \frac{2x}{\sqrt{2x^2 - 1}}$$ **(1 mark)**

b. $f(x)=\dfrac{x}{e^{3x}}$

$f'(x)=\dfrac{e^{3x}\times 1-3e^{3x}\times x}{(e^{3x})^2}$ (quotient rule)

$=\dfrac{e^{3x}-3xe^{3x}}{e^{6x}}$ **(1 mark)**

$f'(1)=\dfrac{e^3-3e^3}{e^6}$

$=\dfrac{-2e^3}{e^6}$

$=\dfrac{-2}{e^3}$

(1 mark)

Question 2

$\log_e(3)+2\log_e(x)=\log_e(4x)$

$\log_e(3)+\log_e(x^2)=\log_e(4x)$ **(1 mark)**

$\log_e(3x^2)=\log_e(4x)$

$3x^2=4x$

$3x^2-4x=0$

$x(3x-4)=0$

$x=0$ or $x=\dfrac{4}{3}$ **(1 mark)**

but $\log_e(x)$ is not defined for $x=0$ so $x=\dfrac{4}{3}$

(1 mark)

Question 3

a. $g(x)=3\log_e(x-2)$

Let $y=3\log_e(x-2)$

Swap x and y for inverse. **(1 mark)**

$x=3\log_e(y-2)$

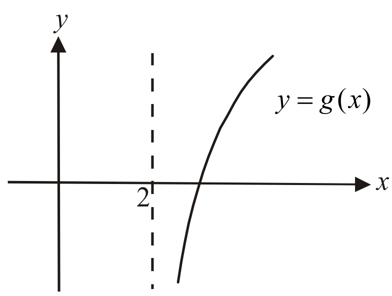
$\dfrac{x}{3}=\log_e(y-2)$

$e^{\frac{x}{3}}=y-2$

$y=2+e^{\frac{x}{3}}$

$d_g=(2,\infty)\quad r_g=R$

So $d_{g^{-1}}=R\qquad r_{g^{-1}}=(2,\infty)$



So $g^{-1}:R\to R,\ g^{-1}(x)=2+e^{\frac{x}{3}}$ **(1 mark)** – correct rule

(1 mark) – correct domain

b. **i.** $h(x)=g^{-1}(g(x))$

$=x$ **(1 mark)**

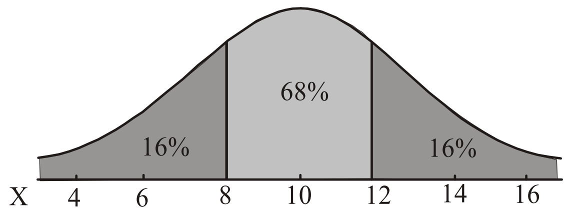
ii. $d_h=d_g$ (since h is a composite function)

$=(2,\infty)$ **(1 mark)**

Question 4

a. Note that since variance = 4, standard deviation $= \sqrt{4} = 2$.

$\Pr(X > 12) = 0.16$



(1 mark)

b. $\Pr(X > 12 | X > 10)$

$= \dfrac{\Pr(X > 12 \cap X > 10)}{\Pr(X > 10)}$ (conditional probability) **(1 mark)**

$= \dfrac{\Pr(X > 12)}{\Pr(X > 10)}$

$= \dfrac{0.16}{0.5}$

$= 0.32$

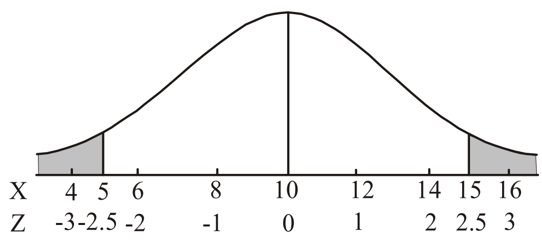
(1 mark)

c. $z = \dfrac{x - \mu}{\sigma}$

$z = \dfrac{5 - 10}{2}$

$z = -2.5$

(1 mark)



Because of the symmetry of the normal curve,
$\Pr(Z > 2.5) = \Pr(X < 5)$
So $a = 2.5$

(1 mark)

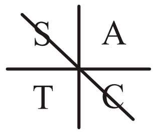
Question 5

$$\sin\left(\frac{x}{2}\right)+\frac{1}{\sqrt{3}}\cos\left(\frac{x}{2}\right)=0$$

$$\sin\left(\frac{x}{2}\right)=-\frac{1}{\sqrt{3}}\cos\left(\frac{x}{2}\right)$$

$$\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}=-\frac{1}{\sqrt{3}}$$

$$\tan\left(\frac{x}{2}\right)=-\frac{1}{\sqrt{3}}$$



(1 mark)

$$\frac{x}{2}=\frac{5\pi}{6}+n\pi,\quad n\in Z$$

(1 mark)

$$x=2\left(\frac{5\pi}{6}+n\pi\right),\quad n\in Z$$

$$x=\frac{5\pi}{3}+2n\pi,\quad n\in Z$$

(1 mark)

(alternative answer $\frac{x}{2}=\frac{-\pi}{6}+n\pi,\ n\in Z$ so $x=\frac{-\pi}{3}+2n\pi,\ n\in Z$)

Question 6

a. We have a binominal distribution where $n=3$ and $p=0.6$.

Method 1

$$\begin{aligned}\Pr(X\geq 2)&=\Pr(X=2)+\Pr(X=3)\\&={}^3C_2(0.6)^2(0.4)^1+{}^3C_3(0.6)^3(0.4)^0\\&=3\times 0.36\times 0.4+0.6^3\\&=0.432+0.216\\&=0.648\end{aligned}$$

(1 mark) – recognition of binominal distribution
(1 mark) – correct answer

Method 2

$$\begin{aligned}\Pr(X\geq 2)&=1-\Pr(X<2)\\&=1-\{\Pr(X=0)+\Pr(X=1)\}\\&=1-\left\{{}^3C_0(0.6)^0(0.4)^3+{}^3C_1(0.6)^1(0.4)^2\right\}\\&=1-(0.4^3+3\times 0.6\times 0.16)\\&=1-(0.064+0.288)\\&=1-0.352\\&=0.648\end{aligned}$$

(1 mark) – recognition of binominal distribution
(1 mark) – correct answer

b. Let the number of orders placed at the drive-through be *n*.

$$\Pr(X \geq 1) = 0.84$$
$$1 - \Pr(X = 0) = 0.84$$ **(1 mark)**

$$1 - {}^nC_0(0.6)^0(0.4)^n = 0.84$$ **(1 mark)**

$$1 - 1 \times 1 \times (0.4)^n = 0.84$$
$$-(0.4)^n = -0.16$$
$$(0.4)^n = 0.16$$
$$n = 2$$

(1 mark)

Two orders need to be placed.

Question 7

We are looking for $\frac{dr}{dt}$, the rate at which the radius of the balloon is changing.

Now, $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ (chain rule) **(1 mark)**

Now, $V = \frac{4}{3}\pi r^3$ (formula sheet)

$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

Also $\frac{dV}{dt} = 2$ (given)

So $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

becomes $\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 2$

$$= \frac{1}{2\pi r^2}$$

When $r = 4$, $\frac{dr}{dt} = \frac{1}{32\pi}$ **(1 mark)**

The radius of the balloon is increasing at the rate of $\frac{1}{32\pi}$ cm/sec.

(1 mark)

Question 8

$g: R\backslash\{0\} \to R,\ g(x) = 1 + \frac{1}{x}$

To Show: $4g(2u) - g(-u) = 3g(u)$

$$\begin{aligned} LHS &= 4g(2u) - g(-u) \\ &= 4\left(1 + \frac{1}{2u}\right) - \left(1 - \frac{1}{u}\right) \end{aligned}$$ **(1 mark)**

$$\begin{aligned} &= 4 + \frac{4}{2u} - 1 + \frac{1}{u} \\ &= 3 + \frac{2}{u} + \frac{1}{u} \\ &= 3 + \frac{3}{u} \\ &= 3\left(1 + \frac{1}{u}\right) \\ &= 3g(u) \\ &= RHS \end{aligned}$$

as required.

(1 mark)

Question 9

$f(x+h) \approx f(x) + h\,f'(x)$

$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$ **(1 mark)**

$h = 0.03$ **(1 mark)**

So $f(x+h) \approx f(x) + h\,f'(x)$

becomes $f(x+0.03) \approx \sqrt{x} + \frac{0.03}{2\sqrt{x}}$

$$\begin{aligned} f(9+0.03) &\approx \sqrt{9} + \frac{0.03}{2 \times \sqrt{9}} \\ &= 3 + \frac{0.03}{6} \\ &= 3 + 0.005 \\ &= 3.005 \end{aligned}$$

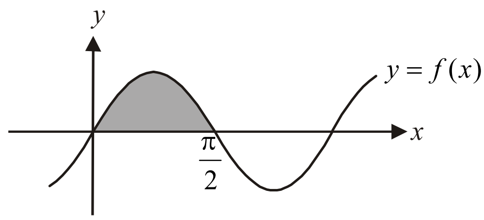
(1 mark)

Question 10

The period of the graph of $y = a\sin(2x)$ is $\frac{2\pi}{2} = \pi$ so the graph intersects the x-axis at the right end of the shaded region at $x = \frac{\pi}{2}$.

(1 mark)

So $\int_0^{\frac{\pi}{2}} a\sin(2x)dx = 4$ **(1 mark)**



$$a\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}} = 4$$ **(1 mark)**

$$-\frac{a}{2}(\cos(\pi) - \cos(0)) = 4$$

$$-\frac{a}{2}(-1-1) = 4$$

$$-\frac{a}{2} \times -2 = 4$$

$$a = 4$$

(1 mark)

Question 11

a. Since the graph of $y = f(x)$ is not smooth at the point where $x = 0$, then $d_{f'} = R \setminus \{0\}$.

(1 mark)

b. $f(x) = 2|x| - 3x^4 + 1$

Method 1

$$f(x) = \begin{cases} 2x - 3x^4 + 1 & \text{if } x \geq 0 \\ -2x - 3x^4 + 1 & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2 - 12x^3 & \text{if } x > 0 \\ -2 - 12x^3 & \text{if } x < 0 \end{cases}$$

(1 mark)
(1 mark)

Method 2

$$f'(x) = \frac{2|x|}{x} - 12x^3 \quad \text{for } x \in R \setminus \{0\}$$

(1 mark) – first term
(1 mark) – second term