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MATHS METHODS (CAS) UNITS 3 & 4
TRIAL EXAMINATION 2
SOLUTIONS
2011

SECTION 1 – Multiple-choice answers

1. D	9. A	17. A
2. C	10. C	18. B
3. C	11. D	19. D
4. D	12. B	20. D
5. B	13. C	21. E
6. C	14. C	22. B
7. A	15. E	
8. B	16. A	

SECTION 1 – Multiple-choice solutions

Question 1

$y = \log_e(2x+1)$
For this function to be defined,
$2x + 1 > 0$

$x > -\frac{1}{2}$

So maximal domain is $x \in \left(-\frac{1}{2}, \infty\right)$.

The answer is D.

Question 2

Sketch the graph of $y = f(x)$, taking particular note of the domain.

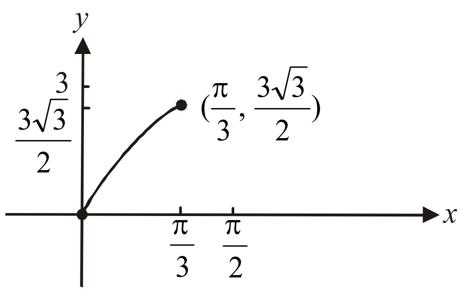
Since $f\left(\frac{\pi}{3}\right) = 3\sin\left(\frac{\pi}{3}\right)$

$= 3 \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}$

$(\approx 2.59...)$

So $r_f = \left[0, \frac{3\sqrt{3}}{2}\right]$



The answer is C.

Question 3

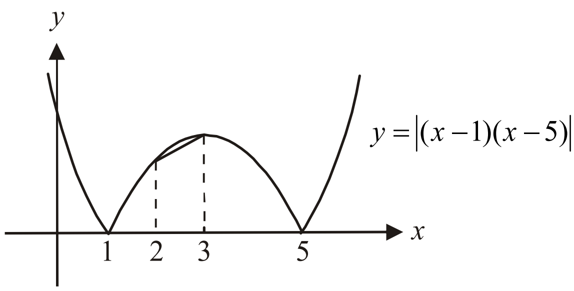
$$\begin{aligned}\text{Approximate area} &= f(2)\times 1 + f(3)\times 1\\ &= \log_e(2)+\log_e(3)\\ &= \log_e(6)\text{ square units}\end{aligned}$$

The answer is C.

Question 4

$$\begin{aligned}\text{Average rate of change} &= \frac{f(3)-f(2)}{3-2}\\ &= \frac{|(3-1)(3-5)|-|(2-1)(2-5)|}{1}\\ &= |2\times -2|-|1\times -3|\\ &= |-4|-|-3|\\ &= 4-3\\ &= 1\end{aligned}$$

A quick sketch tells you that the first two answers are not feasible. We are looking for the gradient of the line segment joining the points where $x = 2$ and $x = 3$ on the graph of $y = |(x-1)(x-5)|$.



The answer is D.

Question 5

$$\begin{bmatrix} 2 & k-2 \\ k+1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For no solutions or infinitely many solutions,

$$\begin{aligned} \det\begin{bmatrix} 2 & k-2 \\ k+1 & 2 \end{bmatrix} &= 0 \\ 2\times 2-(k+1)(k-2) &= 0 \\ 4-(k^2-k-2) &= 0 \\ 4-k^2+k+2 &= 0 \\ k^2-k-6 &= 0 \\ (k-3)(k+2) &= 0 \end{aligned}$$

If $k=3$, we have

$$\begin{aligned} 2x+y&=2 \quad -(1) \\ 4x+2y&=-1 \quad -(2) \end{aligned}$$

$(2)\div 2 \quad 2x+y=-\frac{1}{2}$

We have parallel lines so no solutions for $k=3$.

If $k=-2$, we have

$$\begin{aligned} 2x-4y&=2 \quad -(1) \\ -x+2y&=-1 \quad -(2) \end{aligned}$$

$(1)\div 2 \quad x-2y=1$

$(2)\times -1 \quad x-2y=1$

These equations represent the same line so infinite solutions for $k=-2$.
The answer is B.

Question 6

$y = x - 3$ and $y = x^2 + kx - 1$

$$x^2 + kx - 1 = x - 3$$

$$x^2 + kx - x - 1 + 3 = 0$$

$$x^2 + (k-1)x + 2 = 0 \quad \text{(equation of intersection)}$$

For no solutions; and therefore no points of intersection,

$$\Delta < 0$$

$$b^2 - 4ac < 0$$

$$(k-1)^2 - 4 \times 1 \times 2 < 0$$

Method 1 – using CAS

Solve $(k-1)^2 - 8 < 0$ for k

$1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$

The answer is C.

Method 2 – by hand

Do a quick sketch of $y = (k-1)^2 - 8$.

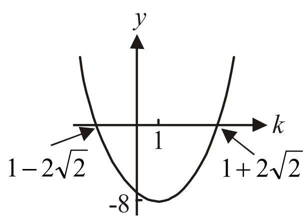
If $(k-1)^2 - 8 = 0$

$$(k-1)^2 = 8$$

$$k - 1 = \pm\sqrt{8}$$

$$k - 1 = \pm 2\sqrt{2}$$

$$k = 1 \pm 2\sqrt{2}$$



So, from the graph, $(k-1)^2 - 8 < 0$

for $1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$

The answer is C.

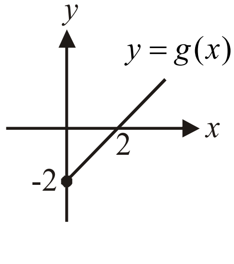
Question 7

$f : [0, \infty) \to R,\ f(x) = x - 1$

$$\begin{aligned} g(x) &= f(f(x)) \\ &= f(x-1) \\ &= (x-1) - 1 \\ &= x - 2 \end{aligned}$$

$$d_g = d_f = [0, \infty)$$

$$r_g = [-2, \infty)$$



So $d_{g^{-1}} = r_g = [-2, \infty)$

The answer is A.

Question 8

$y = \dfrac{e^x - e^{3x}}{x^2}$

Using CAS,

$\dfrac{dy}{dx} = \dfrac{-e^x((3x-2)e^{2x} - x + 2)}{x^3}$

At $x = 2$, $\dfrac{dy}{dx} = \dfrac{-e^6}{2}$.

The answer is B.

Question 9

<u>Method 1</u> – using CAS

Let $y = \sqrt{x-2}$

Swap *x* and *y* for inverse to obtain $x = \sqrt{y-2}$ and solve for *y*.

$y = x^2 + 2$ and $x \geq 0$

So $d_{f^{-1}} = [0, \infty)$

$f^{-1} : [0, \infty) \to R,\ f^{-1}(x) = x^2 + 2$

The answer is A.

<u>Method 2</u> – by hand

$f : [2, \infty) \to R,\ f(x) = \sqrt{x-2}$

Let $y = \sqrt{x-2}$

Swap *x* and *y* for inverse.

$x = \sqrt{y-2}$

$x^2 = y - 2$

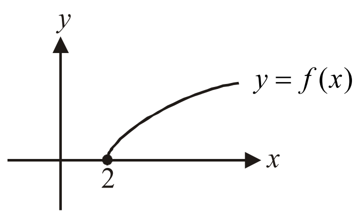
$y = x^2 + 2$

$d_f = [2, \infty)\ \ r_f = [0, \infty)$

So $d_{f^{-1}} = r_f = [0, \infty)$

$f^{-1} : [0, \infty) \to R,\ f^{-1}(x) = x^2 + 2$

The answer is A.



Question 10

$$\text{Average value} = \frac{1}{4-0}\int_0^4 2x\log_e(x^2+2)\,dx$$

$$= 8.6601...$$

The closest answer is 8.66.

The answer is C.

Question 11

This is a binomial distribution with $n = 10$ and $p = \frac{1}{6}$.

$\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$

$\text{binom cdf}\left(10, \frac{1}{6}, 0, 2\right) = 0.775227...$

So $\Pr(X < 3) = 0.775227....$

The closest answer is 0.7752.

The answer is D.

Question 12

$\text{normal cdf}(-\infty, 20, 23, 2) = 0.066807...$

So 6.68…% of people completing the survey did so in less than 20 minutes.

The closest answer is 6.7%.

The answer is B.

Question 13

Method 1

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - (\text{E}(X))^2 \\ &= 0^2 \times 0.4 + 2^2 \times 0.1 + 5^2 \times 0.5 - 2.7^2 \\ &= 5.61\end{aligned}$$

The answer is C.

Method 2

$$\begin{aligned}\text{Var}(X) &= \sum(x - \mu)^2\, p(x) \qquad \text{(from formula sheet)} \\ &= (0 - 2.7)^2 \times 0.4 + (2 - 2.7)^2 \times 0.1 + (5 - 2.7)^2 \times 0.5 \\ &= 2.916 + 0.049 + 2.645 \\ &= 5.61\end{aligned}$$

The answer is C.

Question 14

<u>Method 1</u> – using CAS

Solve $\int_0^a \frac{3\sqrt{x}}{2}\,dx = 0.5$ for a.

$$a = 0.6299$$

The answer is C.

<u>Method 2</u> – by hand

$$\int_0^a \frac{3\sqrt{x}}{2}\,dx = 0.5$$

$$\frac{3}{2}\int_0^a x^{\frac{1}{2}}\,dx = 0.5$$

$$\frac{3}{2}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^a = 0.5$$

$$\left[x^{\frac{3}{2}}\right]_0^a = 0.5$$

$$a^{\frac{3}{2}} - 0 = 0.5$$

$$\left(a^{\frac{3}{2}}\right)^{\frac{2}{3}} = 0.5^{\frac{2}{3}}$$

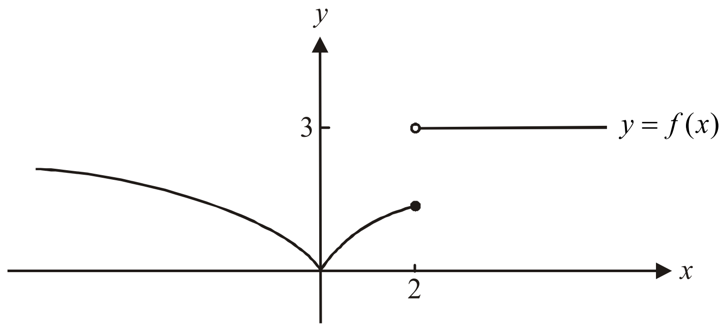
$$a = 0.5^{\frac{2}{3}}$$

$$a = 0.6299...$$

The answer is C.

Question 15

Sketch the function *f*.



The derivative function f' is defined for $x \in R\backslash\{0,2\}$.
The answer is E.

Question 16

Method 1 - using CAS

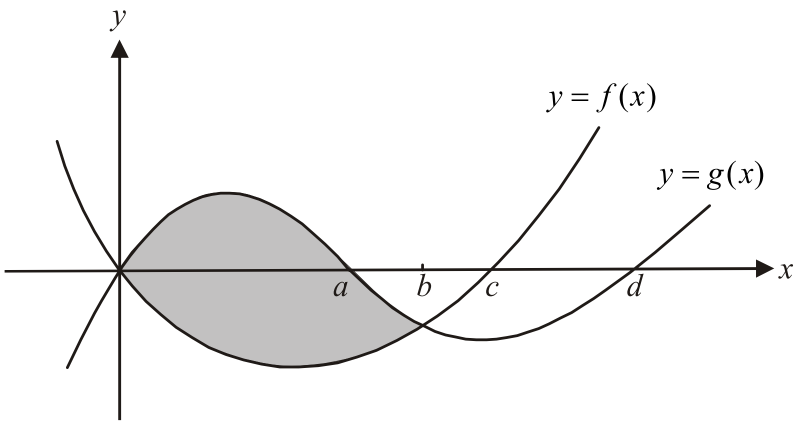
The equation of the tangent to the curve $y = x^{\frac{1}{3}}$ at the point where $x = 8$ is given by
$y = \frac{x}{12} + \frac{4}{3}$.
The answer is A.
Method 2 – by hand

$$y = x^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$
$$= \frac{1}{3\left(\sqrt[3]{x}\right)^2}$$

At $x = 8$
$$\frac{dy}{dx} = \frac{1}{3 \times \left(\sqrt[3]{8}\right)^2}$$
$$= \frac{1}{3 \times 2^2}$$
$$= \frac{1}{12}$$

So $m = \frac{1}{12}$ and $(x_1,\ y_1) = (8,\ 2)$
$$y - y_1 = m(x - x_1)$$
$$y - 2 = \frac{1}{12}(x - 8)$$
$$y = \frac{x}{12} - \frac{8}{12} + 2$$
$$y = \frac{x}{12} + \frac{4}{3}$$
The answer is A.

Question 17



The area required is shaded.
The graphs of $y = f(x)$ and $y = g(x)$ intersect at the points where $x = 0$ and $x = b$. Over this interval $x \in (0, b)$, $g(x) > f(x)$.

So area required $= \int_0^b (g(x) - f(x))dx$.

The answer is A.

Question 18

$y = 4e^{x+1}$

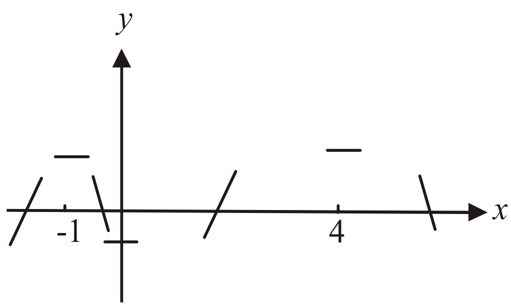
The graph of $y = e^x$ has been dilated from the x-axis by a factor of 4, and translated 1 unit to the left.
The rule could therefore be

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

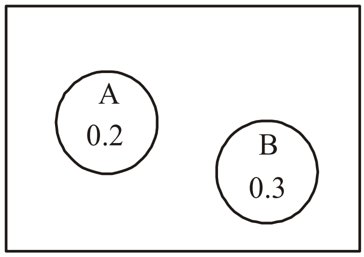
The answer is B.

Question 19

Stationary points occur where $x = -1, 0$ and 4.
The gradient is positive for $x < -1$ and $0 < x < 4$.
The gradient is negative for $-1 < x < 0$ and $x > 4$.
So, there is a local maximum where $x = -1$ and $x = 4$ and a local minimum at $x = 0$.
Only option D is true.
The answer is D.



Question 20



For mutually exclusive events $\Pr(A \cap B) = 0$
so option D is not true.
The other options can be verified using a Venn diagram.
The answer is D.

Question 21

The graph of $y = f(x)$ has been

- reflected in the x-axis
- dilated by a factor of $\frac{1}{2}$ from the y-axis
- translated 1 unit to the right
- translated 3 units up

to become the graph of

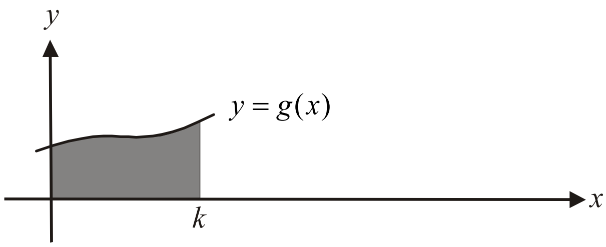
$$\begin{aligned} y &= 3 - f(2x - 2) \\ &= 3 - f(2(x - 1)) \end{aligned}$$

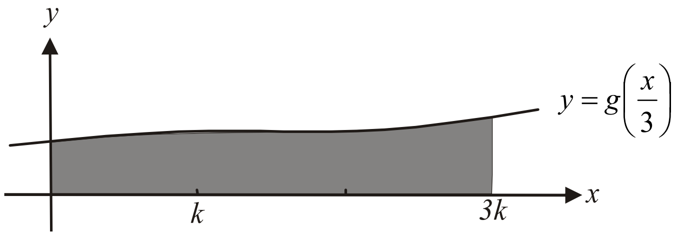
Only graph E shows this.
The answer is E.

Question 22

$$\begin{aligned} 4\int_0^{3k} \left(g\left(\frac{x}{3}\right) - 1 \right) dx &= 4\int_0^{3k} g\left(\frac{x}{3}\right) dx - 4\int_0^{3k} 1\,dx \\ &= 4 \times 3 \times 2k - 4\left[x \right]_0^{3k} \\ &= 24k - 4(3k - 0) \\ &= 12k \end{aligned}$$

Note that the graph of $y = g(x)$ has been dilated by a factor of 3 from the y-axis to become the graph of $y = g\left(\frac{x}{3}\right)$ so if $\int_0^k g(x)dx = 2k$ then $\int_0^{3k} g\left(\frac{x}{3}\right) dx = 3 \times 2k$.





The answer is B.

SECTION 2

Question 1

a. **i.** Method 1 – using CAS

Solve $2\cos\left(\frac{x}{4}\right)+1=0$ for x **(1 mark)**

$$x = \ldots -\frac{-8\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}, \ldots$$

From the graph, we are looking for the second smallest positive value of x; that is, not the first positive x-intercept.

So $d=\frac{16\pi}{3}$.

(1 mark)

Method 2 – by hand

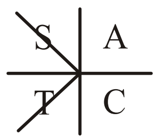
$2\cos\left(\frac{x}{4}\right)+1=0$ **(1 mark)**

$$2\cos\left(\frac{x}{4}\right)=-1$$

$$\cos\left(\frac{x}{4}\right)=-\frac{1}{2}$$

$$\frac{x}{4}=\ldots -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$$

$$x=\ldots -\frac{8\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}, \ldots$$

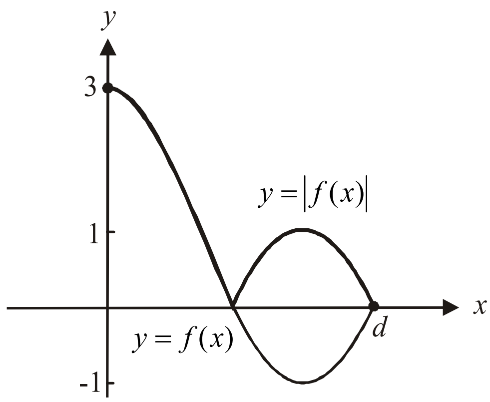


From the graph, we are looking for the second smallest positive value of x; that is, not the first positive x-intercept.

So $d=\frac{16\pi}{3}$.

(1 mark)

ii.

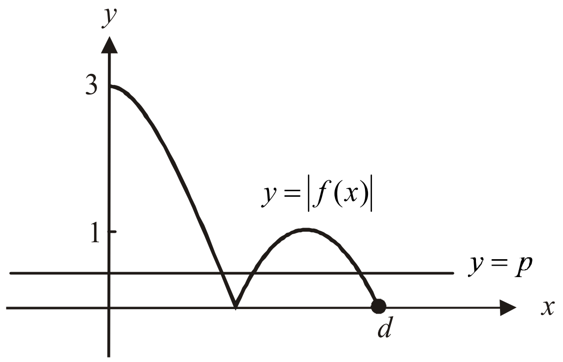


(1 mark)

iii. From the graph, the maximum value of $|f(x)|$ is 3.

(1 mark)

iv. Again from the graph, if $p<1$ and $p>0$ then there are three solutions to the equation $|f(x)|=p$.



If $p=1$, there are two solutions and if $p=0$ there are 2 solutions.
So $p\in(0,\ 1)$ or $\{p: 0<p<1\}$.

(1 mark) – correct values
(1 mark) – endpoints excluded

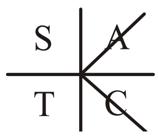
b. Method 1 – using CAS

Solve $2\cos\left(\frac{x}{4}\right)+1=\sqrt{3}+1$ for x. **(1 mark)**

$x=\frac{2\pi(12n+1)}{3}$ or $x=\frac{2\pi(12n-1)}{3}$, $n\in Z$
(1 mark) **(1 mark)**

Method 2 – by hand

$2\cos\left(\frac{x}{4}\right)+1=\sqrt{3}+1$ **(1 mark)**



$\cos\left(\frac{x}{4}\right)=\frac{\sqrt{3}}{2}$

$\frac{x}{4}=\frac{-\pi}{6}+2n\pi$ or $\frac{x}{4}=\frac{\pi}{6}+2n\pi$, $n\in Z$

$x=4\left(\frac{-\pi}{6}+2n\pi\right)$ or $x=4\left(\frac{\pi}{6}+2n\pi\right)$

$x=-\frac{2\pi}{3}+8n\pi$ or $x=\frac{2\pi}{3}+8n\pi$

$x=\pm\frac{2\pi}{3}+8n\pi \quad n\in Z$

(1 mark) –first answer **1 mark)** – second answer

(Note that $x=\frac{-2\pi}{3}+8n\pi$ and $x=\frac{2\pi}{3}+8n\pi$

$=\frac{-2\pi+24n\pi}{3}$ $\quad=\frac{2\pi+24n\pi}{3}$

$=\frac{2\pi(-1+12n)}{3}$ $\quad=\frac{2\pi(1+12n)}{3}$

which confirms the CAS answer.)

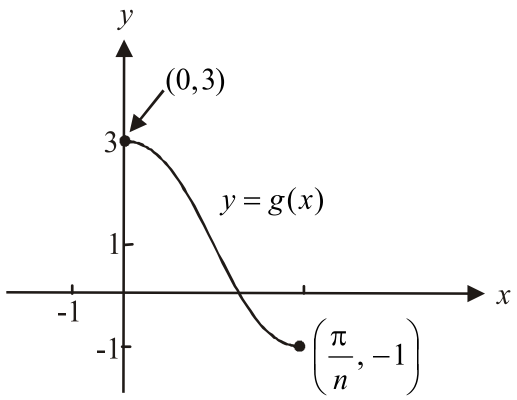
c. **i.** For $y = 2\cos(nx) + 1$,

$\text{period} = \dfrac{2\pi}{n}$

For g^{-1} to exist, g must be 1:1

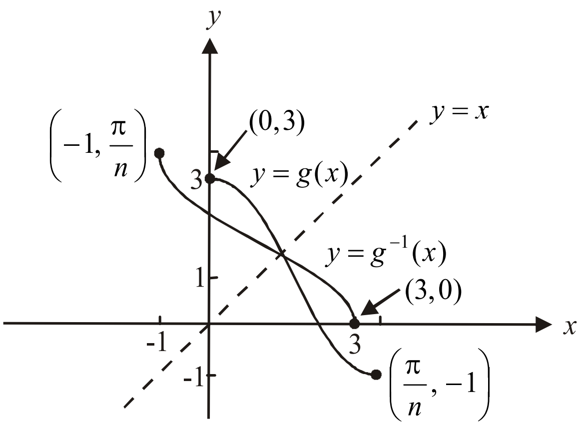
so $q = \dfrac{2\pi}{n} \div 2$

$= \dfrac{\pi}{n}$



(1 mark) – correct value of q
(1 mark) – correct endpoints and shape

ii.



Note that the points $\left(\dfrac{\pi}{n}, -1\right)$ and $\left(-1, \dfrac{\pi}{n}\right)$ can be anywhere along the x-axis and y-axis respectively as long as the two axes are consistent.

(1 mark) – correct endpoints
(1 mark) – correct shape

iii. $g(x) = 2\cos(nx) + 1$

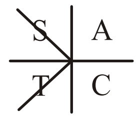
$y = 2\cos(nx) + 1$

Since $T\left(t,\ 1-\sqrt{3}\right)$ lies on g,

$1-\sqrt{3} = 2\cos(nt) + 1$ **(1 mark)**

$-\sqrt{3} = 2\cos(nt)$

$-\dfrac{\sqrt{3}}{2} = \cos(nt)$



$nt = \dfrac{5\pi}{6}$

$n = \dfrac{5\pi}{6t}$

as required. **(1 mark)**

iv. $g(x) = 2\cos(nx) + 1$

$g'(x) = -2n\sin(nx)$ **(1 mark)**

At $T\left(t,\ 1-\sqrt{3}\right)$

$$
\begin{aligned}
g'(x) &= -2n\sin(nt) \\
&= -2 \times \frac{5\pi}{6t} \times \sin\left(\frac{5\pi}{6t} \times t\right) \\
&= \frac{-10\pi}{6t} \times \sin\left(\frac{5\pi}{6}\right) \\
&= \frac{-5\pi}{3t} \times \frac{1}{2} \\
&= \frac{-5\pi}{6t}
\end{aligned}
$$

So the gradient of the normal to the graph of $y = g(x)$ at the point $T\left(t,\ 1-\sqrt{3}\right)$ is $\dfrac{6t}{5\pi}$. **(1 mark)**

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \left(1-\sqrt{3}\right) &= \frac{6t}{5\pi}(x - t) \\
y &= \frac{6t}{5\pi}(x - t) + 1 - \sqrt{3}
\end{aligned}
$$

(1 mark)

Total 18 marks

Question 2

a. Create a transition matrix.

$$\begin{array}{c} \text{this serve} \\ \begin{matrix} F & B \end{matrix} \end{array}$$

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \begin{matrix} F \\ B \end{matrix} \text{ next serve}$$

$$\begin{aligned} \Pr(F\ F\ F\ F) &= 0.4 \times 0.4 \times 0.4 \times 0.4 \\ &= 0.0256 \end{aligned}$$

(1 mark)

b. Method 1 – use the transition matrix

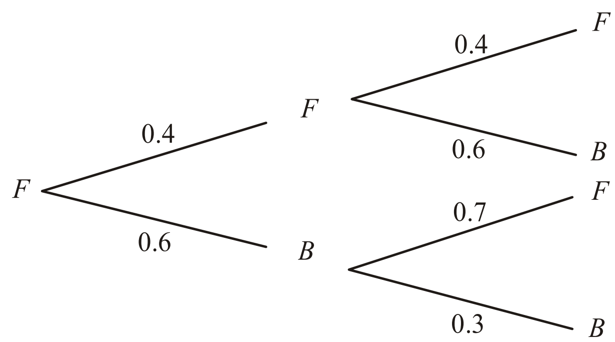
$$\text{next serve} \begin{matrix} F \\ B \end{matrix} \overset{\text{this serve: } F \quad B}{\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

(1 mark)

The probability that the third serve was to her forehand is 0.58.

(1 mark)

Method 2 – use a tree diagram



$\Pr(FFF) + \Pr(FBF)$ **(1 mark)**

$= 1 \times 0.4 \times 0.4 + 1 \times 0.6 \times 0.7$

$= 0.16 + 0.42$

$= 0.58$

(1 mark)

c. $\Pr(FFF) = 1 \times 0.4 \times 0.4 = 0.16$

$\Pr(FBF) = 1 \times 0.6 \times 0.7 = 0.42$

$\Pr(FFB) = 1 \times 0.4 \times 0.6 = 0.24$

$\Pr(FBB) = 1 \times 0.6 \times 0.3 = 0.18$

Set up a distribution table.

x	0	1	2
$\Pr(X = x)$	0.16	$0.42 + 0.24 = 0.66$	0.18

(2 marks)

(Check $0.16 + 0.66 + 0.18 = 1$)

$\text{E}(X) = 0 \times 0.16 + 1 \times 0.66 + 2 \times 0.18$ **(1 mark)**

$= 1.02$

(1 mark)

d. Method 1

Find the steady state.

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5384... \\ 0.4615... \end{bmatrix}$$

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^{21} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5384... \\ 0.4615... \end{bmatrix}$$

Steady state has been reached after 20 serves (maybe earlier).
In the long term 46%(to the nearest whole percent) of serves Sam received were to her backhand.

(1 mark)

Method 2

Let $X_n = 0$ represent Sam receiving a forehand serve.

Let $X_n = 1$ represent Sam receiving a backhand serve.

$$T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$$

So $a = 0.6$ and $b = 0.7$.

For steady state, $\Pr(X_n = 1) = \dfrac{a}{a+b} = \dfrac{0.6}{0.6+0.7} = \dfrac{0.6}{1.3} = 0.46$

In the long term 46% of serves Sam received were to her backhand.

(1 mark)

e. Method 1

The transition matrix for Wednesday is

this serve

$$\begin{array}{c} \quad F \qquad B \\ \begin{bmatrix} p & 1-2p \\ 1-p & 2p \end{bmatrix} \begin{matrix} F \\ B \end{matrix} \end{array} \text{ next serve}$$

Solve

$$\begin{bmatrix} p & 1-2p \\ 1-p & 2p \end{bmatrix}^{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.73 \\ 0.27 \end{bmatrix} \text{ for } p$$

(1 mark)

$p = 0.1$ or $p = 0.9$ **(1 mark)**

but from the transition matrix since

$\Pr(B \text{ this serve} | B \text{ last serve}) = 2p,$

$p \neq 0.9$

So $p = 0.1$

(1 mark)

Method 2

$$\Pr(FFF) + \Pr(FBF) = 0.73$$
$$1 \times p \times p + 1 \times (1-p)(1-2p) = 0.73$$
$$p^2 + 1 - 2p - p + 2p^2 = 0.73$$
$$3p^2 - 3p + 1 = 0.73$$

(1 mark)

$$3p^2 - 3p + 0.27 = 0$$
$$3(p^2 - p + 0.09) = 0$$
$$3(p-0.1)(p-0.9) = 0$$

$p = 0.1$ or $p = 0.9$ **(1 mark)**

but from the transition matrix since

$\Pr(B \text{ this serve} | B \text{ last serve}) = 2p$,

$p \neq 0.9$

So $p = 0.1$

(1 mark)

f. **i.** $\Pr(X < 4) = \int_2^4 \frac{29-4x}{45} dx$

$= \frac{34}{45}$

(1 mark)

ii. $\Pr(X < 3 | X < 4) = \frac{\Pr(X < 3 \cap X < 4)}{\Pr(X < 4)}$

$= \frac{\Pr(X < 3)}{\Pr(X < 4)}$

(1 mark)

$$\Pr(X < 3) = \int_2^3 \frac{29-4x}{45} dx$$
$$= \frac{19}{45}$$

So $\frac{\Pr(X < 3)}{\Pr(X < 4)} = \frac{19}{45} \div \frac{34}{45}$

$= \frac{19}{34}$

(1 mark)
Total 14 marks

Question 3

a. $C(0) = \dfrac{500}{100-0}$

$= 5\text{mg/m}^3$

(1 mark)

b. Since the function is continuous,

$C(p) = \dfrac{500}{100-p}$ and $C(p) = m$.

So $m = \dfrac{500}{100-p}$

(1 mark)

c. From the graph and part **a.**, the minimum value of m is 5 which occurs when $p = 0$.

(1 mark)

From the graph and part **b.**, the maximum value of m must occur when $p = 90$ (since the function is continuous).

So $m = \dfrac{500}{100-90}$

$= 50$

(1 mark)

d. Method 1 – using CAS

$C(t) = \dfrac{500}{100-t}$ for $0 < t < p$

$C'(t) = \dfrac{500}{(t-100)^2}$

(1 mark)

Method 2 – by hand

Let $y = \dfrac{500}{100-t}$

$= \dfrac{500}{u}$ where $u = 100 - t$

$= 500u^{-1}$ $\dfrac{du}{dt} = -1$

$\dfrac{dy}{du} = -500u^{-2}$

$= \dfrac{-500}{u^2}$

$\dfrac{dy}{dt} = \dfrac{dy}{du} \cdot \dfrac{du}{dt}$ (chain rule)

$= \dfrac{-500}{u^2} \times -1$

So $\dfrac{dy}{dt} = \dfrac{500}{(100-t)^2}$

So $C'(t) = \dfrac{500}{(100-t)^2}$ **(1 mark)**

Note $\dfrac{500}{(100-t)^2} = \dfrac{500}{(-1(t-100))^2} = \dfrac{500}{(t-100)^2}$ confirming the CAS answer above.

e. Solve $C'(t)=1$

i.e. $\dfrac{500}{(t-100)^2}=1$ for t.

(1 mark)

$t=77.6393...$ or $t=122.361...$

but $t<90$

So $t=77.64$ minutes (correct to 2 decimal places)

(1 mark)

f. $\text{average concentration} = \dfrac{1}{10-0}\displaystyle\int_0^{10}\dfrac{500}{100-t}dt$ **(1 mark)**

$=5.26803...$

$=5.27\text{mg/m}^3$ (correct to 2 decimal places)

(1 mark)

g. For $0\le t\le 90$ the maximum concentration is m mg/m^3.

From part **b.**, $m=\dfrac{500}{100-p}$.

For Victoria to survive, we require that

$$m<6,$$

Solve $\dfrac{500}{100-p}<6$ for p. **(1 mark)**

$$p<\frac{50}{3} \quad (\text{since } 0\le p<90, \text{ reject } p>100)$$

So in order for Victoria to survive, $p\in\left[0, \dfrac{50}{3}\right)$.

(1 mark)

Total 11 marks

Question 4

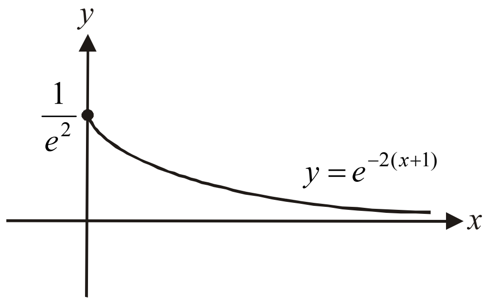
a. **i.** $f(g(x)) = f(2(x+1))$
$= e^{-2(x+1)}$

(1 mark)

ii. $d_{f \circ g} = d_g$
$= [0, \infty)$

(1 mark)

To find $r_{f \circ g}$ sketch the graph of $y = e^{-2(x+1)}$ and restrict the domain to $x \in [0, \infty)$.



y-intercept occurs when $x = 0$

$$y = e^{-2(0+1)}$$
$$= e^{-2}$$
$$= \frac{1}{e^2}$$

So $r_{f \circ g} = \left(0, \frac{1}{e^2}\right]$

(1 mark) – correct left endpoint and bracket
(1 mark) – correct right endpoint and bracket

iii. $\text{Area} = \int_0^1 e^{-2(x+1)}\,dx$ **(1 mark)**

$$= \frac{e^{-4}(e^2 - 1)}{2} \text{ square units}$$

(1 mark)

iv. The graph of $y = e^{-x}$ is

- dilated by a factor of $\frac{1}{2}$ from the y-axis and
- translated 1 unit left

to become the graph of $y = e^{-2(x+1)}$

(1 mark) – description of dilation
(1 mark) – description of translation

b. **i.** Stationary point occurs when $h'(x)=0$. **(1 mark)**

Solve $h'(x)=\left(2a\,e^{-a^2}-2x\,e^{-a^2}\right)e^{2ax-x^2}=0$ for x.
so $x=a$

Since $h(a)=1$, stationary point occurs at $(a,\,1)$.

(1 mark)

ii. $h(x)$ is strictly increasing for $x\in(-\infty,a]$.

(1 mark)

iii. $A=\text{length}\times\text{width}$
$=MN\times NP$
$=2(x-a)\times h(x)$
$=2(x-a)\times e^{-(x-a)^2}$
as required.

(1 mark)

iv. $A=2(x-a)\times e^{-(x-a)^2}$
Maximum occurs when $\frac{dA}{dx}=0$.
Solve $\frac{dA}{dx}=0$ for x. **(1 mark)**

$x=\frac{2a-\sqrt{2}}{2}$ or $x=\frac{2a+\sqrt{2}}{2}$

$x=a-\frac{\sqrt{2}}{2}$ or $x=a+\frac{\sqrt{2}}{2}$

Since $x>a$, (from the diagram) $x=a+\frac{\sqrt{2}}{2}$

(1 mark)

Substitute $x=a+\frac{\sqrt{2}}{2}$ into $A=2(x-a)\,e^{-(x-a)^2}$.

$$A=\sqrt{\frac{2}{e}}$$

Maximum area is $\sqrt{\frac{2}{e}}$ square units.

(1 mark)
Total 15 marks