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Student Name……………………………………

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

TRIAL EXAMINATION 2

2011

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section 1 and Section 2.
Section 1 consists of 22 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 24 of this exam.
Section 2 consists of 4 extended-answer questions.
Section 1 begins on page 2 of this exam and is worth 22 marks.
Section 2 begins on page 11 of this exam and is worth 58 marks.
There is a total of 80 marks available.
All questions in Section 1 and Section 2 should be answered.
Diagrams in this exam are not to scale except where otherwise stated.
Where more than one mark is allocated to a question, appropriate working must be shown.
Students may bring one bound reference into the exam.
Students may bring a CAS calculator into this exam.
A formula sheet can be found on page 23 of this exam.

SECTION 1

Question 1

The maximal domain of the function $y = \log_e(2x+1)$ is

A. $x \in R$

B. $x \in R \setminus \left\{-\frac{1}{2}\right\}$

C. $x \in \left[-\frac{1}{2}, \infty\right)$

D. $x \in \left(-\frac{1}{2}, \infty\right)$

E. $x \in \left[\frac{1}{2}, \infty\right)$

Question 2

The range of the function $f : \left[0, \frac{\pi}{3}\right] \to R,\ f(x) = 3\sin(x)$ is

A. $\left[0, \frac{3}{2}\right]$

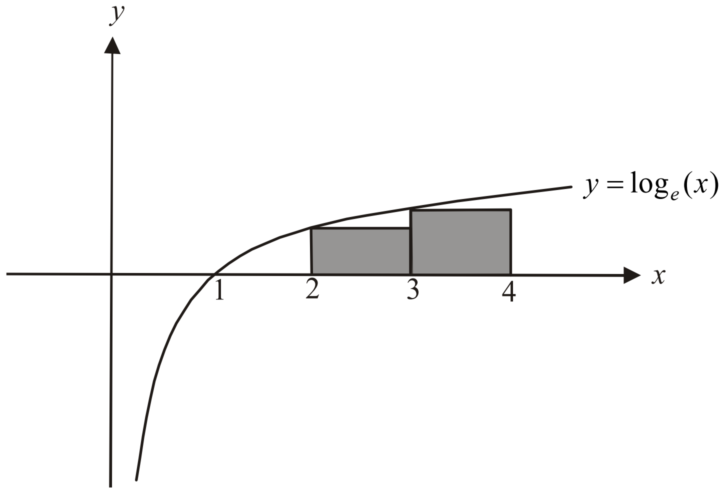
B. $\left[0, \frac{3}{\sqrt{2}}\right]$

C. $\left[0, \frac{3\sqrt{3}}{2}\right]$

D. $[0, 3]$

E. $[0, \pi]$

Question 3



The area under the curve $y = \log_e(x)$ between $x = 1$ and $x = 4$ is approximated by the two shaded rectangles shown above. This approximate area in square units is

A. $\log_e(1.5)$
B. $\log_e(5)$
C. $\log_e(6)$
D. $2\log_e(2)$
E. $3\log_e(2)$

Question 4

The average rate of change of the function $y = |(x-1)(x-5)|$ between $x = 2$ and $x = 3$ is

A. -1
B. 0
C. $\frac{1}{6}$
D. 1
E. 7

Question 5

The simultaneous linear equations

$$2x + (k-2)y = 2$$
$$(k+1)\,x + 2y = -1$$

have no solutions for

A. $k = -2$
B. $k = 3$
C. $k \in \{-2, 3\}$
D. $k \in R$
E. $k \in R \setminus \{-2, 3\}$

Question 6

The graph of $y = x - 3$ and $y = x^2 + kx - 1$ do **not** intersect for

A. $k = 1 \pm 2\sqrt{2}$
B. $-1 - 2\sqrt{2} < k < -1 + 2\sqrt{2}$
C. $1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$
D. $k < -1 - 2\sqrt{2}$ and $k > -1 + 2\sqrt{2}$
E. $k < 1 - 2\sqrt{2}$ and $k > 1 + 2\sqrt{2}$

Question 7

Let $f : [0, \infty) \rightarrow R,\ f(x) = x - 1$ and let $g(x) = f(f(x))$.

The domain of g^{-1}, the inverse function of g, is

A. $[-2, \infty)$
B. $(0, \infty)$
C. $[0, \infty)$
D. $[2, \infty)$
E. R

Question 8

The rate of change of the function $y = \dfrac{e^x - e^{3x}}{x^2}$ with respect to x, at the point where $x = 2$ is

A. e^{-6}
B. $\dfrac{-e^6}{2}$
C. $e^2 + e^{-4}$
D. $e^2 - \dfrac{e^6}{2}$
E. $\dfrac{-e^2(e^4 - 1)}{4}$

Question 9

Let $f : [2, \infty) \rightarrow R,\ f(x) = \sqrt{x - 2}$. The inverse function f^{-1} is given by

A. $f^{-1} : [0, \infty) \rightarrow R,\ f^{-1}(x) = x^2 + 2$
B. $f^{-1} : [0, \infty) \rightarrow R,\ f^{-1}(x) = (x + 2)^2$
C. $f^{-1} : [2, \infty) \rightarrow R,\ f^{-1}(x) = x^2 + 4$
D. $f^{-1} : [2, \infty) \rightarrow R,\ f^{-1}(x) = x^2 + 2$
E. $f^{-1} : [0,\ 2] \rightarrow R,\ f^{-1}(x) = (x + 2)^2$

Question 10

The average value of the function $g(x)=2x\log_e(x^2+2)$ for $0 \le x \le 4$ is closest to

A. 4.33
B. 6.07
C. 8.66
D. 22.62
E. 34.64

Question 11

A fair die is rolled 10 times. The probability that a 4 appears less than 3 times is closest to

A. 0.2093
B. 0.4845
C. 0.6372
D. 0.7752
E. 0.9303

Question 12

The times (in minutes) taken for an on-line survey to be completed are normally distributed with a mean of 23 minutes and a standard deviation of 2 minutes. The percentage of people completing the survey who did so in less than 20 minutes is closest to

A. 6.5%
B. 6.7%
C. 9.25%
D. 65%
E. 67%

Question 13

The discrete random variable *X* has the following probability distribution.

X	0	2	5
$\Pr(X=x)$	0.4	0.1	0.5

Given that the mean of *X* is 2.7, the variance of *X* is

A. 1.64
B. 2.37
C. 5.61
D. 7.29
E. 14.38

Question 14

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\dfrac{3\sqrt{x}}{2} & \text{if } x\in[0,\,1]\\ 0 & \text{otherwise}\end{cases}$$

The median of X is closest to

A. 0.50
B. 0.60
C. 0.63
D. 0.67
E. 1

Question 15

Let $f(x)=\begin{cases}x^{\frac{2}{5}}, & x\in(-\infty,\,2]\\ 3, & x\in(2,\,\infty)\end{cases}$

The derivative function f' is defined for

A. $x\in R$
B. $x\in\{0,2\}$
C. $x\in R\backslash\{0\}$
D. $x\in R\backslash\{2\}$
E. $x\in R\backslash\{0,2\}$

Question 16

The equation of the tangent to the curve with equation $y=x^{\frac{1}{3}}$ at the point $(8,2)$ is given by

A. $y=\dfrac{x}{12}+\dfrac{4}{3}$

B. $y=\dfrac{x}{12}+\dfrac{47}{6}$

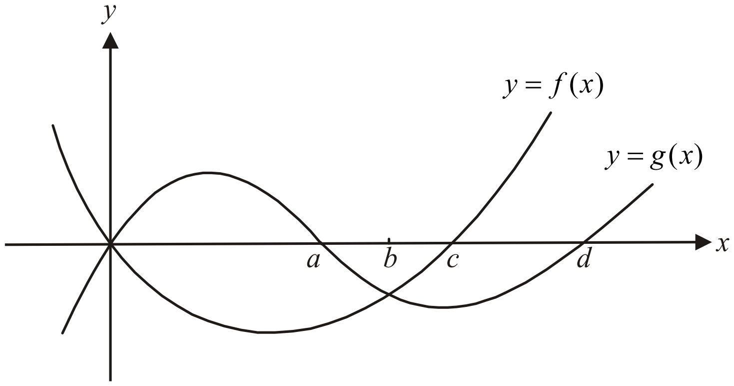
C. $y=\dfrac{x}{6}+\dfrac{2}{3}$

D. $y=\dfrac{x}{6}+\dfrac{23}{3}$

E. $y=\dfrac{4x}{3}-\dfrac{26}{3}$

Question 17

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



The area enclosed by these two graphs is given by

A. $\int_0^b (g(x) - f(x))dx$

B. $\int_0^b (g(x) + f(x))dx$

C. $\int_0^a (g(x))dx - \int_0^b f(x)dx$

D. $\int_0^a (g(x))dx - \int_0^c f(x)dx$

E. $\int_0^d (g(x))dx - \int_0^c f(x)dx$

Question 18

A transformation $T : R^2 \to R^2$, which maps the curve with equation $y = e^x$ onto the curve with equation $y = 4e^{x+1}$ could be given by

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Question 19

The function g is continuous and differentiable for $x \in R$. The function g satisfies the following conditions.

- $f'(x) = 0$ where $x = -1,\ 0$ and 4
- $f'(x) > 0$ where $x < -1$ and $0 < x < 4$
- $f'(x) < 0$ where $-1 < x < 0$ and $x > 4$

It is true to say that the graph of $y = g(x)$ has

A. a stationary point of inflection where $x = 0$
B. a point of inflection where $x = 0$
C. a local maximum where $x = 0$
D. a local minimum where $x = 0$
E. two local minimum

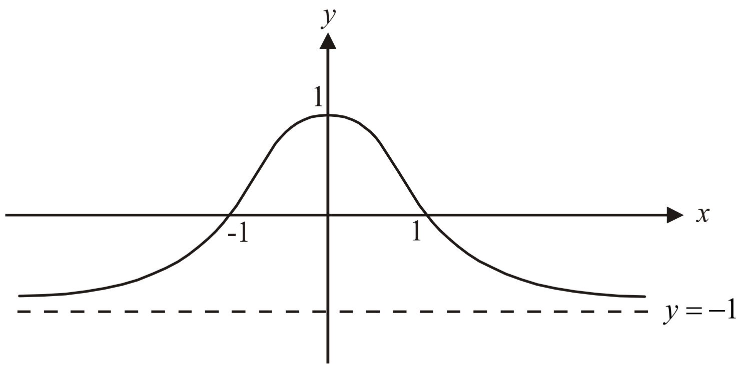
Question 20

For the mutually exclusive events A and B that occur in a sample space, $\Pr(A) = 0.2$ and $\Pr(B) = 0.3$. Which one of the following statements is **not** true?

A. $\Pr(A \cup B) = 0.5$
B. $\Pr(A' \cup B') = 1$
C. $\Pr(A' \cap B) = 0.3$
D. $\Pr(A \cap B) = 0.06$
E. $\Pr(A' \cap B') = 0.5$

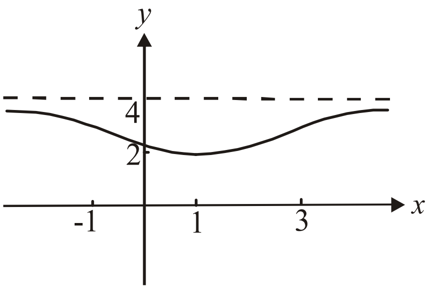
Question 21

The graph of $y = f(x)$ is shown below.

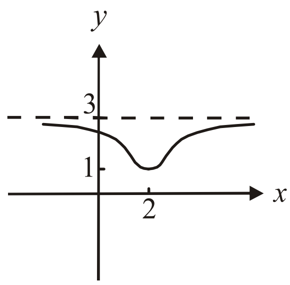


The graph of $y = 3 - f(2x-2)$ is given by

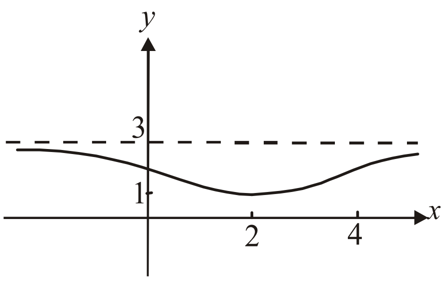
A.



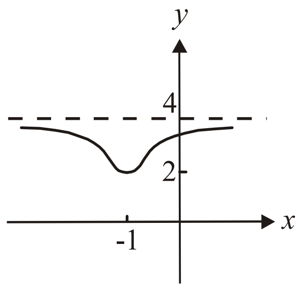
B.



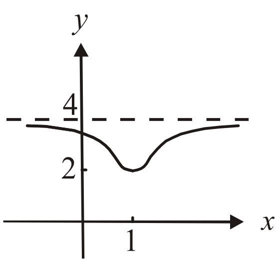
C.



D.



E.



Question 22

The function g is a smooth, continuous function for $x \in R$ and $g(x) \geq 0$ for $x \in [0,\ k]$.
Given that $\int_0^k g(x)dx = 2k$, then $4\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$ is equal to

A. 0
B. $12k$
C. $20k$
D. $12k-4$
E. $24k-4$

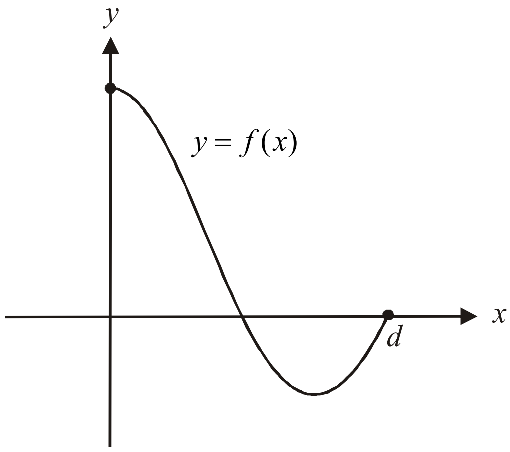
SECTION 2

Answer all questions in this section.

Question 1

a. Let $f:[0,\ d]\rightarrow R,\ f(x)=2\cos\left(\frac{x}{4}\right)+1$ where d is a constant.

The graph of $y=f(x)$ is shown below.



i. Find the value of d.

ii. Sketch the graph of $y=|f(x)|$ on the set of axes shown above.

iii. State the maximum value of $|f(x)|$.

iv. Find the value(s) of p for which the equation $|f(x)|=p$ has three solutions.

$2+1+1+2=6$ marks

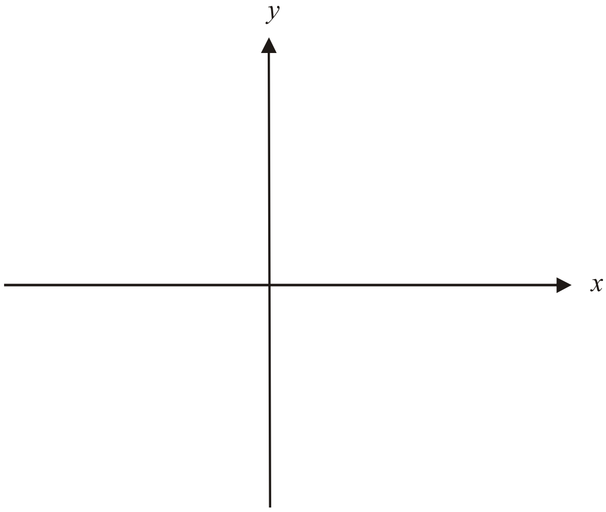
b. Let $h: R \to R,\ h(x) = 2\cos\left(\frac{x}{4}\right) + 1$

Find the general solution for x of the equation $h(x) = \sqrt{3} + 1$

3 marks

c. Let $g:[0,\ q]\to R,\ g(x)=2\cos(nx)+1$, where n is a real constant and q is the largest possible value so that the inverse function g^{-1} exists.

i. Find q in terms of n and hence sketch the graph of $y=g(x)$ on the set of axes below. Label the coordinates of the endpoints clearly. It is not necessary to find the x-intercept.



ii. On the same set of axes sketch the graph of $y=g^{-1}(x)$. Label the coordinates of the endpoints clearly. It is not necessary to find the y-intercept.

The function g passes through the point $T\left(t,\, 1-\sqrt{3}\right)$

iii. Show that $n = \dfrac{5\pi}{6t}$.

iv. Find the equation of the normal to the graph of $y = g(x)$ at the point $T\left(t, 1-\sqrt{3}\right)$ which involves the variables x and y and the parameter t.

$2+2+2+3=9$ marks
Total 18 marks

Question 2

Sam is a tennis player. She has a machine that serves tennis balls to her forehand side or to her backhand side. The machine has settings so that the order in which the forehand serves and backhand serves are delivered can be mixed up. On Tuesday, if Sam received a serve to her forehand then the probability that the next serve would be to her backhand was 0.6. If she received a serve to her backhand then the probability that the next serve would be to her backhand was 0.3.

On Tuesday the first serve that Sam received was to her forehand.

a. Find the probability that the next four serves received by Sam on Tuesday were all to her forehand.

1 mark

b. Find the probability that the third serve received by Sam on Tuesday was to her forehand.

2 marks

c. The first ball after lunch on Tuesday was served to Sam's forehand. For the first three balls served after lunch, what is the expected number to be served to Sam's backhand?

4 marks

d. In the long term what percentage of serves Sam received were to her backhand? Express your answer to the nearest whole percent.

1 mark

On Wednesday, the machine is set so that the probability of Sam receiving a backhand after the previous serve was a backhand is twice as likely as her receiving a forehand after the previous serve was a forehand.

Let the probability of Sam receiving a forehand after the previous serve was a forehand be p.

The machine's first serve on Wednesday is to Sam's forehand and the probability that the third serve is to the forehand is 0.73.

e. Find p.

3 marks

The time, in hours, spent by Sam each week doing weights training is a continuous random variable X with probability density function given by

$$f(x)=\begin{cases}\dfrac{29-4x}{45} & \text{if } \; 2 \le x \le 5\\ 0 & \text{otherwise.}\end{cases}$$

f. **i.** Find the probability that Sam spends less than 4 hours a week doing weights training. Express your answer as an exact value.

ii. Find $\Pr(X<3|X<4)$. Express your answer as an exact value.

$1+2=3$ marks
Total 14 marks

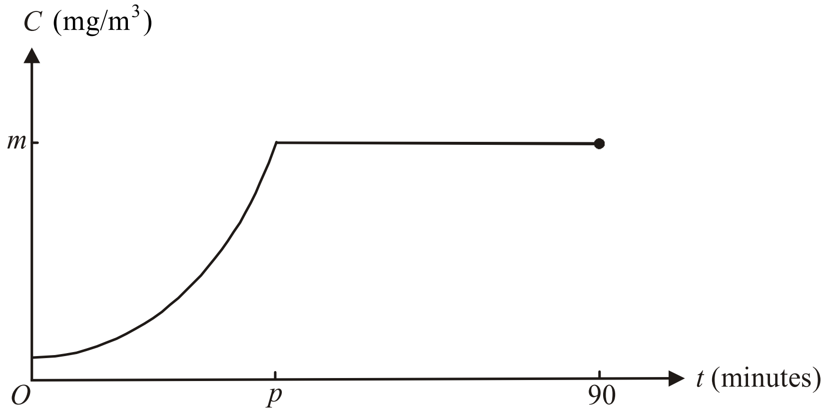
Question 3

Victoria James is a spy.
She is trapped in a space where poisonous gas is leaking.
The concentration C, in mg/m^3, of the gas t minutes after Victoria became trapped is given by the continuous function

$$C(t)=\begin{cases}\dfrac{500}{100-t}, & 0\le t\le p\\ m, & p<t\le 90\end{cases}$$

where m and p are constants.

A graph of the function is shown below.



a. What is the initial concentration of the gas in mg/m^3?

1 mark

b. Find an expression for m in terms of p.

1 mark

c. Find the minimum and maximum values of m.

2 marks

d. Find the function $C'(t)$ for $0 < t < p$.

1 mark

e. If the rate at which the concentration of the gas is increasing was 1 mg/m^3 per minute, find the value of *t*. Express your answer in minutes correct to 2 decimal places.

2 marks

f. If $p = 10$, find the average concentration of the gas between $t = 0$ and $t = p$, correct to 2 decimal places.

2 marks

If the concentration of the gas reaches 6 mg/m^3 then a human cannot survive.

g. Given that Victoria is trapped for 90 minutes in this space, find the possible values of *p* in order for her to survive.

2 marks

Total 11 marks

Question 4

a. Consider the functions

$$f : R \to R, \quad f(x) = e^{-x}$$

$$\text{and} \quad g : [0, \infty) \to R, \quad g(x) = 2(x+1)$$

i. Write down the rule for $f(g(x))$.

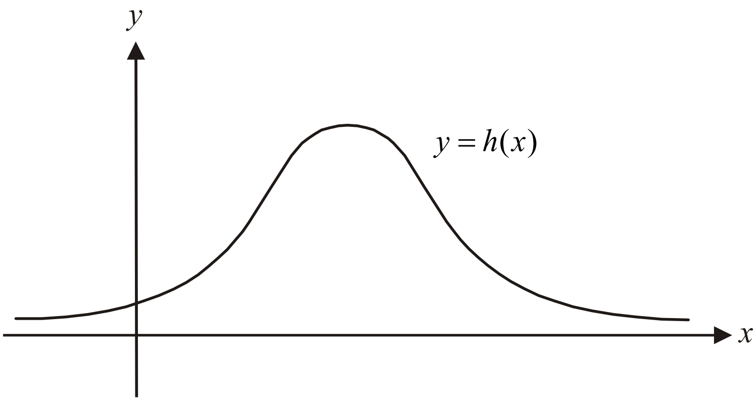
ii. Find the domain and range of $f(g(x))$. Express values as exact values where appropriate.

iii. Find the area enclosed by the function $y = f(g(x))$, the x and y axes and the line with equation $x = 1$. Express your answer as an exact value.

iv. Describe the two transformations that the graph of $y = f(x)$ undergoes to become the graph of $y = f(g(x))$.

$1+3+2+2=8$ marks

b. The graph of the function $h: R \to R,\ h(x) = e^{-(x-a)^2}$ where a is a constant and $a > 0$ is shown below.



i. Show that the stationary point of this graph occurs at $(a,1)$.

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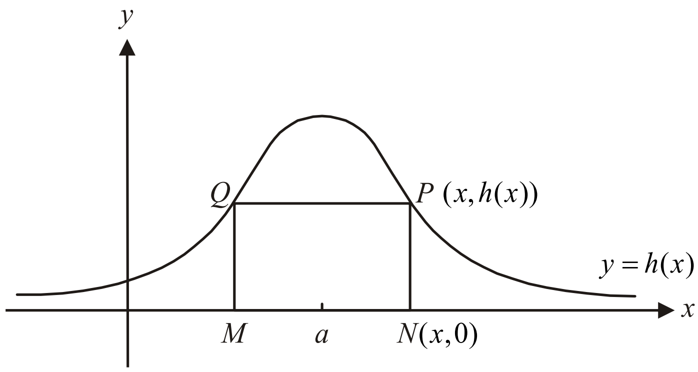
ii. Find the values of x for which $h(x)$ is strictly increasing.

The diagram below shows the graph of $y = h(x)$ where $h(x) = e^{-(x-a)^2}$ and the rectangle *MNPQ*.



The sidelength *MN* lies on the *x*-axis, points *P* and *Q* lie on the graph of $y = h(x)$ and the line $x = a$ bisects the rectangle.

The coordinates of points *N* and *P* are $(x, 0)$ and $(x,\ h(x))$ respectively.

iii. Show that the area *A*, in square units, of the rectangle is given by

$$A = 2(x-a) \times e^{-(x-a)^2}.$$

iv. Find the maximum area of rectangle *MNPQ*. Express your answer as an exact value.

$2+1+1+3 = 7$ marks
Total 15 marks

Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e|x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		**mean**	**variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma x\, p(x)$	$\sigma^2 = \Sigma (x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

MATHEMATICAL METHODS (CAS)

TRIAL EXAMINATION 2

MULTIPLE- CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ▬ C D E

The answer selected is B. Only one answer should be selected.

1.	A	B	C	D	E	12.	A	B	C	D	E
2.	A	B	C	D	E	13.	A	B	C	D	E
3.	A	B	C	D	E	14.	A	B	C	D	E
4.	A	B	C	D	E	15.	A	B	C	D	E
5.	A	B	C	D	E	16.	A	B	C	D	E
6.	A	B	C	D	E	17.	A	B	C	D	E
7.	A	B	C	D	E	18.	A	B	C	D	E
8.	A	B	C	D	E	19.	A	B	C	D	E
9.	A	B	C	D	E	20.	A	B	C	D	E
10.	A	B	C	D	E	21.	A	B	C	D	E
11.	A	B	C	D	E	22.	A	B	C	D	E