INSIGHT

Trial Exam Paper

2011

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details

SECTION 1

Question 1
Where k is a real constant, the simultaneous linear equations

$$kx - 5y = 0$$
$$7x - (k+2)y = k$$

have no solution, provided that

A. $k = 5$
B. $k = -7$
C. $k = 5,\ -7$
D. $5 < k < 7$
E. $k \neq 5,\ -7$

Answer is C

Solution
To have no solution, the lines need to be two parallel lines. When $k = 5,\ -7$ the gradients are the same but the y-intercept is different, therefore have no solution.

For equation 1: $y = \dfrac{kx}{5}$.

For equation 2: $y = \dfrac{7x}{k+2} - \dfrac{k}{k+2}$.

Question 2
At the point (2, 3) on the graph of the function with the rule $y = 2(x-2)^5 + 3$,

A. there is a local maximum.

B. there is a local minimum.

C. there is a stationary point of inflection.

D. the gradient is not defined.

E. there is a point of discontinuity.

Answer is C

Solution
The graph is a power of 5, which means that it behaves like a cubic graph. Therefore, it has a stationary point of inflection at (2, 3).

Question 3

The maximal domain, D, of the function $f : D \to R$, with the rule $f(x) = \log_e(a - 2x)$ for $a > 0$, is

A. R^+

B. $(-\infty, \frac{a}{2}]$

C. $(-\infty, \frac{a}{2})$

D. $(-\infty, a)$

E. $(-\infty, a]$

Answer is C

Solution

$$\begin{aligned} f(x) &= \log_e(a - 2x) \\ &= \log_e(-2x + a) \\ &= \log_e -2\left(x - \frac{a}{2}\right) \end{aligned}$$

So the graph has undergone a reflection in the y-axis and a translation to the right of $\frac{a}{2}$ units.

This means the domain is $(-\infty, \frac{a}{2})$.

Alternatively:

$$\begin{aligned} a - 2x &> 0 \\ -2x &> -a \\ x &< \frac{a}{2} \end{aligned}$$

Question 4

The range of the function $f:\left[\frac{\pi}{6},\frac{\pi}{2}\right)\to R,\ \ f(x)=3\cos(2x)$ is

A. $\left[\frac{\pi}{6},\frac{\pi}{2}\right)$

B. $[-3, 3]$

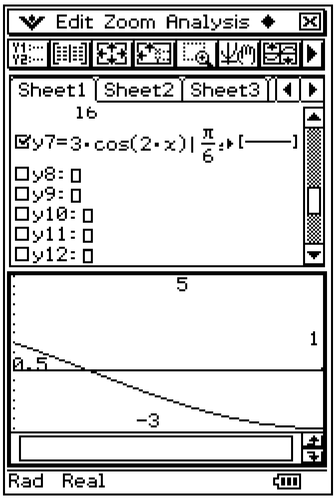
C. $(-3, 0]$

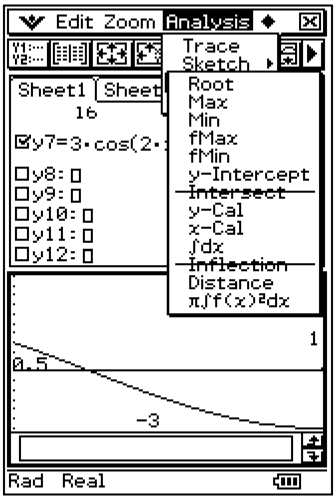
D. $(-3, 1.5]$

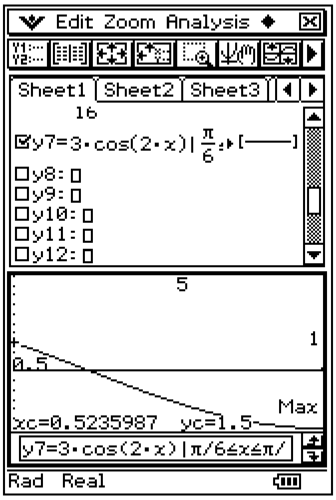
E. $(-3, 1.5)$

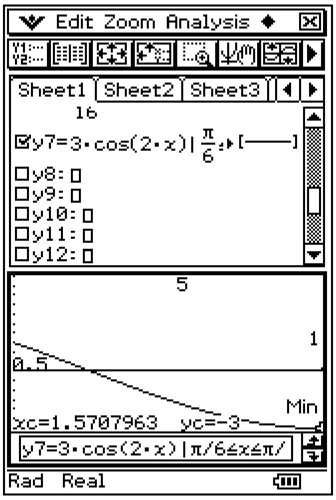
Answer is D

Solution
Graph the function and use fMin and fMax from the G-solve menu.









Question 5

The inverse of the function $f: R^{+} \to R,\ f(x) = \dfrac{1}{\sqrt{x}} + a$ for $a \in R$ is

A. $f^{-1}:[a,\infty) \to R$, where $f^{-1}(x) = \dfrac{1}{(x-a)^2}$

B. $f^{-1}:(a,\infty) \to R$, where $f^{-1}(x) = \dfrac{1}{(x-a)^2}$

C. $f^{-1}:R^{+} \to R$, where $f^{-1}(x) = \dfrac{1}{(x-a)^2}$

D. $f^{-1}:[a,\infty) \to R$, where $f^{-1}(x) = \dfrac{1}{x^2} - a$

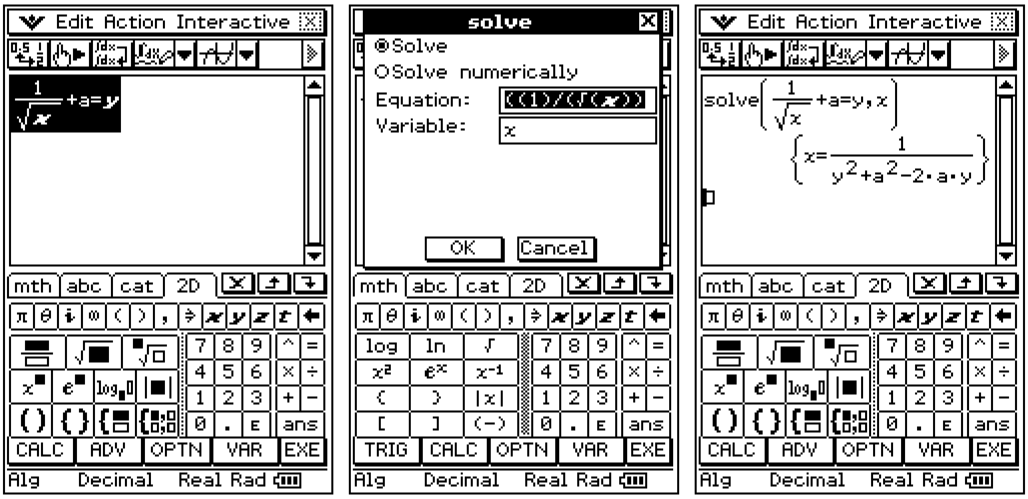
E. $f^{-1}:(a,\infty) \to R$, where $f^{-1}(x) = \dfrac{1}{x^2} - a$

Answer is B

Solution

Using the fact that the domain of the inverse is the range of the original, the domain of the inverse is (a,∞).

To find the rule use CAS:



This gives the rule for the inverse as $f^{-1}(x) = \dfrac{1}{(x-a)^2}$.

Hence, the *Answer is B*

Question 6
The continuous random variable X has a normal distribution with mean 60 and standard deviation 7. If the random variable Z has the standard normal distribution, then the probability that X is less than 46 is equal to

A. $\Pr(Z > 2)$

B. $\Pr(Z > -2)$

C. $\Pr(Z > 74)$

D. $\Pr(X > 2)$

E. $\Pr(X > -2)$

Answer is A

Solution

$$\begin{aligned}\Pr(X < 46) &= \Pr\left(Z < \frac{46-60}{7}\right)\\ &= \Pr(Z < -2)\\ &= \Pr(Z > 2), \text{ using symmetry}\end{aligned}$$

Question 7

For $y = e^{3x}\sin(3x)$ the rate of change of y with respect to x when $x = \frac{\pi}{3}$ is

A. 0

B. -3

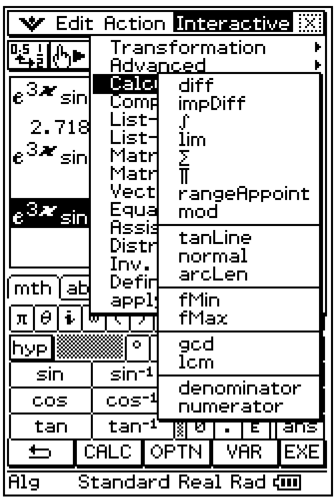
C. $-3e^{3\pi}$

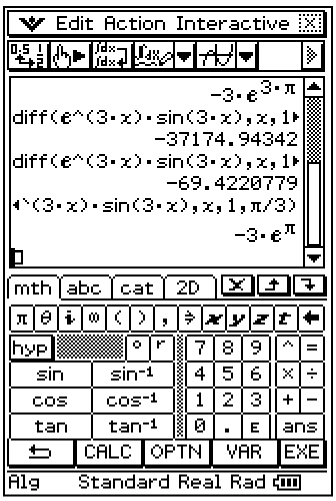
D. $-3e^{\pi}$

E. $-3e^{\frac{\pi}{3}}$

Answer is D

Use CAS, giving:





Question 8

According to a survey, 20% of primary school students at a particular primary school like the colour red. If 10 students from this primary school are selected at random, the probability that at least 6 of them like the colour red is

A. 0.0064

B. 0.0009

C. 0.9936

D. 0.0055

E. 0.9945

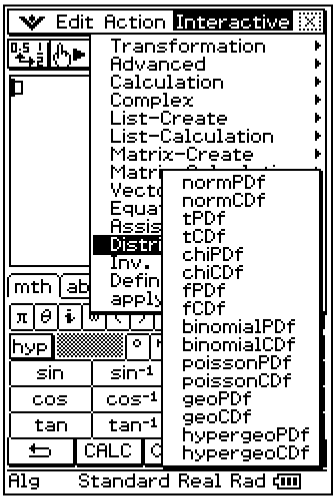
Answer is A

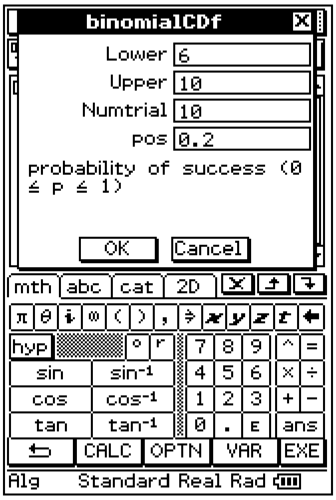
Solution

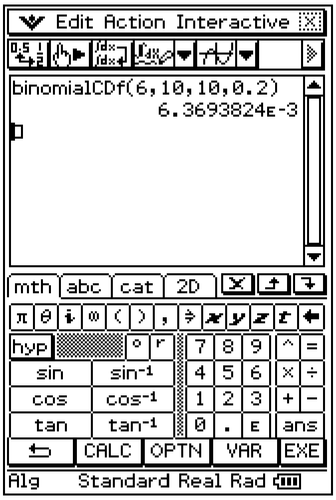
$X \sim Bi(n=10, p=0.2)$

$\Pr(X \geq 6) = 0.0064$

Use CAS:







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Question 9

A transformation $T: R^2 \to R^2$ that maps the curve with the equation $y = \cos(x)$ onto the curve with the equation $y = 1 - 2\cos(3x + \pi)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{-\pi}{3} \\ 1 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{-\pi}{3} \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

Answer is A

Solution

Looking at the x terms, when $y = \cos(x)$ becomes $y = 1 - 2\cos(3x + \pi)$, x changes to $3x + \pi$.
The matrix in option A allows this to happen.

Expanding the matrix equation gives $x' = \frac{x}{3} - \frac{\pi}{3}$. Rearranging to make x the subject gives $3x' + \pi = x$, as required.

Question 10

Let X be a discrete random variable with a binomial distribution. The mean of X is 2 and the variance is 1.6. The values of n (number of independent trials) and p (the probability of each trial) are

A. $p = 2$ and $n = 20$

B. $p = 0.8$ and $n = 10$

C. $p = 0.2$ and $n = 10$

D. $p = 0.02$ and $n = 100$

E. $p = 2$ and $n = 100$

Answer is C

Solution

If X is a binomial, then $\mu = np$ and $\sigma^2 = npq$.

Substituting in the values gives $2 = np$ and $1.6 = npq$, therefore $q = 0.8$.

$\Rightarrow p = 0.2$ and $n = 10$

Question 11
For the function $f: R \to R, \quad f(x) = (x+a)^2(b-x)$, the subset of R for which the gradient of f is negative is

A. $(-\infty, -a)$

B. $\left(-a, \dfrac{2b-a}{3}\right)$

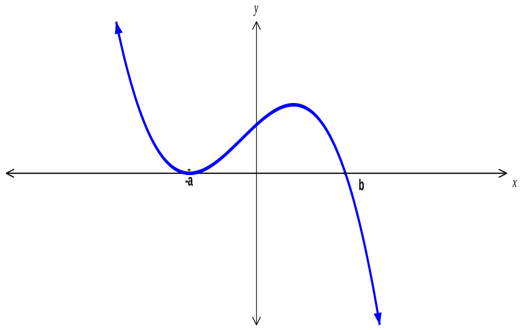
C. $(-\infty, -a) \cup (b, \infty)$

D. $(-a, b)$

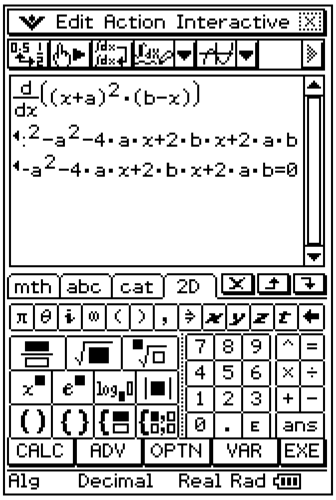
E. $(-\infty, -a) \cup \left(\dfrac{2b-a}{3}, \infty\right)$

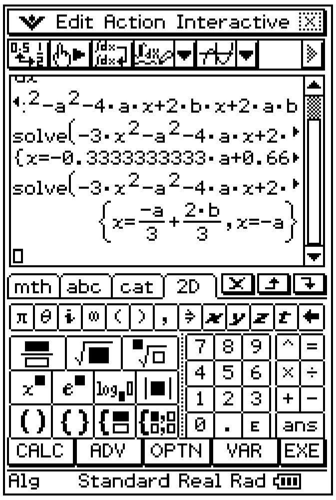
Answer is E

Solution
Graph is a negative cubic and a quick sketch gives



Using CAS, the graph has turning points at $(-a, 0)$ and $\left(\dfrac{2b-a}{3}, 0\right)$.





So the gradient is negative from $(-\infty, -a) \cup \left(\dfrac{2b-a}{3}, \infty\right)$.

Question 12
The discrete random variable X has a probability distribution as shown.

x	0	1	2	3
$\Pr(X = x)$	$4a$	$2a$	$3a$	a

The median of X is

A. $6a$

B. $5a$

C. 1

D. $\frac{a}{6}$

E. 2

Answer is C

Solution
Summing the probabilities to 1 gives $4a + 2a + 3a + a = 1$, so $a = 0.1$.
The median occurs when the sum of the probabilities is 0.5. This occurs during the interval of $X = 1$, so the median is 1.

Question 13
The tangent at the point (2, 3) on the graph of the curve $y = f(x)$ has the equation $y = 2x - 1$.
Therefore, the equation of the tangent at the point (2, –3) on the curve $y = -f(x)$ has the equation

A. $y = -2x - 1$

B. $y = -2x + 1$

C. $y = -2x - 3$

D. $y = -\frac{1}{2}x + 1$

E. $y = -\frac{1}{2}x - 1$

Answer is B

Solution
The graph has been reflected across the x-axis and, likewise, the tangent is also reflected, giving the equation $y = -(2x - 1) = -2x + 1$.

Question 14

If $\int_{-1}^{2} f(x)\,dx = 4$, then $\int_{-1}^{2} (3 - f(x))\,dx$ is equal to

A. –1

B. 7

C. 1

D. –4

E. 5

Answer is E

Solution

$$\begin{aligned}\int_{-1}^{2}(3 - f(x))\,dx &= \int_{-1}^{2} 3\,dx - \int_{-1}^{2} f(x)\,dx \\ &= [3x]_{-1}^{2} - 4 \\ &= 6 + 3 - 4 \\ &= 5\end{aligned}$$

Question 15

For $y = \frac{1}{(1 - f(x))}$, $\frac{dy}{dx}$ is equal to

A. $\log_e(1 - f(x))$

B. $-\log_e(1 - f(x))$

C. $\frac{-1}{(1 - f(x))^2}$

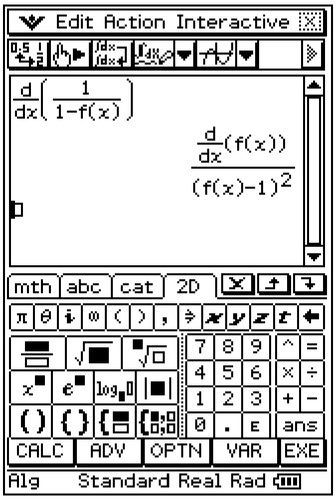
D. $\frac{f'(x)}{(1 - f(x))^2}$

E. $\frac{-f(x)}{(1 - f(x))^2}$

Answer is D

Solution

Use CAS:



Question 16

The average value of the function $f: R\setminus\{5\} \to R,\ f(x) = \dfrac{1}{5-x}$ over the interval $[1,\ k]$ is $\dfrac{1}{2}\log_e 2$. Hence, the value of k is

A. $\dfrac{1}{9}$

B. e^3

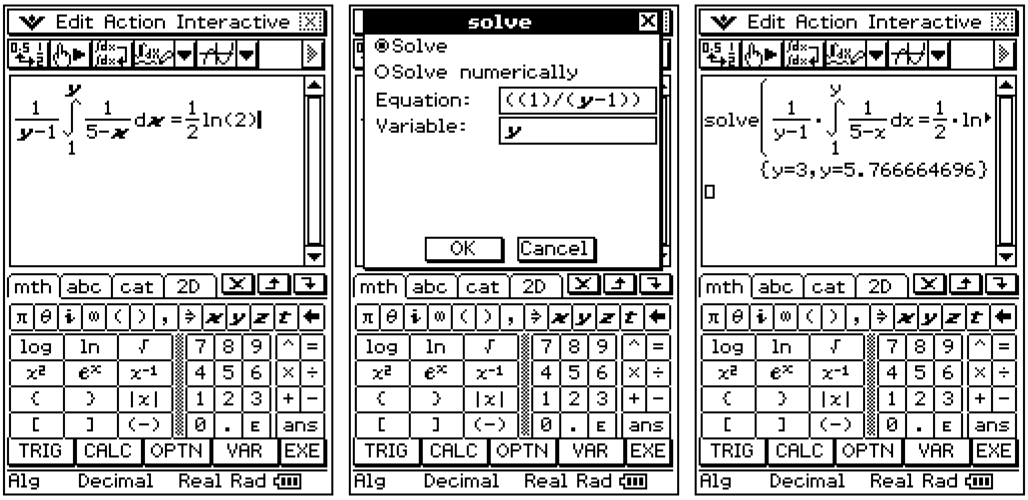
C. –3

D. 3

E. 7

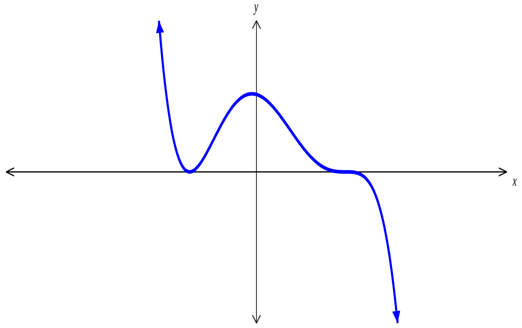
Answer is D

Solution
Use CAS and solve.



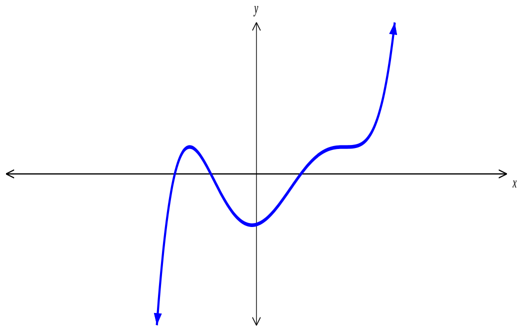
Question 17

The graph of $y = f(x)$ is shown below.

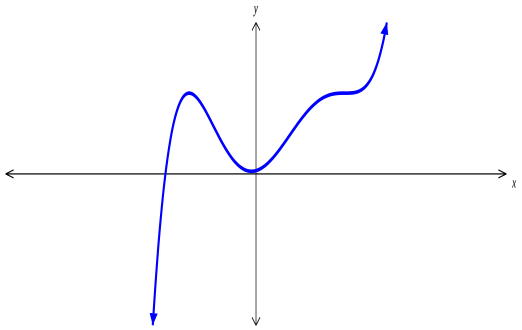


The graph that best represents $f(-x)+2$ is

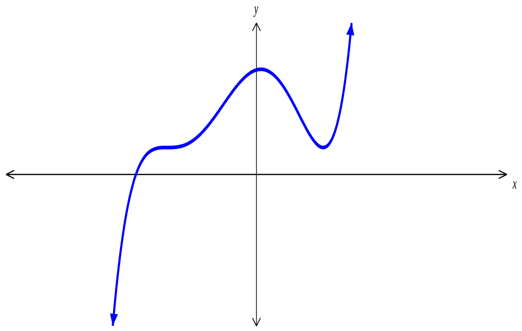
A.



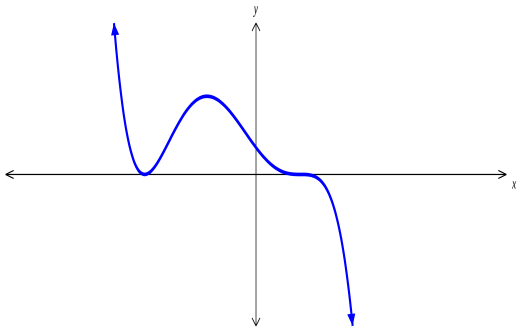
B.



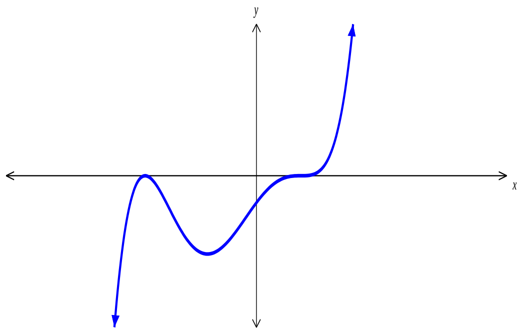
C.



D.



E.



Answer is C

Solution

The transformation $f(-x)+2$ means the graph has been reflected in the y-axis and shifted up 2 units. This is represented by graph C.

Question 18

The exact area of the region bounded by the curve $f(x)=e^{2x}-2e^{x}-3$, the x-axis and the y-axis is

A. $3\log_e 3$

B. $4-3\log_e 3$

C. $4\log_e 3$

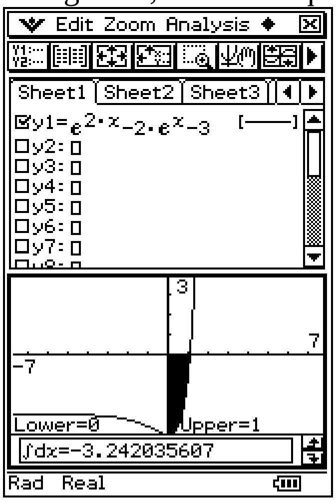
D. $4-3\log_e 3$

E. 3

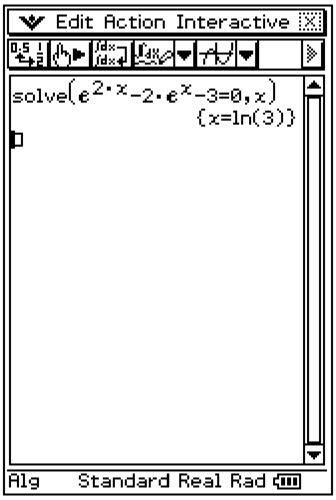
Answer is A

Solution

Using CAS, the area required can be seen as the shaded region.



Use the main menu to find the exact value of the x-intercept.



So, the area is:

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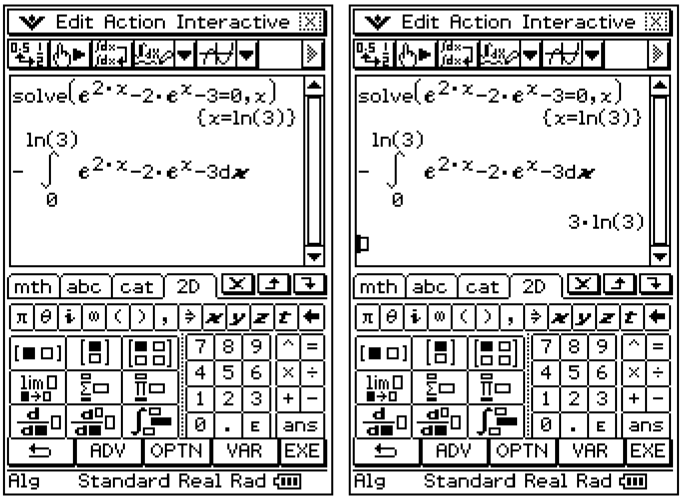


Question 19

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases} k\sin(x), \text{ if } \ 0<x<\pi \\ 0, \text{ otherwise} \end{cases}$$

The $\Pr(X \geq k)$ is closest to

A. $\dfrac{\cos\left(\frac{1}{2}\right)}{2}$

B. $\dfrac{1-\sin\left(\frac{1}{2}\right)}{2}$

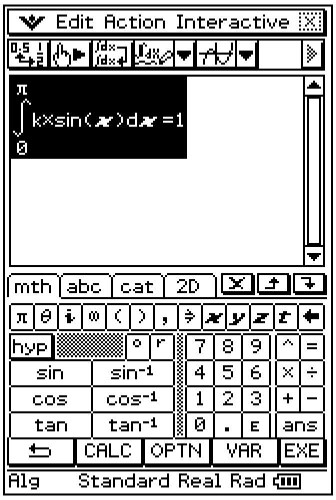
C. $\dfrac{\sin\left(\frac{1}{2}\right)}{2}$

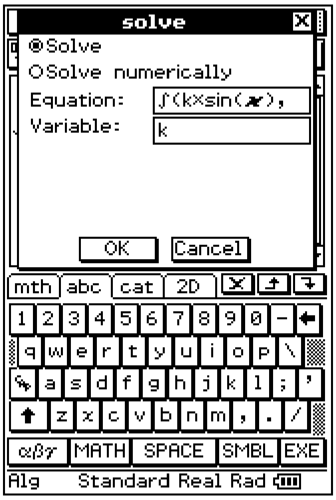
D. $\dfrac{1-\cos\left(\frac{1}{2}\right)}{2}$

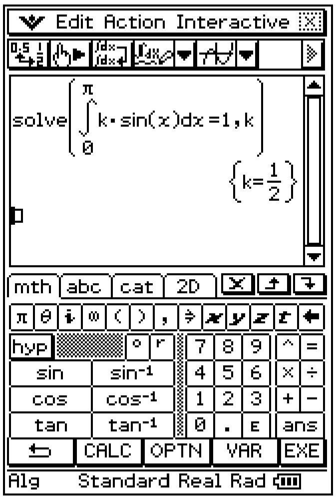
E. $\dfrac{\cos\left(\frac{1}{2}\right)+1}{2}$

Answer is E

Solution
Using CAS gives the value of $k = 0.5$.

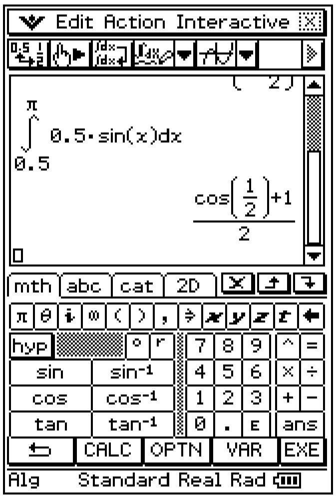






So the $\Pr(X \geq k) = \int_{0.5}^{\pi} 0.5\sin(x)\,dx$.

Again using CAS, this is:



Question 20
The interval [0, 4] is divided into n equal subintervals by the points

$x_0, x_1, ..., x_{n-1}, x_n,$ where $0 = x_0 < x_1, < ...... < x_{n-1} < x_n = 4.$

Let $\delta x = x_i - x_{i-1}$ for $i = 1, 2, 3....., n.$

Then $\lim_{\delta x \to 0} \sum_{i=1}^{n} (x_i^2 \delta x)$ is equal to

A. $\frac{64}{3}$

B. 64

C. 16

D. 8

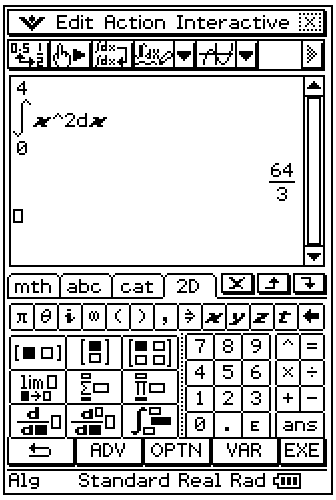
E. 21

Answer is A

Solution
This is the formal definition for finding the integral.

$$\lim_{\delta x \to 0} \sum_{i=1}^{n} (x_i^2 \delta x) = \int_0^4 x^2 \, dx$$

Using CAS, this is:



Question 21
The velocity of a car moving in a straight line in metres per second is given by

$$v(t) = 8 - \frac{1}{2}t^2, t \geq 0.$$

Hence, the distance, in metres, travelled in the first 5 seconds is

A. –12.5

B. 23.5

C. 21.3

D. 19.2

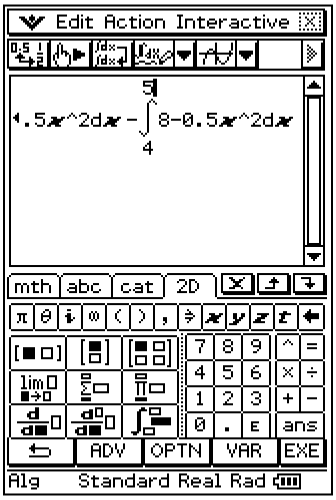
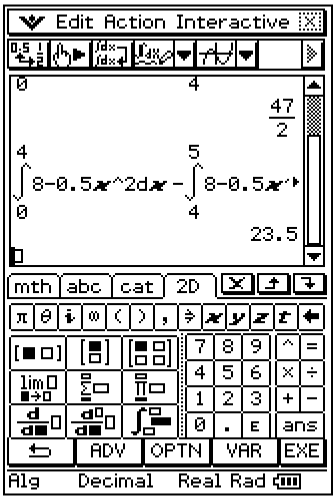
E. 11.3

Answer is B

Solution
The distance travelled is the area bounded by the graph and the x-axis from $x = 0$ to $x = 5$. The graph has an x-intercept at $t = 4$, so the distance is calculated by:

$$\text{Distance} = \int_0^4 (8 - \frac{1}{2}t^2)\, dt - \int_4^5 (8 - \frac{1}{2}t^2) dt$$

Using CAS, this is:

Question 22

Let $f(x) = a - b\sin(x)$, where a and b are real numbers and $b > 0$.

Then $f(x) > 0$ for all real values of x if

A. $a > b$

B. $b > a$

C. $-a > b$

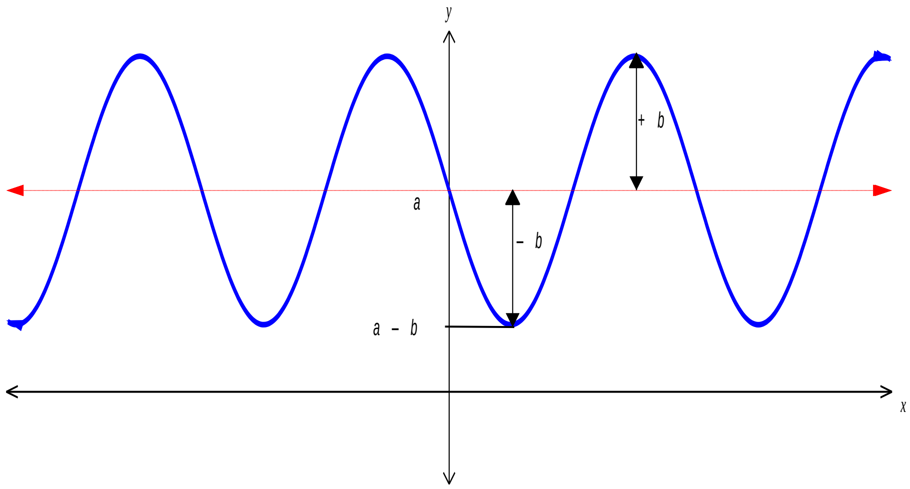
D. $b > -a$

E. $-a < b$

Answer is A

Solution

Graph is a negative sin curve with a minimum value of $a - b$.



We want $a - b > 0$, so $a > b$.

END OF SECTION 1

SECTION 2

Question 1

Hugo wishes to compete in the State-wide Junior Schools Athletics carnival. His chosen event is discus.

The longest throwing distance recorded by his school is 51.60 metres.
The current State-wide junior schools record is 55.80 metres.

To be selected to compete he must throw at least his own school's record.

Hugo knows that the distance X, in metres, he can throw the discus follows a normal distribution with a mean of 50.55 metres and a standard deviation of 3.90.

a. Complete the following table. Give probabilities, correct to 3 decimal places

Distance thrown	Probability
Better than own school record	
Better than own school record but less than State-wide record	
Better than State-wide record	

3 marks

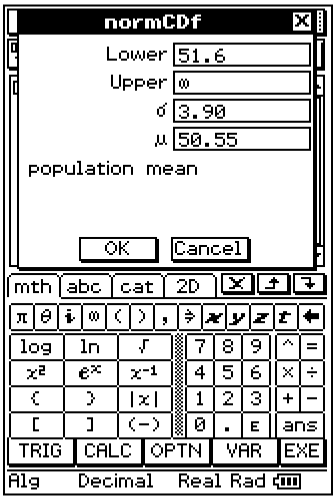
Solution

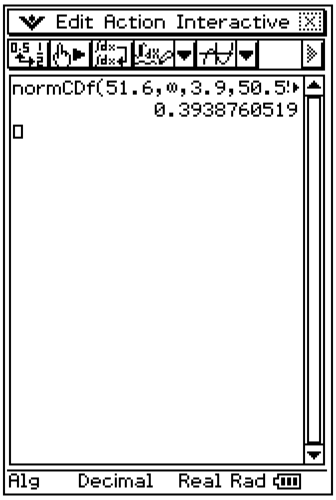
$X \sim N(\mu = 50.55, \sigma = 3.90)$

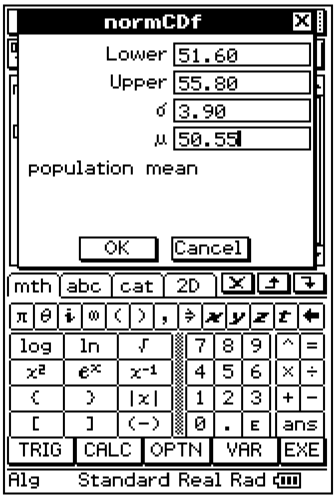
$\Pr(X > 51.60) = 0.394$

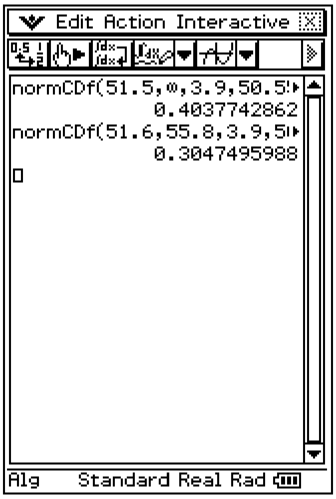
$\Pr(51.60 < X < 55.80) = 0.305$

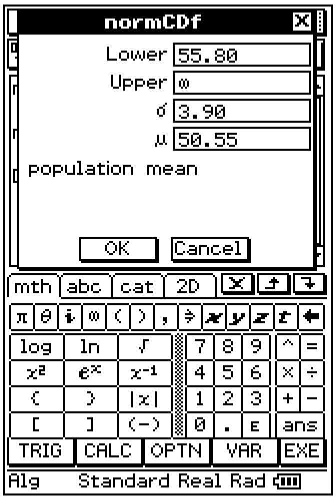
$\Pr(X > 55.80) = 0.089$

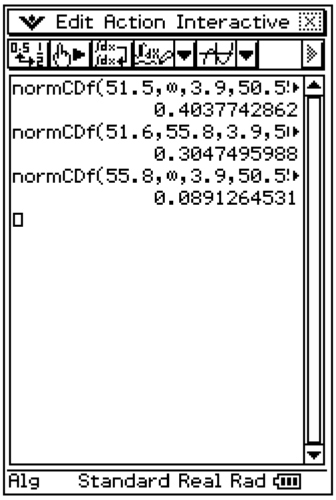












Mark allocation

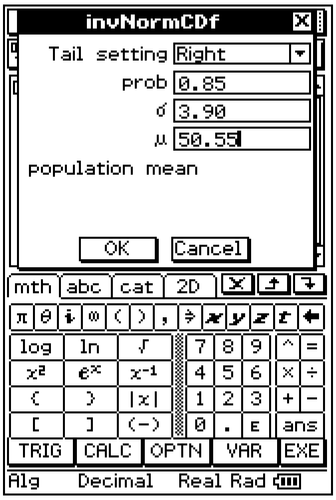
- 1 mark for each correct answer.

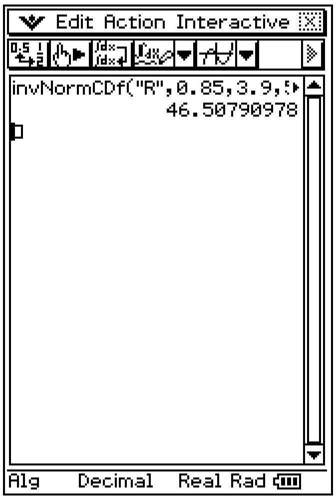
b. 85% of Hugo's discus throws travel at least A metres. Find the value of A, correct to 2 decimal places.

2 marks

Solution

$\Pr(X > A) = 0.85$





So A = 46.51 m.

Mark allocation

- 1 method mark for setting up $\Pr(X > A) = 0.85$.
- 1 answer mark for finding the correct value of A.

c. On a particular attempt, Hugo throws the discus a distance that is less than the State-wide record. What is the probability that the distance thrown is farther than the school's record? Give your answer correct to 2 decimal places.

2 marks

Solution

This question involves conditional probability.

$$\Pr(X > 51.60 \mid X < 55.80) = \frac{\Pr(X > 51.60 \cap X < 55.80)}{\Pr(X < 55.80)} = \frac{\Pr(51.60 < X < 55.80)}{\Pr(X < 55.80)}$$

$$= \frac{0.305}{(1-0.089)} \quad \text{(Answers from part } \mathbf{a}.)$$

$$= 0.33$$

Mark allocation

- 1 mark for recognising that the problem is one of conditional probability.
- 1 mark for the correct answer.

d. During practice, Hugo throws five discuses. What is the probability that more than three of them are farther than the school's record? Give your answer correct to 2 decimal places.

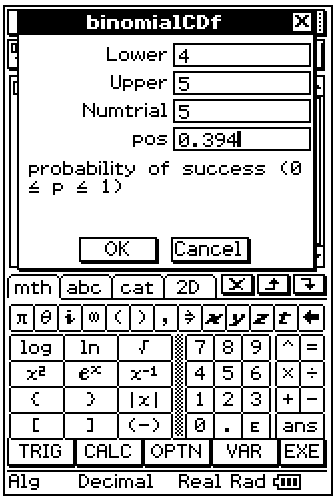
2 marks

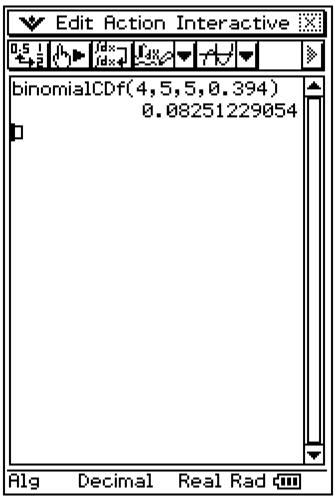
Solution

Let Y = Number of throws farther than the school's record

$Y \sim Bi(n = 5,\ p = 0.394)$

$\Pr(Y > 3) = 0.09$





So the probability that more than three of five throws are farther than the school's record is 0.08.

Mark allocation

- 1 method mark for recognising that the problem is one of binomial probability.
- 1 answer mark for the correct probability.

Riley is another competitor trying out for the State-wide athletics carnival.

The distance he is able to throw the discus is normally distributed with a mean of 51.20 metres and a standard deviation of σ.

e. He knows from past experience that 20% of his throws reach the State-wide record distance. Find the value of the standard deviation, σ, correct to 2 decimal places.

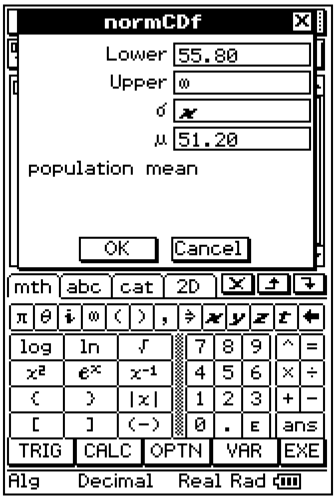
2 marks

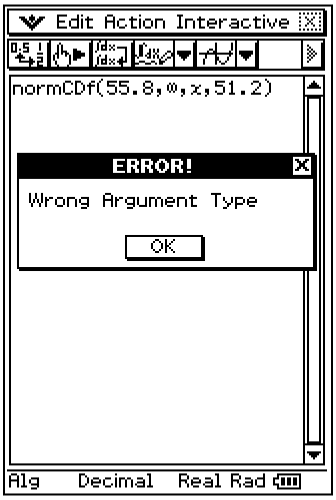
Solution

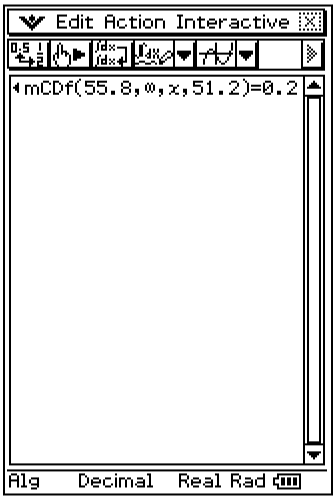
Let X_R = Riley's discus distance

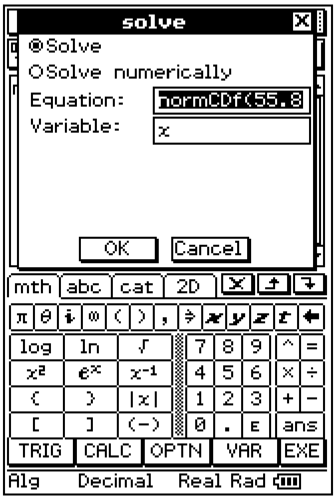
$X_R = N(\mu = 51.20, \sigma)$

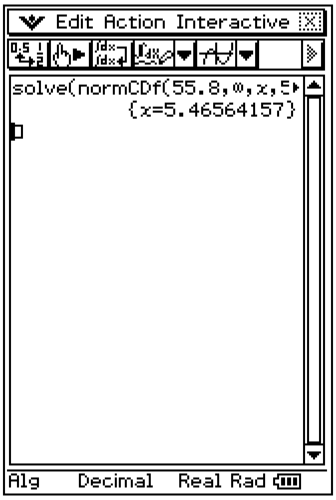
$\Pr(X > 55.80) = 0.2$











So the standard deviation for Riley's distribution is $\sigma = 5.47$.

Mark allocation

- 1 method mark for setting up equation with unknown standard deviation

  $X_R = N(\mu = 51.20, \sigma)$

  $\Pr(X > 55.80) = 0.2$
- 1 answer mark for finding correct standard deviation.

On the day of competition, Marcus, a discus competitor from another school, knows that his chances of throwing better than the State-wide junior schools' record depend only on what he has thrown at the previous attempt. If he throws farther than the State-wide record on one attempt, then there is a 60% chance that he'll throw farther than the State-wide record on the next attempt. If he throws less than the State-wide record on one attempt, then there is a 30% chance he'll throw farther than the State-wide record on the next attempt.

f. State the transition matrix for this information.

1 mark

Solution

Let F = throws farther and L = throws less.

$$\begin{array}{cc} & \begin{array}{cc} F & L \end{array} \\ \begin{array}{c} F \\ L \end{array} & \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array}$$

Mark allocation

- 1 answer mark for correct matrix.

g. If Marcus throws farther than the State-wide record on his first attempt, what is the probability that he throws farther than the State-wide record on exactly two of his next four attempts? Give your answer correct to 4 decimal places.

3 marks

Solution

There are ${}^4C_2 = 6$ ways that he can throw farther on exactly two of the next four occasions.

These are:

FFLL = $0.6 \times 0.6 \times 0.4 \times 0.7 = 0.1008$

FLFL = $0.6 \times 0.4 \times 0.3 \times 0.4 = 0.0288$

FLLF = $0.6 \times 0.4 \times 0.7 \times 0.3 = 0.0504$

LFFL = $0.4 \times 0.3 \times 0.6 \times 0.4 = 0.0288$

LFLF = $0.4 \times 0.3 \times 0.4 \times 0.3 = 0.0144$

LLFF = $0.4 \times 0.7 \times 0.3 \times 0.6 = 0.0504$

The total of these probabilities is 0.2736.

Mark allocation

- 1 method mark for identifying six possibilities.
- 1 method mark for at least two correct calculations.
- 1 answer mark for correct probability.

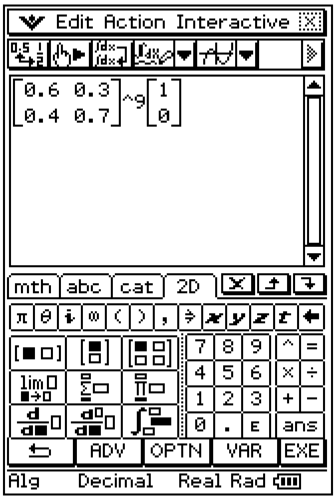
h. What is the probability that Marcus throws farther than the State-wide record on his tenth attempt, if he throws farther than the State-wide record on his first attempt? Give your answer correct to 3 decimal places.

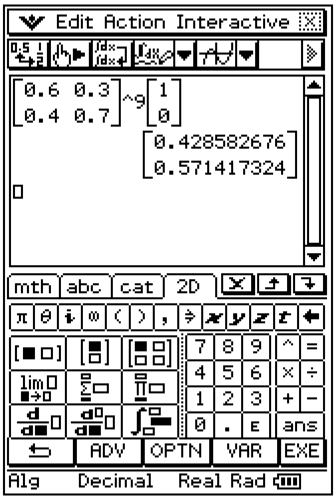
1 mark

Solution

There are nine transitions to go from Marcus' first attempt to his tenth attempt, so we calculate the matrix multiplication:

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}^9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$





Marcus has a probability of 0.429 of throwing farther than the State-wide record on his tenth attempt.

Mark allocation

- 1 answer mark for the correct answer.

i. To be eligible for National selection, Marcus has to have thrown at least 650 throws that are farther than the State-wide record, from 1200 attempts. Based on his current performance, is this possible for Marcus? Explain.

2 marks

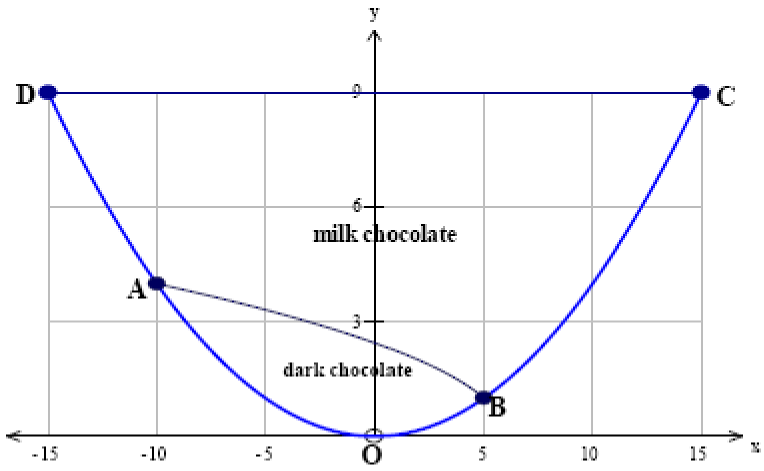
Solution
Let's look at the long-term probability.

One technique is to consider $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ the values on the non-leading diagonal, which give the probability associated with change. Marcus has a probability of $\frac{0.3}{0.3+0.4}$, so $\frac{3}{7}$ or 0.4286 of his attempts long term should be successful. The requirement is for 650 out of 1200; i.e. 0.5417, so he falls short of reaching this requirement.

Mark allocation
- 1 method mark for finding long-term probability.
- 1 answer mark for correct conclusion.

Total = 18 marks

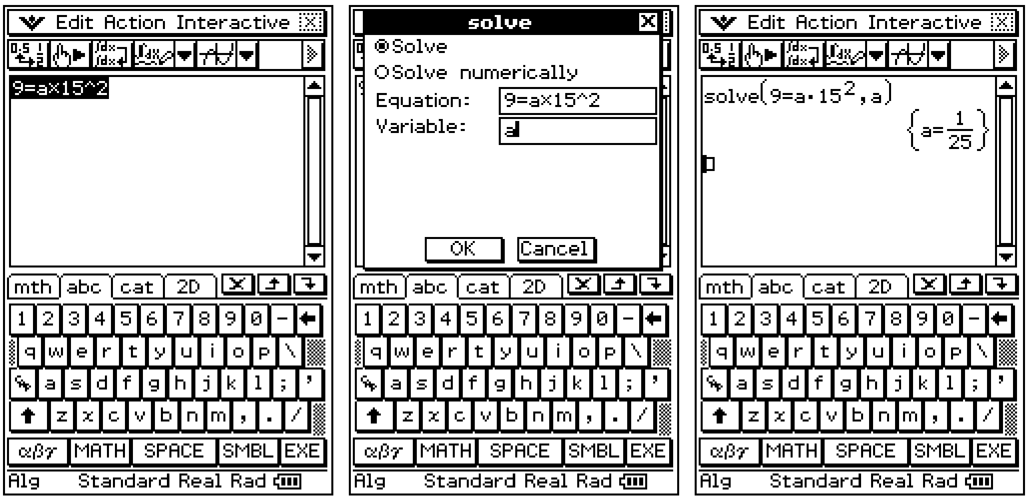
Question 2
A vertical section of a chocolate mousse in a bowl is illustrated in the diagram below. The diagram shows the arrangement of dark chocolate and milk chocolate in the mousse.
The axes *OX* and *OY* are as indicated. All measurements are in centimetres.



a. The chocolate mousse is placed in a paraboli-shaped bowl and the outer curved edge is described by the equation $y = ax^2$. Show that the value of $a = \frac{1}{25}$.

1 mark

Solution
The graph passes through (15, 9), so when $x = 15,\ y = 9$. This gives:



$$9 = a \times 15^2$$
$$\therefore\ a = \frac{1}{25}$$

Mark allocation
- 1 method mark for using a correct coordinate that leads to the correct answer.

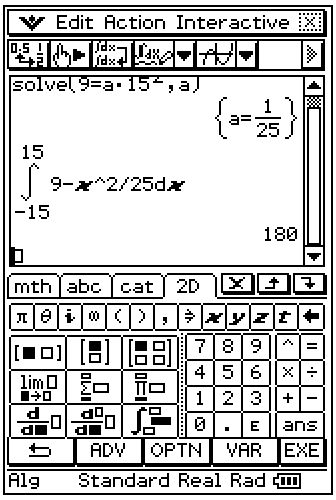
SECTION 2 – Question 2 – continued
TURN OVER

b. The parabolic-shaped bowl is filled to the top with mousse. Find the cross-sectional area of the mousse (both milk and dark), correct to the nearest cm^2.

2 marks

Solution

$$\int_{-15}^{15} 9-\frac{x^2}{25}\ dx = 180$$



Mark allocation

- 1 mark for stating the integral correctly.
- 1 mark for the correct answer.

c. The curve *AB* separates the dark chocolate from the milk chocolate. The equation of the curve *AB* is given by $y = \sqrt{6-x}$.

i. Find the coordinates of points *A* and *B*.

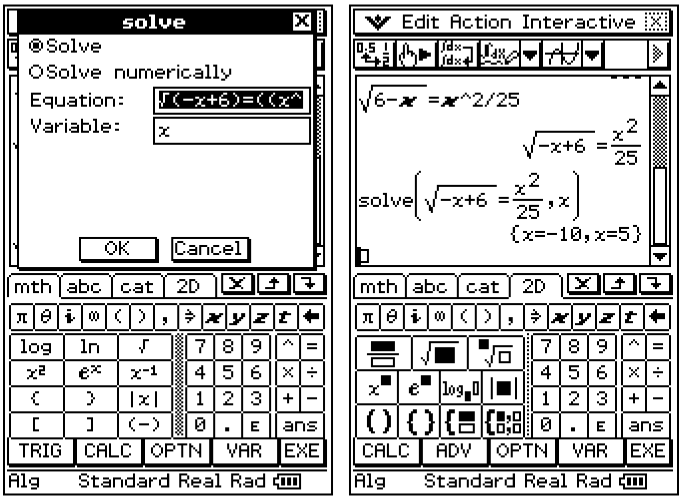
2 marks

Solution

We need to equate $y = \sqrt{6-x}$ with $y = \frac{x^2}{25}$.

So $\sqrt{6-x} = \frac{x^2}{25}$.

Using CAS, this gives:



So the coordinates are *A*(–10, 4) and *B*(5, 1).

Mark allocation

- 1 mark for equating $y = \sqrt{6-x}$ with $y = \frac{x^2}{25}$.
- 1 mark for correctly finding both coordinates. (Must be presented as coordinates!)

ii. Hence, write an expression that will determine the cross-sectional area of the dark chocolate.

1 mark

Solution

$$\int_{-10}^{5} \left(\sqrt{6-x} - \frac{x^2}{25} \right) dx$$

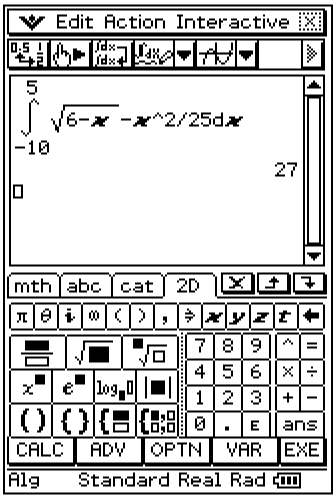
Mark allocation

- 1 mark for stating the derivative correctly. (*Note:* Students must include the brackets and *dx* in their answer.)

iii. What is the cross-sectional area of the dark chocolate?

1 mark

Solution



The area is 27 cm^2.

Mark allocation

- 1 mark for the correct area.

iv. Find the ratio of milk chocolate to dark chocolate mousse in the bowl. State the ratio in simplest form.

1 mark

Solution

The area of milk chocolate is 180 – 27= 153.
The area of dark chocolate is 27.
So the ratio is 153 : 27 or 51 : 19.

Mark allocation

- 1 mark for giving the correct simplified ratio.

v. It is decided that the ratio of milk chocolate to dark chocolate mousse needs to be 1 : 1. This is to be achieved by altering the amount of milk chocolate mousse that is added. The milk chocolate mousse will still form an upper horizontal level but will not reach to the top of the bowl. Find the new height of the mousse in the bowl.

3 marks

Solution

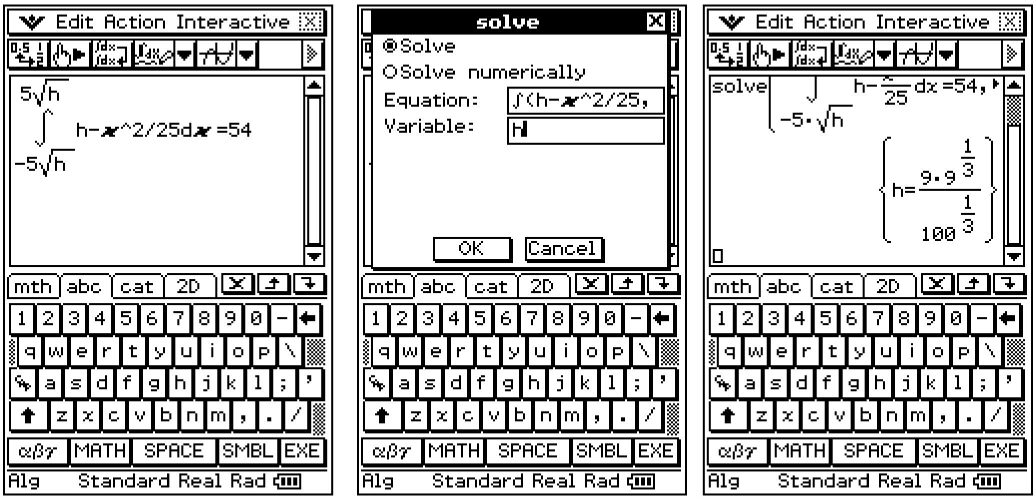
To be in a 1 : 1 ratio, we need the area of the milk chocolate mousse to equal the area of the dark chocolate mousse, which is 27 cm^2. So the total area will be 54 cm^2.

Let the height of the mousse in the bowl be h.

The intersection points with the curve are at $\frac{x^2}{25} = h$, so $x = \pm 5\sqrt{h}$.

This means $\int_{-5\sqrt{h}}^{5\sqrt{h}} \left(h - \frac{x^2}{25} \right) dx = 54$.

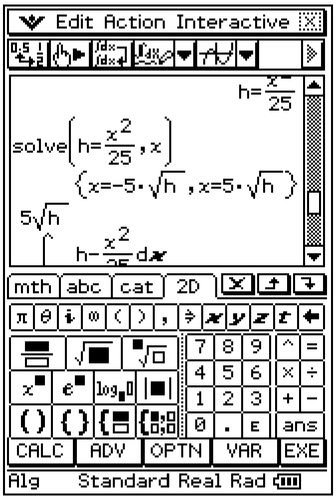
Solve using CAS:

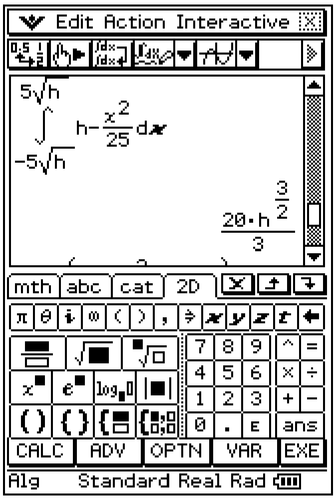


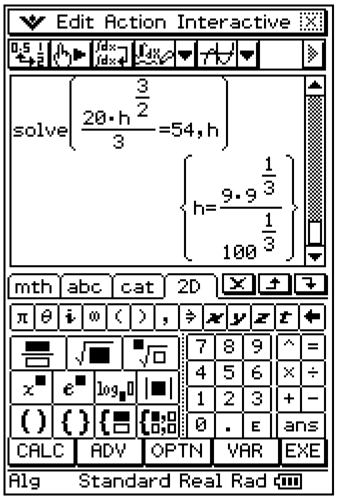
This gives $h = 9\sqrt[3]{0.09}$ or equivalent exact form.

Mark allocation

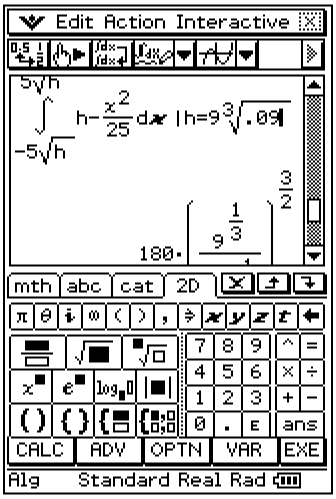
- 1 mark for finding terminals of $x = \pm 5\sqrt{h}$.
- 1 mark for setting integral equal to 54.
- 1 mark for the correct answer (must be exact).

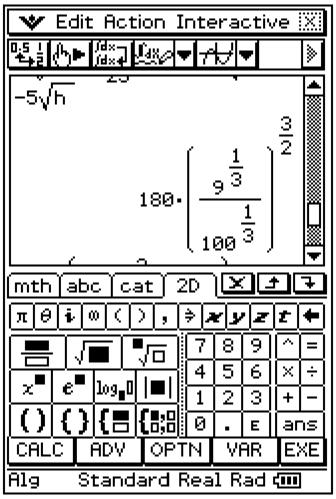

h= x²/25
solve(h=x²/25, x)
{x=-5·√h, x=5·√h}
∫ from -5√h to 5√h of h - x²/25 dx


∫ from -5√h to 5√h of h - x²/25 dx
20·h^(3/2) / 3


solve(20·h^(3/2)/3 = 54, h)
{h = 9·9^(1/3) / 100^(1/3)}

Check:


∫ from -5√h to 5√h of h - x²/25 dx | h=9∛.09


180·(9^(1/3) / 100^(1/3))^(3/2)

d. The volume of mousse in the bowl is given by $V = \frac{\pi}{3125}h^4$, where h is the height of the mousse, in cm.
Chocolate mousse is added to the bowl as a liquid and is poured into the bowl at a rate of 5 cm^3 per minute. Find the rate at which the height of the chocolate mousse is increasing when the volume is $\frac{\pi}{625}$ cm^3.

3 marks

Solution
This is a related rates question.

We have $\frac{dV}{dt} = 5$ and we want $\frac{dh}{dt}$ when $v = \frac{\pi}{625}$.

Using the equation given of $V = \frac{\pi}{3125}h^4$ when $V = \frac{\pi}{625}$, we have $h = \sqrt[4]{5}$.

We make use of the rate equation $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.

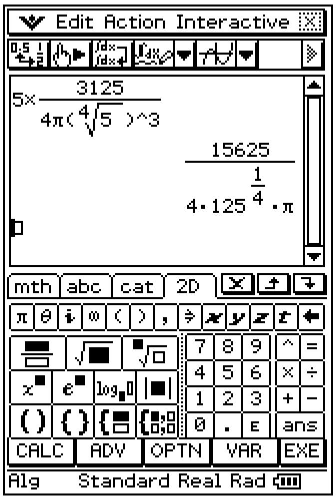
To get $\frac{dh}{dV}$, we differentiate $V = \frac{\pi}{3125}h^4$ with respect to h, giving us $\frac{dV}{dh} = \frac{4\pi}{3125}h^3$.

This gives $\frac{dh}{dV} = \frac{3125}{4\pi h^3}$.

So $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$$= 5 \times \frac{3125}{4\pi h^3}$$

When $h = \sqrt[4]{5}$, this gives:



$$\frac{dh}{dt} = \frac{15625}{\pi\, 4\, \sqrt[4]{125}} \text{ cm/min}$$

Mark allocation

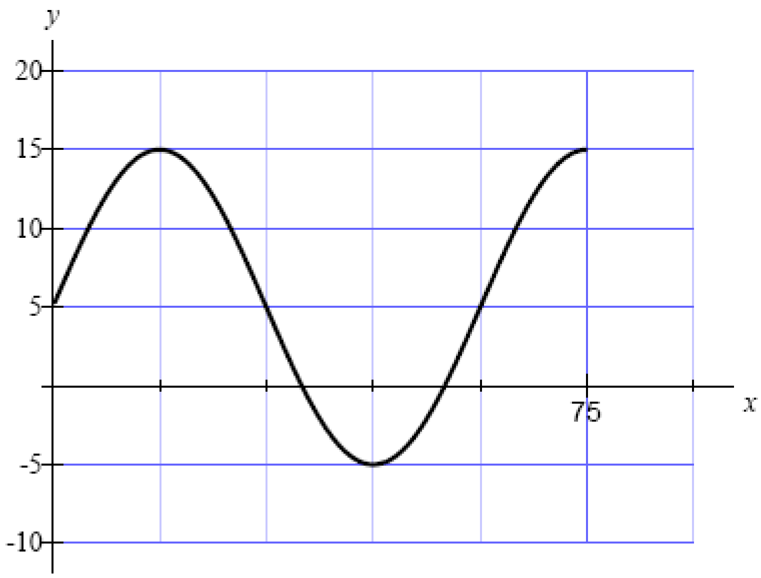
- 1 mark for finding the value of *h*.
- 1 mark for setting up the rate equation correctly.
- 1 mark for the correct answer.

Total = 14 marks

Question 3

Let $f:[0,75] \rightarrow R,\ f(x) = 5 + a\sin(bx)$.

The graph of $y = f(x)$ is shown below.



a. **i.** State the value of *a*.

1 mark

Solution

a is the amplitude; so a = 10.

Mark allocation

- 1 mark for giving the correct amplitude.

ii State the period and, hence, show that $b = \frac{\pi}{30}$.

2 marks

Solution

Period = 60; so $\frac{2\pi}{n} = 60,$ where $n = \frac{\pi}{30}$.

Therefore, $b = \frac{\pi}{30}$.

Mark allocation

- 1 mark for giving the correct period.
- 1 mark for equation leading to $b = \frac{\pi}{30}$.

b. State the interval of values for which the graph of f is strictly **decreasing**.

1 mark

Solution
The function is strictly decreasing from the maximum to the minimum turning point. The maximum is at x = 15 and the minimum is at x = 45, so the graph is decreasing for $x \in [15, 45]$.

Mark allocation
- 1 mark for the correct answer.

c. Let $g: R \to R,\ g(x) = |x|$

i. Find the rule for $g(f(x))$.

1 mark

Solution

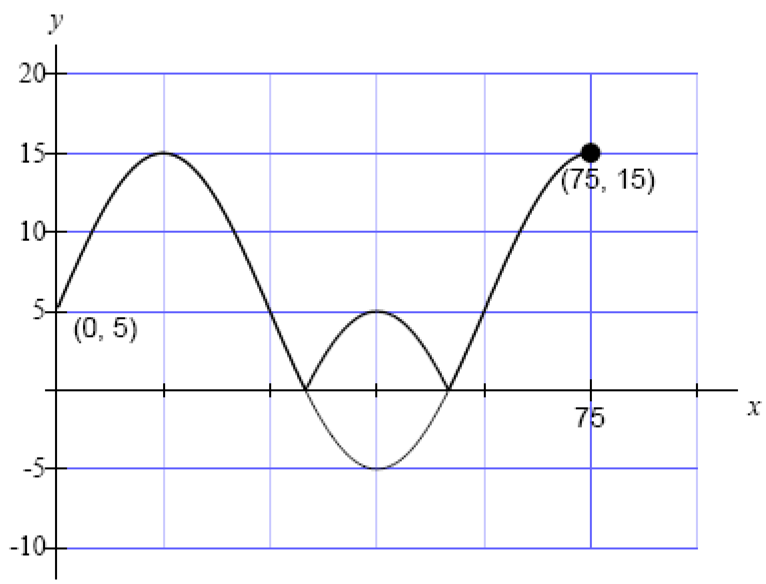
$$g(f(x)) = \left| 5 + 10\sin\left(\frac{\pi x}{30}\right) \right|$$

Mark allocation
- 1 mark for the correct answer.

ii. On the axes of the graph above, sketch the graph of $y = g(f(x))$, labelling endpoints as coordinates.

2 marks

Solution

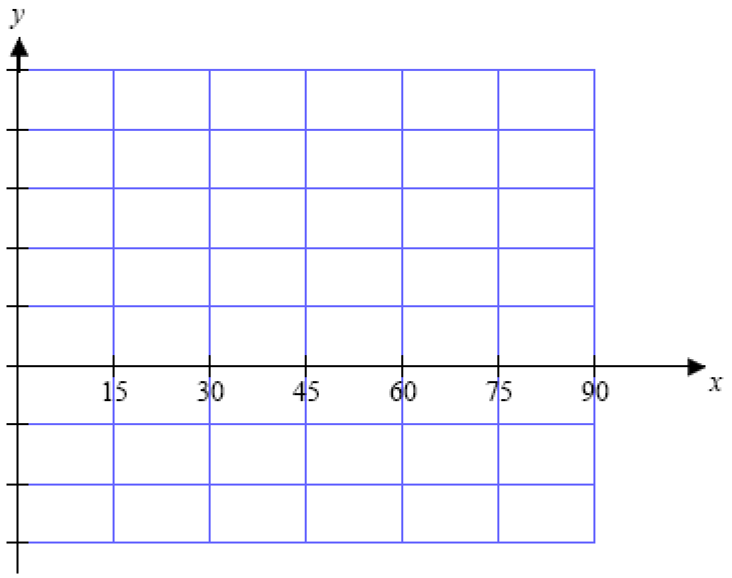


Mark allocation
- 1 mark for labelling endpoints correctly.
- 1 mark for graph drawn correctly with cusps.

SECTION 2 – Question 3 – continued

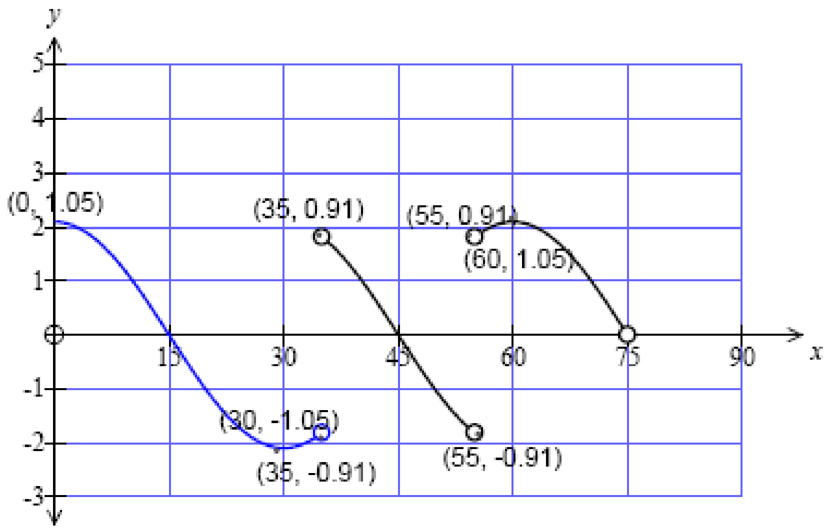
Let $h(x)$ be the derivative of $g(f(x))$ with respect to x.

iii. Sketch the graph of $h(x)$ on the axes provided below. Label endpoints as coordinates, correct to 2 decimal places.



2 marks

Solution



Mark allocation

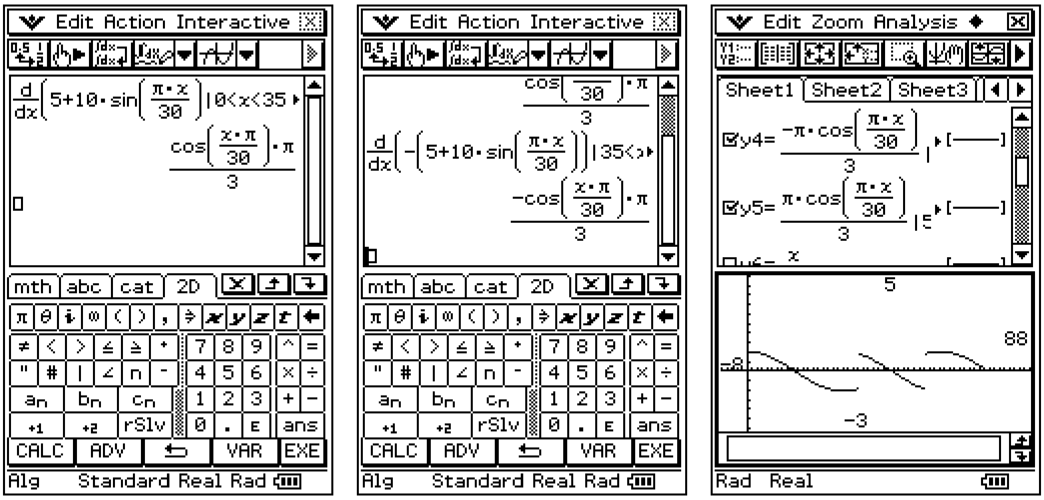
- 1 mark for shape and showing three parts of graph.
- 1 mark for labelling open endpoints and endpoints correctly.

Tip
Differentiate the function as a hybrid function.

$$f(x)=\begin{cases}5+10\sin\left(\dfrac{\pi x}{30}\right), & x\in[0,35]\cup[55,75]\\ -5-10\sin\left(\dfrac{\pi x}{30}\right), & x\in[35,55]\end{cases}$$

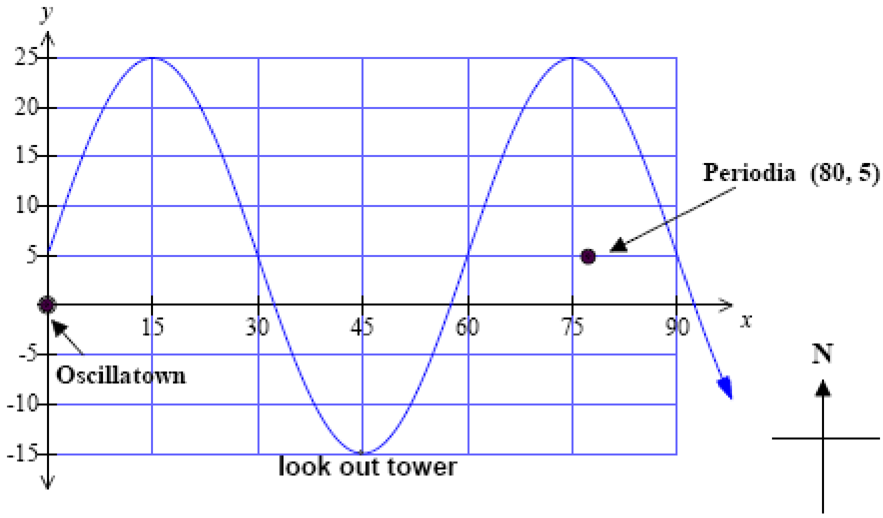
This gives $f'(x)=\begin{cases}\dfrac{\pi}{3}\cos\left(\dfrac{\pi x}{30}\right), & x\in(0,35)\cup(55,75)\\ -\dfrac{\pi}{3}\cos\left(\dfrac{\pi x}{30}\right), & x\in(35,55)\end{cases}$

Now use CAS to sketch the derivative function as a hybrid function.



The newly elected State government is proposing to build a man-made river through a national park. One proposal from an architectural firm has the man-made river weaving a path as shown in the diagram below. The equation of the path of the river is given by
$y = 5 + 20\sin\left(\frac{\pi x}{30}\right)$.

Two towns, Oscillatown and Periodia, are marked on the map. All distances are in kilometres.



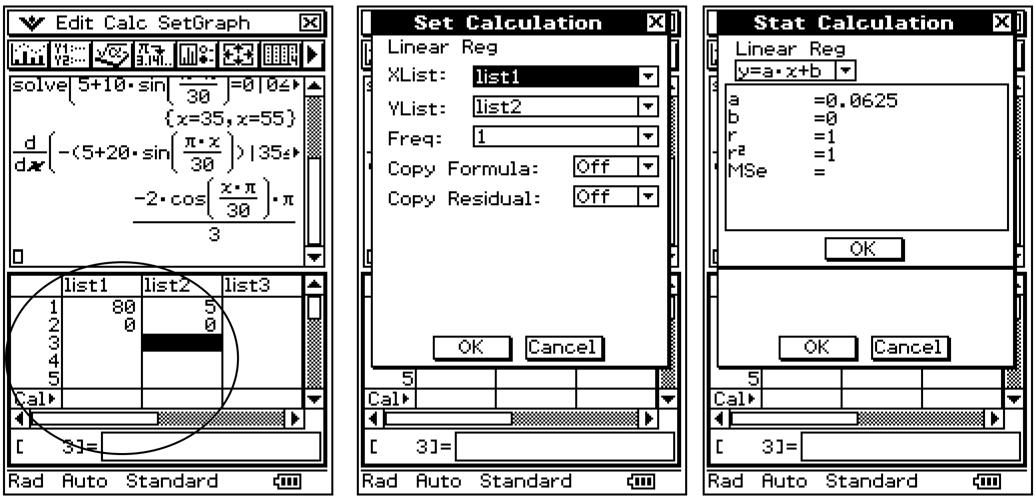
d. The plan includes the idea of building a straight road that must pass through Oscillatown and Periodia. What will be the length of the road, which will be built to the north of the river between Oscillatown and Periodia? Give your answer to 2 decimal places.

2 marks

Solution

Equation of the line joining the towns is $y = \frac{x}{16}$.

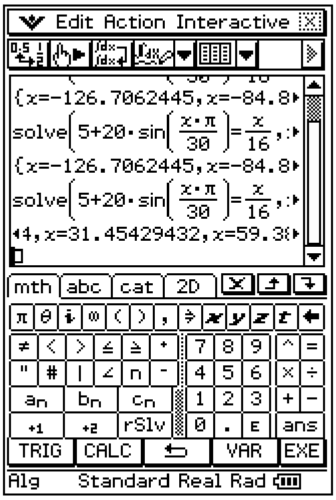
Using CAS, this can be found using the statistics menu and finding linreg.



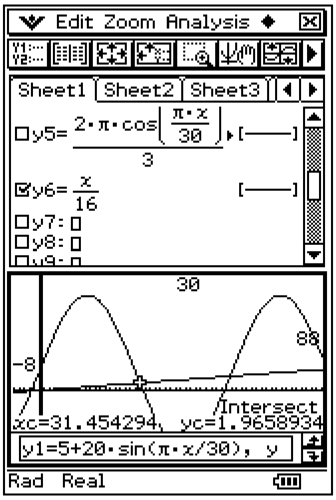
Or, alternatively, this can be found by hand.

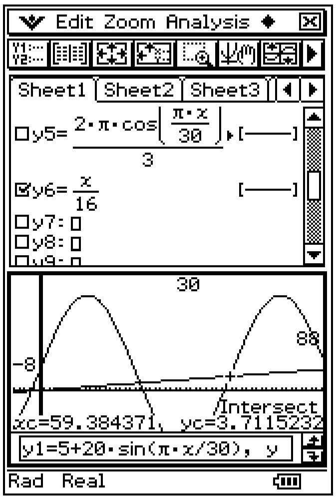
Gradient of $OP = \dfrac{5}{80} = \dfrac{1}{16}$, so $y = \dfrac{x}{16}$ is the equation between the two towns.

We need to find the points of intersection. This can be done by solving the two equations.

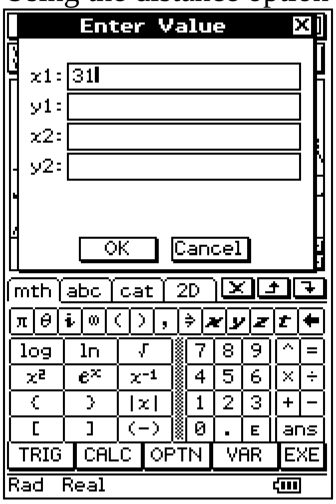


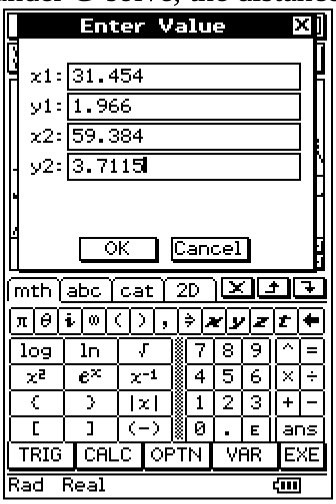
The curves intersect at x = 31.454 and x = 59.384.

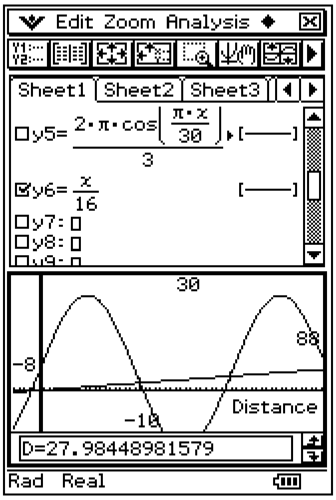




Using the distance option under G-solve, the distance can be found:







Therefore, the length of road joining the two towns and is north of the river will be 27.98 km.

Mark allocation

- 1 mark for finding the coordinates of the points of intersection.
- 1 mark for the finding the correct distance.

e. The State government has rejected this proposal, stating that sightseers will not get to see the river adequately. Instead, it requests that the river follows a similar model but that it weave through at least three complete cycles before passing by the lookout tower at point (45, –15).

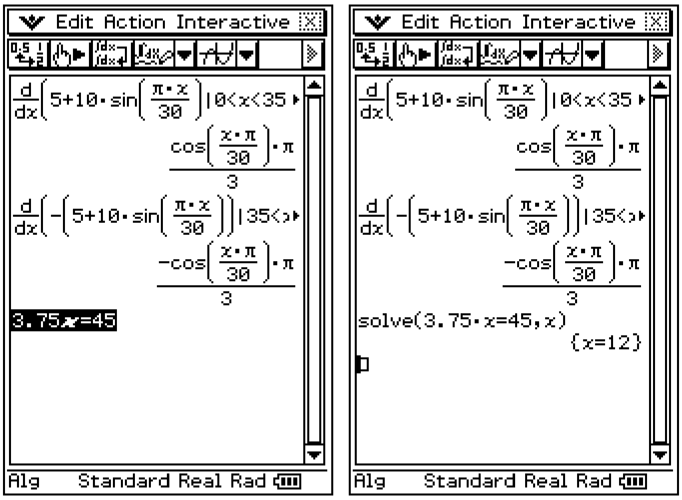
The rule for the equation of the river's path in the new proposal is $y = 5 + 20\sin(kx)$.

Find the smallest value of k that meets the new requirements.

2 marks

Solution

The lookout tower is ¾ of the way through the cycle. So, to have completed three full cycles and then pass the lookout tower means that we want 3¾ of the period to equal 45 km.



So, period is 12.

Therefore, $\frac{2\pi}{k} = 12$, $k = \frac{\pi}{6}$.

Mark allocation

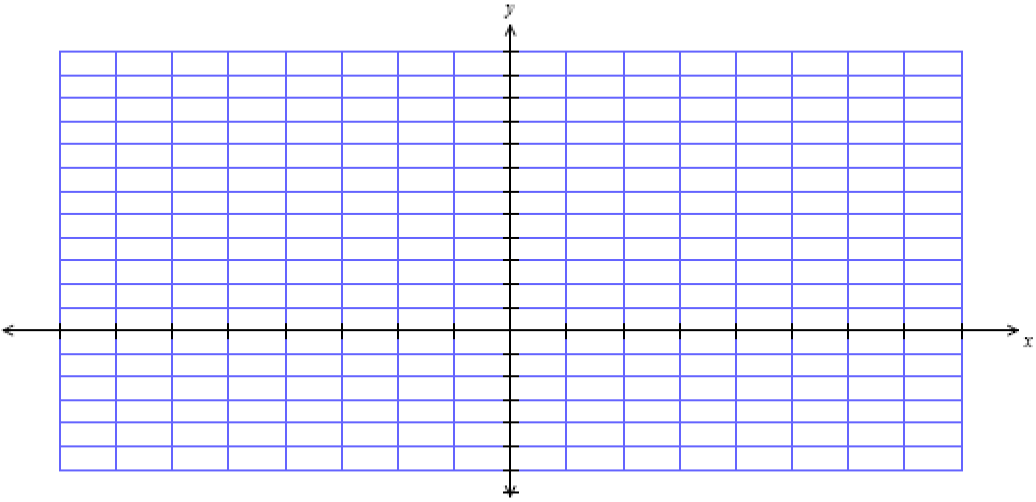
- 1 mark for working with 3¾ cycles.
- 1 mark for answer of $k = \frac{\pi}{6}$.

Total = 13 marks

Question 4

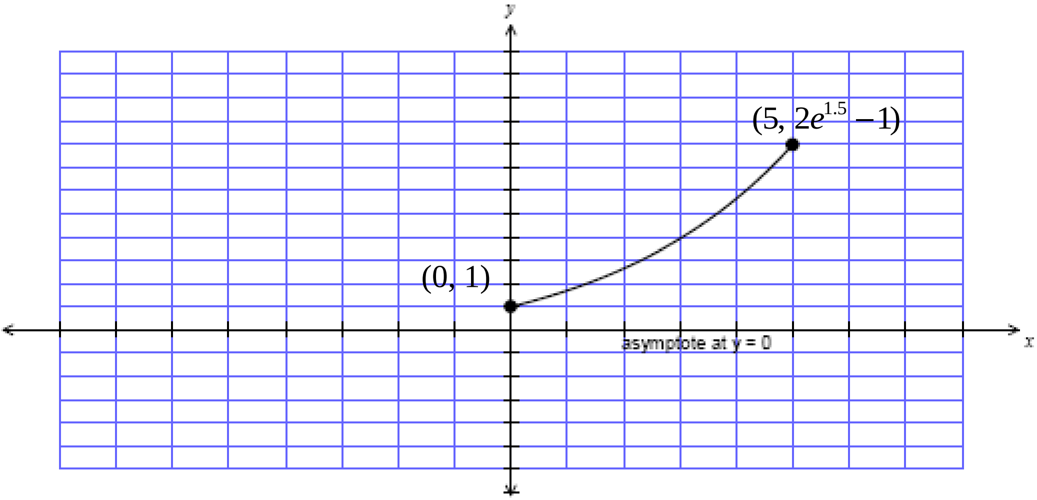
Let $f:[0,5]\rightarrow R$, where $f(x)=2e^{0.3x}-1$.

a. Sketch the graph of $y=f(x)$ on the axes provided below. Label any asymptotes with their equations and the endpoints as exact coordinates.



2 marks

Solution



Mark allocation

- 1 mark for the shape.
- 1 mark for labelling endpoints and the asymptote correctly.

b. State the range of the function $f(x)$.

1 mark

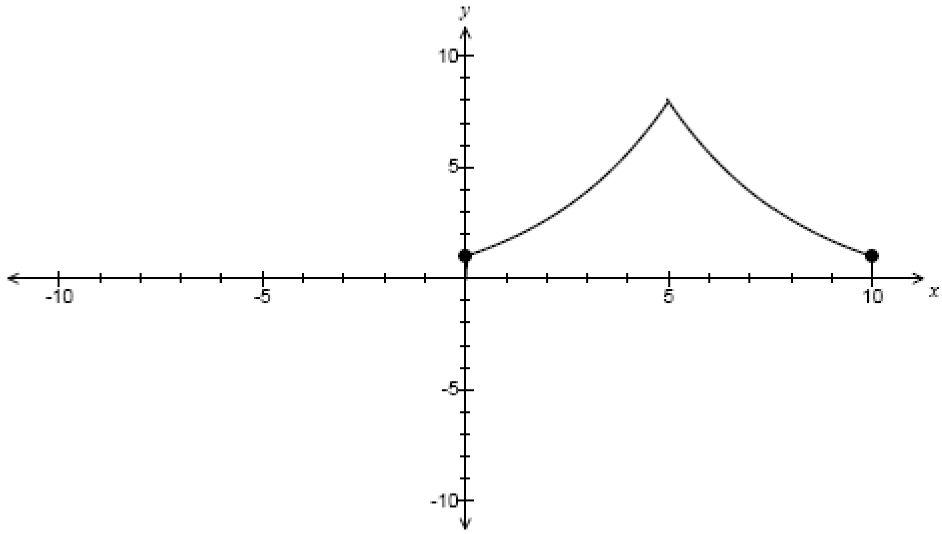
Solution

$[1, 2e^{1.5} - 1]$

Mark allocation

- 1 mark for the correct answer.

The function $f(x)$ forms part of an obstacle in a computer game. The aim of the computer game is to jump 'Bruno', the character in the game, over the obstacles and off to safety before 'Borat', the enemy character, can attack him. The complete obstacle is illustrated below. All measurements are in centimetres.



The obstacle is symmetrical about the line $x = 5$, with the equation of $y = g(x)$ being a 'mirror image' of $f(x)$.

c. State the transformations to produce $y = g(x)$ from $y = f(x)$.

2 marks

Solution

Reflection occurs in the y-axis.
Translation to the right of 10 units.

Mark allocation

- 1 mark for stating reflection correctly.
- 1 mark for stating translation correctly.

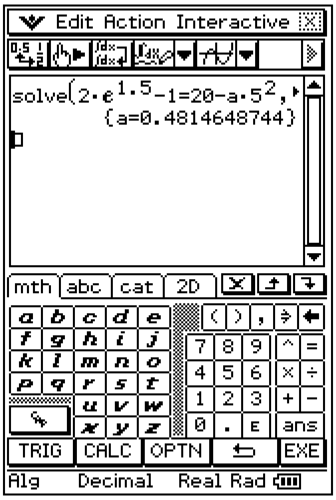
When Bruno jumps, he jumps according to the rule $y = 20 - ax^2$.
Find the values of *a*, correct to 4 decimal places, that would see Bruno

d. land on the left side of the obstacle.

2 marks

Solution

The right-most point on this part of the obstacle occurs at $(5,\ 2e^{1.5} - 1)$, so let the curve with the equation $y = 20 - ax^2$ pass through this.



So, $a \in [0.4815, \infty]$.

Mark allocation

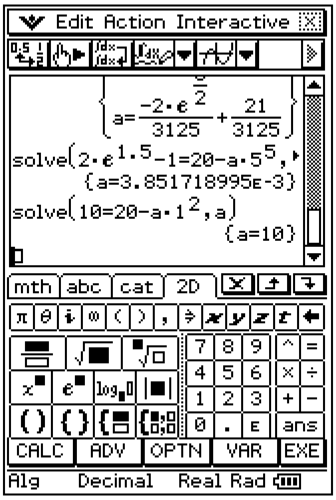
- 1 mark for substituting in $(5,\ 2e^{1.5} - 1)$.
- 1 mark for finding the correct interval of values.

e. jump completely over the obstacle.

2 marks

Solution

To completely clear the obstacle, the curve needs to pass to the right of point (10, 1).
Therefore, substitute (10, 1) into the equation and solve for *a*.



So, $a \in [0, 10]$.

Mark allocation

- 1 mark for substituting in (10, 1).
- 1 mark for finding the correct interval of values.

When the computer game player hits the 'random' button, Bruno jumps so that his basic equation $y = 20 - x^2$ is transformed according to a randomly selected matrix. On one particular occasion, the matrix is $\begin{bmatrix} 0.2 & 0 \\ 0 & 0.9 \end{bmatrix}$.

f. Find the equation of the transformed jump.

2 marks

Solution

$$\left.\begin{aligned} x' &= 0.2x \\ y' &= 0.9y \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{x'}{0.2} \\ y &= \frac{y'}{0.9} \end{aligned}$$

So $y = 20 - x^2$ becomes $\dfrac{y'}{0.9} = 20 - \left(\dfrac{x'}{0.2}\right)^2$.

So $y = 0.9\left(20 - \dfrac{x^2}{0.04}\right)$ or $y = 18 - 22.5x^2$.

SECTION 2 – Question 4 – continued
TURN OVER

Mark allocation

- 1 mark for constructing the transposed version of x and y:

$$\left.\begin{array}{l} x' = 0.2x \\ y' = 0.9y \end{array}\right\} \Rightarrow \begin{array}{l} x = \dfrac{x'}{0.2} \\ y = \dfrac{y'}{0.9} \end{array}$$

- 1 mark for stating final equation.

In general, the random matrix is described by $\begin{bmatrix} k & 0 \\ 0 & 0.9 \end{bmatrix}$, $k > 0$. This matrix transforms the basic equation $y = 20 - x^2$ whenever the random button is pressed.

g. Bruno needs to clear the obstacle and must start his jump from the left side of the obstacle. Find the closest distance he can stand to commence his jump and clear the obstacle when affected by the general random matrix.

2 marks

Solution

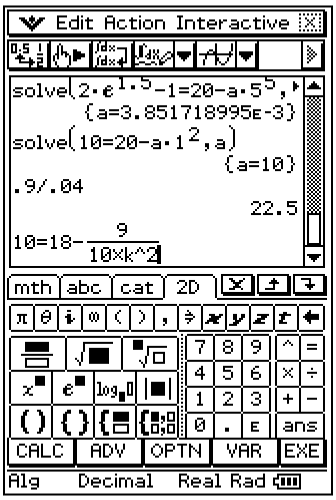
General equation for the transformed equation is:

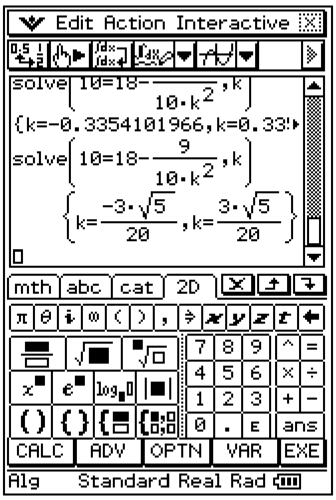
$$\left.\begin{array}{l} x' = kx \\ y' = 0.9y \end{array}\right\} \Rightarrow \begin{array}{l} x = \dfrac{x'}{k} \\ y = \dfrac{y'}{0.9} \end{array}$$

So $y = 20 - x^2$ becomes $\dfrac{y'}{0.9} = 20 - \left(\dfrac{x'}{k}\right)^2$ or $y = 18 - \dfrac{9x^2}{10k^2}$.

To clear the obstacle, Bruno must pass to the right of point (10, 1).

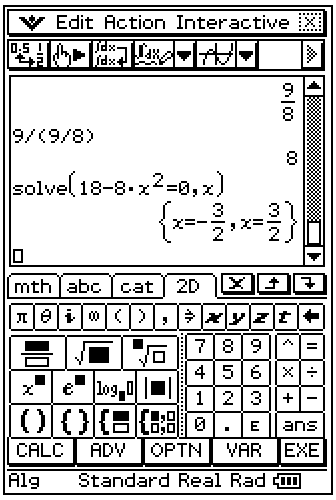
So use CAS to substitute these values into the general equation:





So if $k = \frac{3\sqrt{5}}{20}$, then $y = 18 - \frac{9x^2}{10 \times \left(\frac{3\sqrt{5}}{20}\right)^2} = 18 - 8x^2$.

Therefore, the x-intercepts of $y = 18 - 8x^2$ occurs at $18 - 8x^2 = 0$.



So the closest Bruno can stand and still clear the obstacle is 1.5 cm to its left.

Mark allocation

- 1 mark for obtaining the general equation of the transformed curve.
- 1 mark for the final answer.

Total = 13 marks

END OF WORKED SOLUTIONS BOOK