Year 2011

VCE

Mathematical Methods CAS Solutions Trial Examination 2

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SECTION 1

ANSWERS

1	**A**	B	C	D	E
2	**A**	B	C	D	E
3	A	**B**	C	D	E
4	A	B	C	**D**	E
5	A	**B**	C	D	E
6	A	B	C	D	**E**
7	**A**	B	C	D	E
8	A	B	**C**	D	E
9	A	**B**	C	D	E
10	A	B	C	D	**E**
11	A	B	C	**D**	E
12	A	**B**	C	D	E
13	A	B	**C**	D	E
14	**A**	B	C	D	E
15	**A**	B	C	D	E
16	A	**B**	C	D	E
17	A	B	C	**D**	E
18	A	B	C	D	**E**
19	A	B	C	**D**	E
20	A	B	**C**	D	E
21	A	B	**C**	D	E
22	A	B	C	D	**E**

SECTION 1

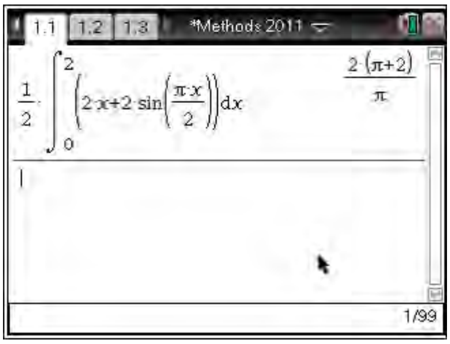
Question 1 **Answer A**

$|x-2k|<k$

$-k<x-2k<k$

$k<x<3k$ or $(k,3k)$

Question 2 **Answer A**



average value $\bar{y}=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$\bar{y}=\frac{1}{2-0}\int_0^2\left(2x+2\sin\left(\frac{\pi x}{2}\right)\right)dx$$

$$\bar{y}=\frac{1}{2}\left[x^2-\frac{4}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^2$$

$$\bar{y}=\frac{1}{2}\left[\left(4-\frac{4}{\pi}\cos(\pi)\right)-\left(0-\frac{4}{\pi}\cos(0)\right)\right]=\frac{1}{2}\left[4+\frac{8}{\pi}\right]$$

$$\bar{y}=2+\frac{4}{\pi}=\frac{2(\pi+2)}{\pi}$$

Question 3 **Answer B**

$$y=\frac{ax+b}{x+c}=a+\frac{b-ac}{x+c}$$

so $y=a$ is a horizontal asymptote and $x=-c$ is a vertical asymptote.

Question 4 **Answer D**

$f(x)=x^2$, $g(x)=\dfrac{1}{x}$ and $h(x)=\cos(x)$

$$\frac{d}{dx}(\tan(x))=\frac{1}{\cos^2(x)}$$

now $g\left(f\left(h(x)\right)\right)=g\left(f\left(\cos(x)\right)\right)=g\left(\cos^2(x)\right)=\dfrac{1}{\cos^2(x)}$

so $\dfrac{d}{dx}(\tan(x))=\dfrac{1}{\cos^2(x)}=g\left(f\left(h(x)\right)\right)$

Question 5 **Answer B**

$f(x)=\sqrt{x-a}+\sqrt{b-x}$

the domain requires $x-a \ge 0$ and $b-x \ge 0$

that is $x \ge a$ and $x \le b$ since $b > a > 0$

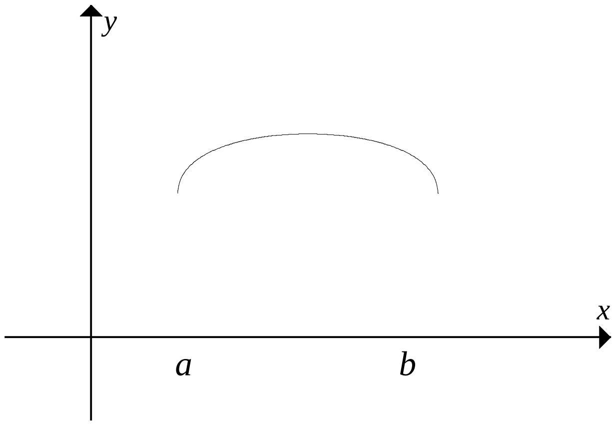
the domain is $[a,b]$

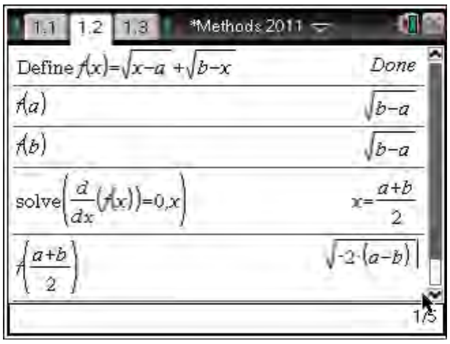
now $f(a)=f(b)=\sqrt{b-a}$

$f'\left(\frac{a+b}{2}\right)=0 \Rightarrow$ turning point at $x=\frac{a+b}{2}$

and $f\left(\frac{a+b}{2}\right)=\sqrt{2(b-a)}$

the range is $\left[\sqrt{b-a}\,,\sqrt{2(b-a)}\right]$





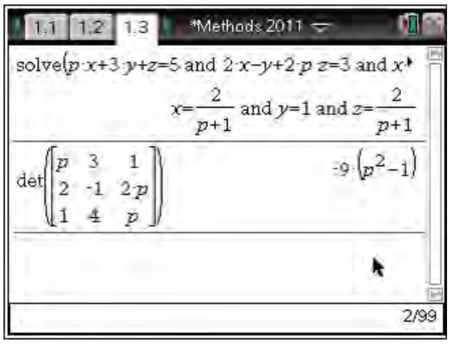
Question 6 **Answer E**

$$\Delta=\begin{vmatrix} p & 3 & 1 \\ 2 & -1 & 2p \\ 1 & 4 & p \end{vmatrix}=-9\left(p^2-1\right)$$

solving using CAS gives

$x=\frac{2}{p+1}$ $y=1$ and $z=\frac{2}{p+1}$

Since $\Delta=0 \Rightarrow p=\pm 1$, there is no unique solution when $p^2=1$,



there is a unique solution when $p^2 \ne 1$. When $p=-1$ there is no solution and when $p=1$ there is an infinite number of solutions.

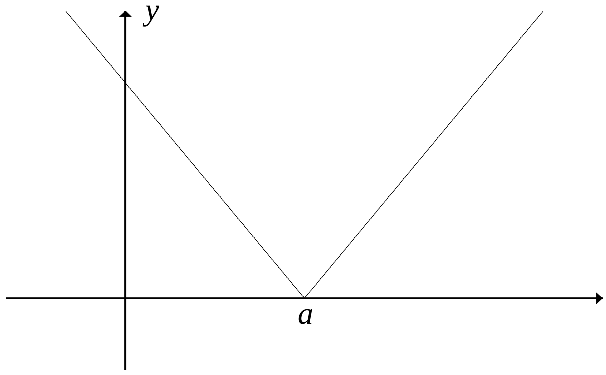
Only option **E.** is correct.

Question 7 **Answer A**

When the point $(2,-3)$ is reflected in the *x*-axis, it becomes, the point $(2,3)$, when it is translated one unit, to the left parallel to the *x*-axis, or away from the *y*-axis, it becomes $(1,3)$, finally it is translated one unit up parallel to the *y*-axis or away from the *x*-axis, it becomes $(1,4)$ under $y = 1 - f(x+1)$.

Question 8 **Answer C**

The function is continuous at $x = a$, all other options are true.



Question 9 **Answer B**

$$z = 3$$
$$x + y = 5 \quad \text{rewrite the equations as}$$
$$y - x = -1$$

$$\begin{aligned} x + y &= 5 \\ x - y &= 1 \\ z &= 3 \end{aligned} \quad \text{in matrix from these become} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

Question 10 **Answer E**

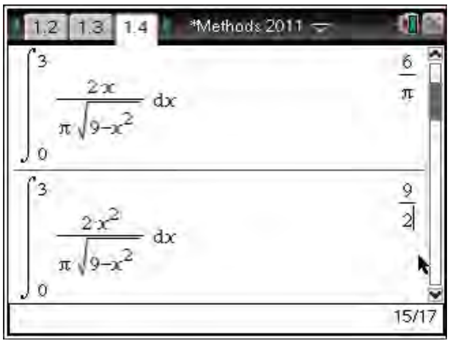
Since it is a probability density function

$$\int_0^3 \frac{2}{\pi\sqrt{9-x^2}}\,dx = 1$$

$$E(X) = \int_0^3 \frac{2x}{\pi\sqrt{9-x^2}}\,dx = \frac{6}{\pi}$$

$$E(X^2) = \int_0^3 \frac{2x^2}{\pi\sqrt{9-x^2}}\,dx = \frac{9}{2} \quad \text{by CAS}$$

$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{9}{2} - \left(\frac{6}{\pi}\right)^2 \approx 0.85$$



Question 11 **Answer D**

$$\frac{dy}{dx} = 4e^{-\frac{x}{2}} \quad \Rightarrow \quad y = \int 4e^{-\frac{x}{2}}\,dx$$

$$y = -8e^{-\frac{x}{2}} + c \quad \text{now when } x = 0 \quad y = 0$$

$$0 = -8 + c \quad \Rightarrow \quad c = 8$$

$$y = 8\left(1 - e^{-\frac{x}{2}}\right)$$

Question 12 **Answer B**

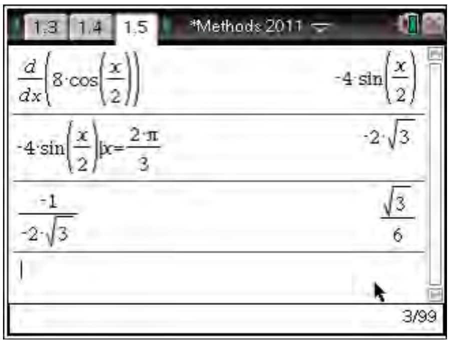
Let $f(x) = e^{-x}$ Now $\dfrac{1}{e^{0.99}} = e^{-0.99} = e^{-(1-0.01)}$

with $x = 1$ and $h = -0.01$,

using $f(x+h) \approx f(x) + hf'(x)$

$$\frac{1}{e^{0.99}} = f(1) - 0.01f'(1)$$

Question 13 **Answer C**



$$y = 8\cos\left(\frac{x}{2}\right) \quad \Rightarrow \quad \frac{dy}{dx} = -4\sin\left(\frac{x}{2}\right)$$

gradient of the tangent $m_T = \left.\dfrac{dy}{dx}\right|_{x=\frac{2\pi}{3}} = -4\sin\left(\dfrac{\pi}{3}\right) = -2\sqrt{3}$

gradient of the normal $m_N = \dfrac{1}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

Question 14 **Answer A**

$f(x) = \dfrac{x^2}{g(x)}$ using the quotient rule

$$f'(x) = \frac{2xg(x) - x^2g'(x)}{\left[g(x)\right]^2}$$

$f'(3) = \dfrac{6g(3) - 9g'(3)}{\left[g(3)\right]^2}$ now $g(3) = 2$ and $g'(3) = 1$

$$f'(3) = \frac{6\times 2 - 9\times 1}{2^2} = \frac{3}{4}$$

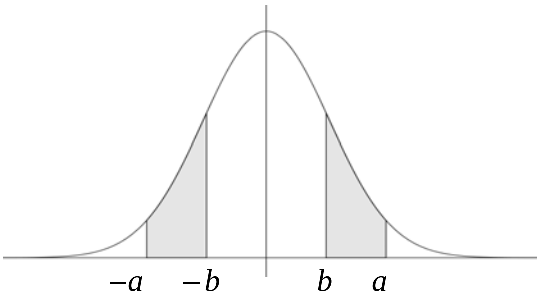
Question 15 **Answer A**

$$\int_2^0 \left(2x - f(x)\right) dx$$

$$= \left[x^2\right]_2^0 - \int_2^0 f(x)\,dx = (0-4) + \int_0^2 f(x)\,dx = -4 + 2 = -2$$

Question 16 **Answer B**

$$\Pr(-a < Z < -b)$$
$$= \Pr(b < Z < a)$$
$$= \Pr(Z < a) - \Pr(Z < b)$$
$$= \left(1 - \Pr(Z > a)\right) - \left(1 - \Pr(Z > b)\right)$$
$$= (1 - A) - (1 - B)$$
$$= B - A$$



Question 17 **Answer D**

The shaded area, with the *x*-axis is

$$A = \int_a^b (y_2 - y_1)\,dx \quad \text{with} \quad a = 0 \quad b = 1 \quad y_2 = 2 \text{ and } y_1 = x^2 + 1$$

$$A = \int_0^1 \left(1 - x^2\right) dx$$

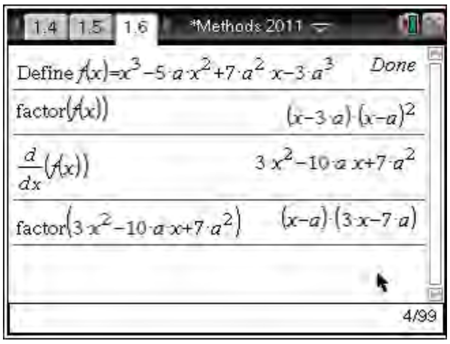
however this is none of the alternatives,
the area with the *y*-axis, is

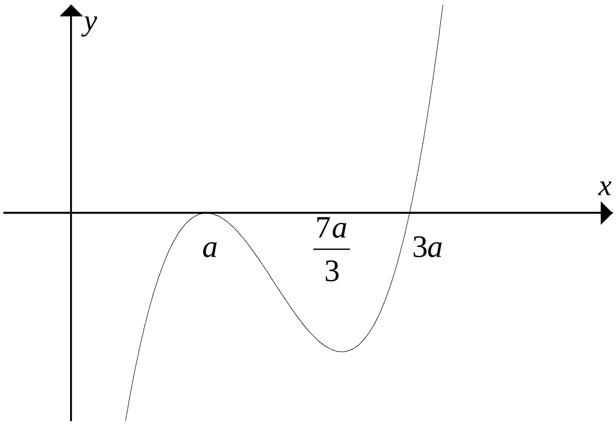
$$A_y = \int_c^d x\,dy \quad \text{with} \quad c = 1 \text{ and } d = 2 \quad y = x^2 + 1$$

$$\Rightarrow x^2 = y - 1 \quad \text{and} \quad x = \sqrt{y-1} \quad \text{since} \quad x > 0$$

$$A = \int_1^2 \sqrt{y-1}\,dy = \int_1^2 \sqrt{x-1}\,dx \quad \text{using dummy variable property.}$$

Question 18 **Answer E**





$f(x) = x^3 - 5ax^2 + 7a^2x - 3a^3 = (x-a)^2(x-3a)$

$f'(x) = 3x^2 - 10ax + 7a^2 = (x-a)(3x-7a)$

there are turning points at $x = a$ and $x = \frac{7a}{3}$,

for the function to be one-one, the only correct option is the restricted interval $\left(\frac{7a}{3}, \infty\right)$

Question 19 **Answer D**

Since A and B are independent events, $\Pr(A \cap B) = \Pr(A)\Pr(B) = ab$

	A	A'	
B	ab	$b - ab$	b
B'	$a - ab$	$1 - a - b + ab$	$1 - b$
	a	$1 - a$	

$$\begin{aligned}\Pr(A' \cup B') &= \Pr(A') + \Pr(B') - \Pr(A' \cap B') \\ &= (1-a) + (1-b) - (1-a-b+ab) \\ &= 1 - ab\end{aligned}$$

Question 20 **Answer C**

Three right rectangles, each of width $h=\dfrac{\pi}{6}$

x	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
$y=\sin(x)$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	1

The shaded area of the three rectangles is $A=\dfrac{\pi}{6}\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}+1\right)=\dfrac{\pi}{6}\left(\dfrac{3+\sqrt{3}}{2}\right)=\dfrac{\pi\left(3+\sqrt{3}\right)}{12}$

Question 21 **Answer C**

$X \sim \text{Bi}(n,p)$

$\Pr(\text{ more than two }) = \Pr(X>2)$

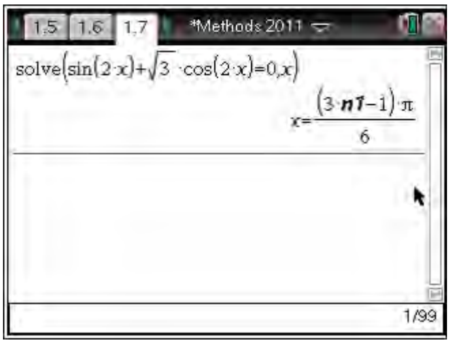
$=1-\left[\Pr(X=0)+\Pr(X=1)+\Pr(X=2)\right]$

$=1-\left[q^n+npq^{n-1}+\dfrac{n(n-1)}{2}p^2q^{n-2}\right]$

$=1-\left(0.7^{10}+10\times0.7^9\times0.3+45\times0.7^8\times0.3^2\right)$

$\Rightarrow n=10 \quad , \quad q=0.7 \quad \text{and} \quad p=0.3$

Question 22 **Answer E**



One solution when $n=0$ is $x=-\dfrac{\pi}{6}$ so that

$2x=-\dfrac{\pi}{3}$ and $\tan(2x)=-\sqrt{3}$

$\dfrac{\sin(2x)}{\cos(2x)}=-\sqrt{3}$ or $\sin(2x)=-\sqrt{3}\cos(2x)$

or $\sin(2x)+\sqrt{3}\cos(2x)=0$

so the general solution of

$\sin(2x)+\sqrt{3}\cos(2x)=0$ is $x=\dfrac{n\pi}{2}-\dfrac{\pi}{6}=\dfrac{(3n-1)\pi}{6}$ where $n\in Z$

$a=1$ and $b=\sqrt{3}$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i. 2 marks, for correct transformations

- reflection in the *x*-axis
- dilation by a factor of 4 parallel to the *y*-axis (or away from the *x*-axis)
- translation by 3 units to the right parallel to the *x*-axis, (or away from the *y*-axis)
- translation by 4 units up and parallel to the *y*-axis, (or away from the *x*-axis)

ii. $m: \quad y = 4 - \dfrac{4}{(x-3)^2}$ interchanging *x* and *y*

$m^{-1}: \quad x = 4 - \dfrac{4}{(y-3)^2}$ M1

$$\frac{4}{(y-3)^2} = 4 - x$$

$$(y-3)^2 = \frac{4}{4-x}$$

$$y - 3 = \frac{\pm 2}{\sqrt{4-x}}$$

Since the range of m^{-1} is $(-\infty, 3)$, the same as the domain of *m*, we must take the negative, so $y = 3 - \dfrac{2}{\sqrt{4-x}}$ A1

Now the domain of m^{-1} is the same as the range of *m*, that is $(-\infty, 4)$.

To state the function, we need to state both the domain and the rule.

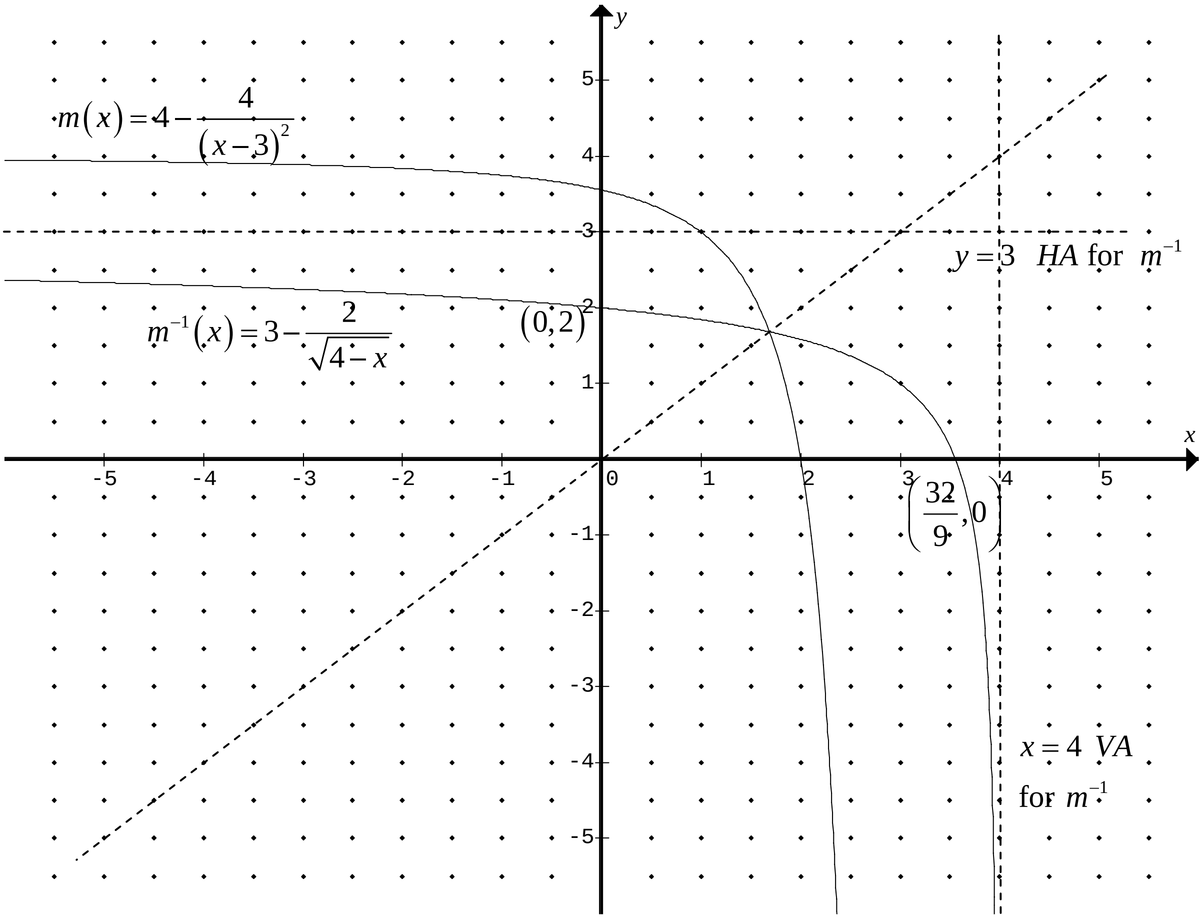
$m^{-1}: (-\infty, 4) \to R$, $m^{-1}(x) = 3 - \dfrac{2}{\sqrt{4-x}}$ A1

iii. the graph of m^{-1} crosses the *x*-axis at $\left(\dfrac{32}{9}, 0\right)$, since $m(0) = 4 - \dfrac{4}{9} = \dfrac{32}{9}$

and crosses the *y*-axis at $(0, 2)$, since $m^{-1}(0) = 3 - \dfrac{2}{\sqrt{4}} = 2$ A1

for the graph of $m(x)$ $x = 3$ is a vertical asymptote and $y = 4$ is a horizontal asymptote, so for the graph of $m^{-1}(x)$ $y = 3$ is a horizontal asymptote and $x = 4$ is a vertical asymptote. A1

correct graph, shape, reflection in the line $y = x$, and the intersection of *m* and m^{-1} must be on the line $y = x$. A1



b.i. since $f(0)=\frac{32}{9}=3\frac{5}{9}$, $f(2)=0$ and $g(2)=g(10)=0$ and $g(6)=\frac{32}{9}$

amplitude $\Rightarrow b=\frac{32}{9}$**,** the phase shift is 2 units, to the right, so that $c=2$ A1

the sine squared wave is half a cycle $\Rightarrow T=\frac{\pi}{n}=8 \Rightarrow n=\frac{\pi}{8}$ A1

the graph of h is the reflection in the line, $x=6$ $h(10)=0$ and $h(12)=\frac{32}{9}$

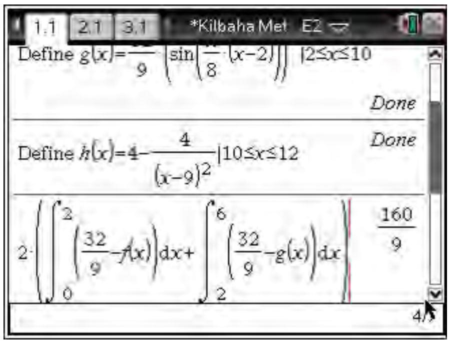
$h(x)=4-\frac{4}{(x-9)^2}$ so that $p=4$, $r=-4$ and $s=9$ A1

ii. using symmetry, in terms of two definite integrals, the area between the curves

$$A=2\left[\int_0^2\left(\frac{32}{9}-\left(4-\frac{4}{(x-3)^2}\right)\right)dx+\int_2^6\left(\frac{32}{9}-b\sin^2\left(n(x-c)\right)\right)dx\right]$$ A2

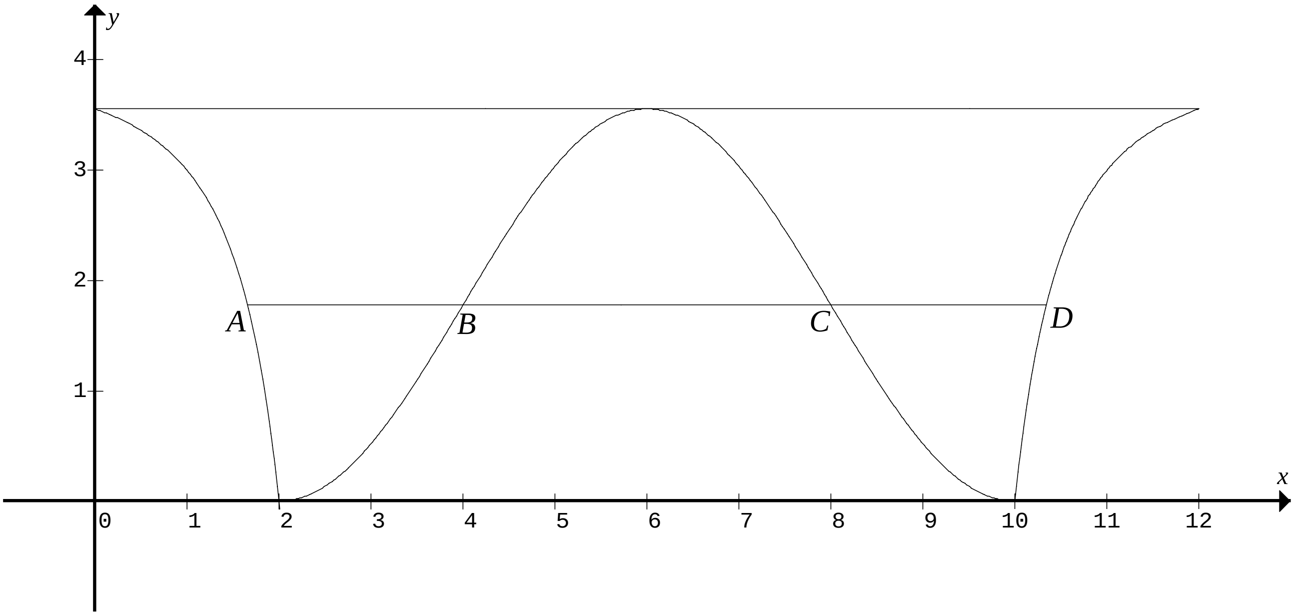
or alternatively, other equivalent answers are possible.

$$A = 2\left[\int_0^2 \left(\frac{4}{(x-3)^2} - \frac{4}{9}\right)dx + \int_2^6 \left(\frac{32}{9}\cos^2\left(\frac{\pi}{8}(x-2)\right)\right)dx\right]$$



iii. $A = \frac{160}{9} = 17\frac{7}{9}$ metres^2 using CAS A1

iv.



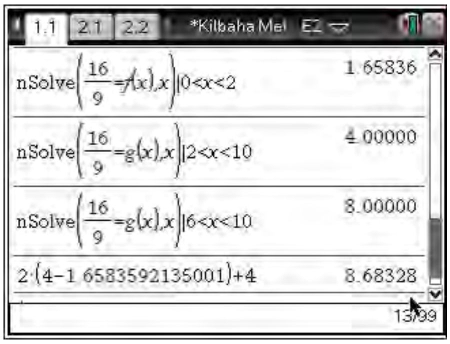
the line through *ABCD* is half the maximum value, that is $y = \frac{16}{9}$, solving $\frac{16}{9} = 4 - \frac{4}{(x-3)^2}$ with $0 < x < 2$

gives $x = 1.65836$ solving

$\frac{16}{9} = \frac{32}{9}\sin^2\left(\frac{\pi}{8}(x-2)\right)$ with $2 < x < 10$

gives $x = 4$ and $x = 8$

$A\left(1.6584, \frac{16}{9}\right)$ $B\left(4, \frac{16}{9}\right)$ $C\left(8, \frac{16}{9}\right)$



the length of $AD = 2(4 - 1.65836) + 4$ or alternatively $= 2(6 - 1.65836)$

length *ABCD* is 8.683 metres A1

Question 2

a.i.
$$\begin{array}{c} \\ C \\ S \end{array}\begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.65 & 0.55 \\ 0.35 & 0.45 \end{bmatrix} \end{array} \quad \text{Let } A = \begin{bmatrix} 0.65 & 0.55 \\ 0.35 & 0.35 \end{bmatrix}$$

$C \to C\ \ 0.65 \quad C \to S\ \ 0.35 \quad S \to C\ \ 0.55 \quad S \to S\ \ 0.45$

$\Pr(\text{Coles once}) = \Pr(CSSS) = 0.35 \times 0.45^2 = 0.0709$ A1

ii. $\Pr(\text{Coles twice}) = \Pr(CCSS) + \Pr(CSCS) + \Pr(CSSC)$ M1

$= 0.65 \times 0.35 \times 0.45 + 0.35 \times 0.55 \times 0.35 + 0.35 \times 0.45 \times 0.55$

$= 0.2564$ A1

iii. $\Pr(\text{Coles 3 times}) = \Pr(CCCS) + \Pr(CCSC) + \Pr(CSCC)$

$= 0.65^2 \times 0.35 + 0.65 \times 0.35 \times 0.55 + 0.35 \times 0.55 \times 0.65$

$= 0.3918$ A1

$\Pr(\text{Coles 4 times}) = \Pr(CCCC)$

$= 0.65^3$

$= 0.2746$ A1

Number of times at Coles	1	2	3	4
Probability	$\frac{567}{8000} = 0.0709$	$\frac{2051}{8000} = 0.2564$	$\frac{637}{1600} = 0.3918$	$\frac{2197}{8000} = 0.2746$

Expected number of times at Coles

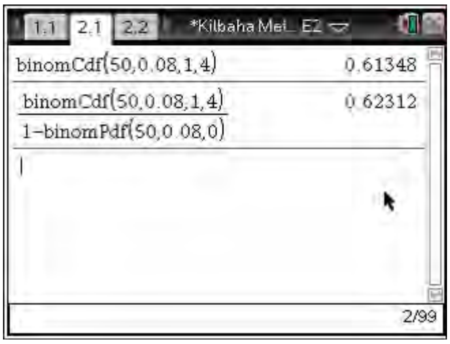
$$E(C) = 1 \times \frac{567}{8000} + 2 \times \frac{2051}{8000} + 3 \times \frac{637}{1600} + 4 \times \frac{2197}{8000}$$

$$E(C) = \frac{5753}{2000}$$ A1

b. Now as $n \to \infty \quad A^n \to \begin{bmatrix} 0.6\dot{1} & 0.6\dot{1} \\ 0.3\dot{8} & 0.3\dot{8} \end{bmatrix}$ or $\dfrac{0.55}{0.55 + 0.35} = 0.6\dot{1} = \dfrac{11}{18}$

so the steady state probability that they go to Coles is $\dfrac{11}{18}$ A1

c.i. $X \sim Bi\left(n = 50,\, p = 0.08\right)$

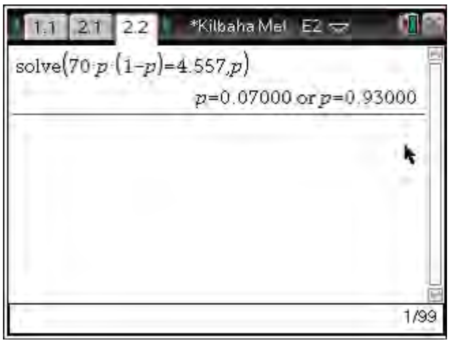


$$\Pr\left(X < 5 \mid X \ge 1\right) = \frac{\Pr\left(1 \le X \le 4\right)}{\Pr\left(X \ge 1\right)}$$ M1

$$= \frac{0.61348}{1 - 0.01547}$$

$= 0.6231$ A1

ii. $X \sim Bi\left(n = 70,\, p = ?\right)$



$\operatorname{var}\left(X\right) = npq = 70\,p\left(1 - p\right) = 4.557$

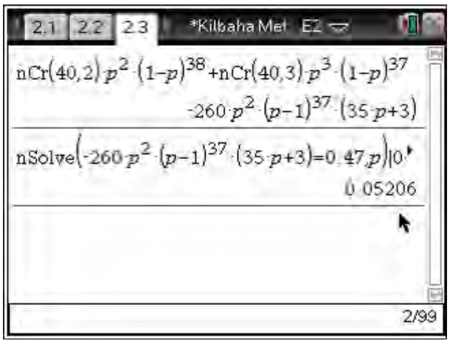
solving for p since

$0 < p < 1$ gives $p = 0.07$ or 0.93 M1

Since $E\left(X\right) < 5$

$p = 0.07$ A1

iii. $X \sim Bi\left(n = 40,\, p = ?\right)$



$\Pr\left(X = 2\right) + \Pr\left(X = 3\right) = 0.47$

$$\binom{40}{2} p^2 \left(1 - p\right)^{38} + \binom{40}{3} p^3 \left(1 - p\right)^{37} = 0.47$$

$\Rightarrow -260\,p^2\left(p - 1\right)^{37}\left(35p + 3\right) = 0.47$ M1

solving numerically using CAS,

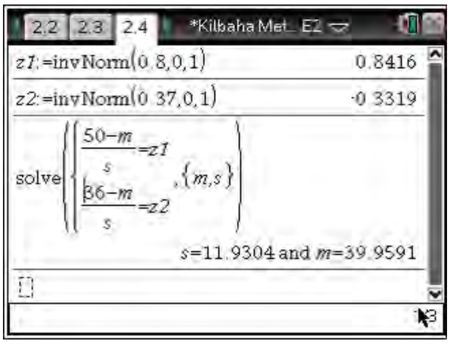
with $0 < p < 1$

$\Rightarrow \; p = 0.0521$ A1

d. X is the time in minutes spent shopping, $X \sim N\left(\mu = ?, \sigma^2 = ?\right)$

$(1) \quad \Pr(X > 50) = 0.2$ M1

$(2) \quad \Pr(X < 36) = 0.37$



$(1) \Rightarrow \dfrac{50-\mu}{\sigma} = 0.842$

$(2) \Rightarrow \dfrac{36-\mu}{\sigma} = -0.332$

$(1) \quad 50 - \mu = 0.842\sigma$

$(2) \quad 36 - \mu = -0.332\sigma$ M1

now subtract equations $(1)-(2)$

$14 = 1.174\ \sigma$

$\sigma = 12$ minutes A1

substituting gives

$\mu = 40$ minutes A1

Question 3

a. arc length $l = r\theta$ but $l = 2\pi r$ circumference of base circle of the cone

$2\pi r = 12\theta \quad \Rightarrow (1)\ r = \dfrac{6\theta}{\pi}$ A1

Pythagoras $\Rightarrow (2)\ h^2 + r^2 = 12^2 = 144$

$(2)\ h^2 = 144 - r^2 = 144 - \left(\dfrac{6\theta}{\pi}\right)^2 = 144 - \dfrac{36\theta^2}{\pi^2}$ M1

$h^2 = \dfrac{36}{\pi^2}\left(4\pi^2 - \theta^2\right)$ so that $h = \dfrac{6}{\pi}\sqrt{4\pi^2 - \theta^2}$ since $h > 0$

Now volume of cone $V = \dfrac{1}{3}\pi r^2 h$

M1

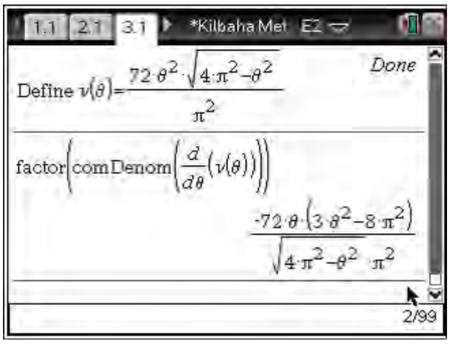
$V = \dfrac{\pi}{3}\left(\dfrac{36\theta^2}{\pi^2}\right)\dfrac{6}{\pi}\sqrt{4\pi^2 - \theta^2}$

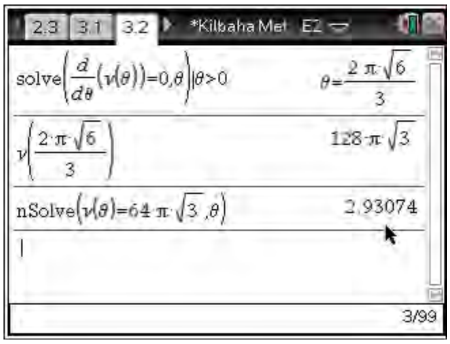
$V = V(\theta) = \dfrac{72\theta^2}{\pi^2}\sqrt{4\pi^2 - \theta^2}$ shown

b. $\dfrac{dV}{d\theta} = \dfrac{72\theta\left(8\pi^2 - 3\theta^2\right)}{\pi^2\sqrt{4\pi^2 - \theta^2}}$ by CAS A1

for max/min $\dfrac{dV}{d\theta} = 0$ solving, since $\theta > 0 \quad \Rightarrow \quad \theta = \dfrac{2\pi\sqrt{6}}{3}$ by CAS A1

$V_{\max} = V\left(\dfrac{2\pi\sqrt{6}}{3}\right) = 128\pi\sqrt{3}$ cm^3 by CAS A1





c. Numerically solving $V(\theta) = \dfrac{1}{2}V_{\max} \quad \Rightarrow \quad 64\pi\sqrt{3} = \dfrac{72\theta^2}{\pi^2}\sqrt{4\pi^2 - \theta^2}$ A1

for θ since $0 < \theta < \pi \quad \Rightarrow \theta = 2.93$ A1

d. r and h are now the radius and height respectively of the mousse in the cone,

given that $\dfrac{dV}{dt} = -0.5$ cm^3/sec find $\dfrac{dh}{dt}$ when $h = 4$ cm

$V = \dfrac{1}{3}\pi r^2 h$ and $\tan(\alpha) = \dfrac{r}{h} = \sqrt{2} \quad \Rightarrow r = \sqrt{2}\,h$ substituting A1

$V = \dfrac{2\pi h^3}{3} \quad \Rightarrow \quad \dfrac{dV}{dh} = 2\pi h^2$ M1

$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{-0.5}{2\pi h^2}$

when $h = 4$ $\dfrac{dh}{dt} = -\dfrac{1}{64\pi}$ cm/sec or falling at a rate of $\dfrac{1}{64\pi}$ cm/sec A1

Question 4

a. $y = f(x) = \frac{1}{x} \quad R\left(r, \frac{1}{r}\right)$

$\frac{dy}{dx} = f'(x) = -\frac{1}{x^2} \quad f'(r) = -\frac{1}{r^2}$ A1

the equation of the tangent at *R* is

$y - \frac{1}{r} = -\frac{1}{r^2}(x - r) = -\frac{x}{r^2} + \frac{1}{r}$

$y = -\frac{x}{r^2} + \frac{2}{r}$

$m = -\frac{1}{r^2}$ and $c = \frac{2}{r}$ A1

b. $P\left(p, \frac{1}{p}\right)$, $Q\left(q, \frac{1}{q}\right)$ $R\left(r, \frac{1}{r}\right)$

Since *M* is the midpoint of *PQ* $M\left(\frac{1}{2}(p+q), \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right)$ A1

gradient $OM = \frac{\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)}{\frac{1}{2}(p+q)} = \frac{\frac{p+q}{pq}}{p+q} = \frac{1}{pq}$ A1

gradient $OR = \frac{\frac{1}{r}}{r} = \frac{1}{r^2} =$ gradient $OM = \frac{1}{pq}$

so that $r^2 = pq$ shown A1

c. gradient $PQ = \frac{\frac{1}{q} - \frac{1}{p}}{q - p} = \frac{\frac{p-q}{pq}}{q-p} = -\frac{1}{pq}$ A1

$f'(r) = -\frac{1}{r^2} = -\frac{1}{pq}$ from **a.** and **b.** A1

so the tangent to the curve at *R*, is parallel to the line segment joining *P* and *Q*.

d. $A=\int_p^q \frac{1}{x}\,dx$

$A=\left[\log_e|x|\right]_p^q=\log_e(q)-\log_e(p)$ since $q>p>0$

$A=\log_e\left(\frac{q}{p}\right)$ A1

e. Area $=\int_p^r \frac{1}{x}\,dx$

Area $=\left[\log_e|x|\right]_p^r=\log_e(r)-\log_e(p)$ since $q>r>p>0$ A1

Area $=\log_e\left(\frac{r}{p}\right)$ now from **b.** since $r=\sqrt{pq}$

Area $=\log_e\left(\frac{\sqrt{pq}}{p}\right)=\log_e\left(\frac{\sqrt{q}}{\sqrt{p}}\right)=\log_e\left(\frac{q}{p}\right)^{\frac{1}{2}}$ A1

Area $=\frac{1}{2}\log_e\left(\frac{q}{p}\right)=\frac{1}{2}A$ A1

f. The line OP is $y=\frac{x}{p^2}$ for $0\le x\le p$, the line OQ is $y=\frac{x}{q^2}$ for $0\le x\le q$ A1

the area between the curves is

$B=\int_0^p\left(\frac{x}{p^2}-\frac{x}{q^2}\right)dx+\int_p^q\left(\frac{1}{x}-\frac{x}{q^2}\right)dx$

$g(x)=\frac{x}{p^2}-\frac{x}{q^2}$ and $h(x)=\frac{1}{x}-\frac{x}{q^2}$ A1

g. $B=\left[\frac{x^2}{2p^2}-\frac{x^2}{2q^2}\right]_0^p+\left[\log_e|x|-\frac{x^2}{2q^2}\right]_p^q$

$B=\frac{p^2}{2p^2}-\frac{p^2}{2q^2}+\left[\left(\log_e(q)-\frac{q^2}{2q^2}\right)-\left(\log_e(p)-\frac{p^2}{2q^2}\right)\right]$ M1

$B=\frac{1}{2}-\frac{p^2}{2q^2}+\log_e\left(\frac{q}{p}\right)-\frac{1}{2}+\frac{p^2}{2q^2}$

$B=\log_e\left(\frac{q}{p}\right)=A$ A1

END OF SECTION 2 SUGGESTED ANSWERS