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MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2012

Question 1

a.

$$y = \frac{e^{2x}}{x^2+3}$$

$$\frac{dy}{dx} = \frac{(x^2+3)\times 2e^{2x} - 2xe^{2x}}{(x^2+3)^2} \qquad \text{(quotient rule)}$$

(1 mark)

Avoid "further engagement" errors by leaving this fraction as it is. No like terms are obtained by expanding the brackets in the numerator.

b.

$$f(x) = 2x\tan\left(\frac{x}{3}\right)$$

$$f'(x) = 2x \times \frac{1}{3}\sec^2\left(\frac{x}{3}\right) + 2\tan\left(\frac{x}{3}\right) \qquad \text{(product rule)}$$

$$f'(\pi) = \frac{2\pi}{3}\sec^2\left(\frac{\pi}{3}\right) + 2\tan\left(\frac{\pi}{3}\right)$$ **(1 mark)**

$$= \frac{2\pi}{3} \times \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} + 2\tan\left(\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} \times \frac{1}{\left(\frac{1}{2}\right)^2} + 2\sqrt{3}$$

$$= \frac{2\pi}{3} \times 1 \div \frac{1}{4} + 2\sqrt{3}$$

$$= \frac{2\pi}{3} \times 1 \times \frac{4}{1} + 2\sqrt{3}$$

$$= \frac{8\pi}{3} + 2\sqrt{3}$$

(1 mark)

Question 2

a. $\int \sin(3x-1)\,dx$

$= -\frac{1}{3}\cos(3x-1)$

(1 mark)

Note, because we are looking for 'an' antiderivative, the constant of antidifferentiation c does not have to be included because "an" antiderivative could be the case where $c = 0$.
Marks would not be lost though if c was included.

b. $\int_0^2 \frac{1}{2x+1}\,dx = \frac{1}{2}\int_0^2 \frac{2}{2x+1}\,dx$

$= \left[\frac{1}{2}\log_e|2x+1|\right]_0^2$ **(1 mark)**

$= \frac{1}{2}\log_e|4+1| - \frac{1}{2}\log_e|1|$

$= \frac{1}{2}\log_e(5)$ since $\log_e(1) = 0$

(1 mark)

Question 3

a. $f(x) = \log_e(x), \quad x > 0$

$f\left(\frac{1}{u}\right) = \log_e\left(\frac{1}{u}\right), \quad u > 0$

$= \log_e(u^{-1})$

$= -1\log_e(u)$

$= -f(u)$

as required.

(1 mark)

b. Given $2f(u) = f(2v) + f(3v)$ and $u, v \in R^+$,

$2\log_e(u) = \log_e(2v) + \log_e(3v)$ **(1 mark)**

$\log_e(u^2) = \log_e(6v^2)$

$u^2 = 6v^2$

So, $u = \pm\sqrt{6}v$ but $u, v \in R^+$

so $u = \sqrt{6}v$

(1 mark)

Question 4

a. $g: R \to R,\ g(x) = 3 + \sin\left(\frac{2x}{3}\right)$

$$\text{period} = \frac{2\pi}{n} \text{ where } n = \frac{2}{3}$$
$$= 2\pi \div \frac{2}{3}$$
$$= 2\pi \times \frac{3}{2}$$
$$= 3\pi$$

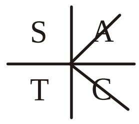
(1 mark)

The amplitude of the function is 1 and the graph of $y = \sin\left(\frac{2x}{3}\right)$ is translated 3 units up to become the graph of $y = 3 + \sin\left(\frac{2x}{3}\right)$.

The range of g is therefore $[2, 4]$.

(1 mark)

b. $\cos(3x) = \frac{\sqrt{3}}{2}, \quad x \in R$



$$3x = -\frac{\pi}{6} + 2n\pi, \quad \frac{\pi}{6} + 2n\pi, \quad n \in Z$$
$$3x = 2n\pi \pm \frac{\pi}{6}$$
$$x = \frac{2n\pi}{3} \pm \frac{\pi}{18}$$

(1 mark) – correct base angles
(1 mark) – correct use of $2n\pi$
(1 mark) – correct answer

Question 5

x	0	1	2	3
$\Pr(X = x)$	0.2	0.1	0.4	p

a. $0.2 + 0.1 + 0.4 + p = 1$

$p = 0.3$ **(1 mark)**

b. $\Pr(X \le 1 | X < 3) = \frac{0.2 + 0.1}{0.2 + 0.1 + 0.4}$ **(1 mark)**

$$= \frac{0.3}{0.7}$$
$$= \frac{3}{7}$$

(1 mark)

c. $E(X) = 0 \times 0.2 + 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.3$ **(1 mark)**

$$= 0.1 + 0.8 + 0.9$$
$$= 1.8$$

(1 mark)

Question 6

$f: R \to R, f(x) = 1 - e^{2(x-1)}$

Let $y = 1 - e^{2(x-1)}$

Swap x and y for inverse.

$x = 1 - e^{2(y-1)}$ **(1 mark)**

Rearrange

$$x - 1 = -e^{2(y-1)}$$

$$1 - x = e^{2(y-1)}$$

$$\log_e(1-x) = 2(y-1)$$

$$\frac{1}{2}\log_e(1-x) = y - 1$$

$$y = 1 + \frac{1}{2}\log_e(1-x)$$

So $f^{-1}(x) = 1 + \frac{1}{2}\log_e(1-x)$

(1 mark)

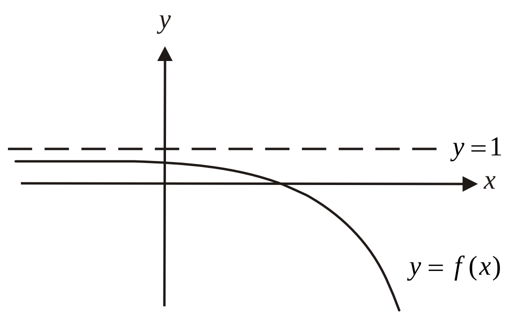
To find the domain of f^{-1},

Method 1

$d_{f^{-1}} = r_f$

Do a quick sketch of $y = f(x)$.

So $r_f = (-\infty, 1)$ and so $d_{f^{-1}} = (-\infty, 1)$



(1 mark)

Method 2

Since $f^{-1}(x) = 1 + \frac{1}{2}\log_e(1-x)$,

$1 - x > 0$ for the log function to be defined

$-x > -1$

$x < 1$

So $d_{f^{-1}} = (-\infty, 1)$

(1 mark)

Note that to find the inverse function f^{-1} you must give the domain and the rule.

Question 7

Method 1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

where (x', y') is the image point

$x' = 2x + 0y + 1$ $\qquad$ $y' = 0x + 4y - 3$

$\quad = 2x + 1$ $\qquad$ $= 4y - 3$

(1 mark)

Rearrange $\quad x' - 1 = 2x$ $\qquad$ $y' + 3 = 4y$

$x = \frac{x'-1}{2}$ $\qquad$ $y = \frac{y'+3}{4}$

Substitute into $y = \frac{1}{x}$

$$\frac{y'+3}{4} = 1 \div \frac{x'-1}{2}$$

$$\frac{y'+3}{4} = 1 \times \frac{2}{x'-1}$$

$$y'+3 = \frac{8}{x'-1}$$

$$y' = \frac{8}{x'-1} - 3$$

So the image equation is $y = \frac{8}{x-1} - 3$ **(1 mark)**

RE-READ THE QUESTION!

Since $y = \frac{a}{x+b} + c, \quad a = 8, \ b = -1$ and $c = -3$.

(1 mark)

Method 2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

We have a dilation by a factor 2 from the *y*-axis

so $y = \frac{1}{x}$ becomes $y = \frac{1}{x/2} = \frac{2}{x}$.

We have a dilation by a factor of 4 from the *x*-axis

so $y = \frac{2}{x}$ becomes $\frac{y}{4} = \frac{2}{x}$ so $y = \frac{8}{x}$. **(1 mark)**

We have a translation of 1 unit to the right and 3 units down

so $y = \frac{8}{x}$ becomes $y + 3 = \frac{8}{x-1}$ so $y = \frac{8}{x-1} - 3$

So the image equation is $y = \frac{8}{x-1} - 3$ **(1 mark)**

RE-READ THE QUESTION!

Since $y = \frac{a}{x+b} + c, \quad a = 8, \ b = -1$ and $c = -3$. **(1 mark)**

Question 8

$\Pr(A)=\frac{1}{3}, \quad \Pr(B)=\frac{1}{2}$

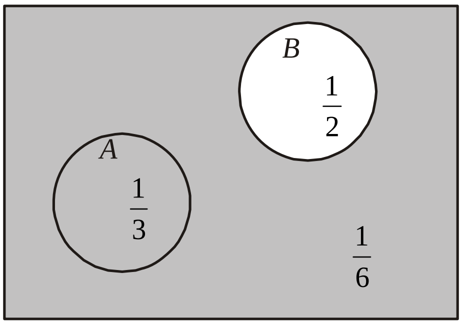
a. If A and B are mutually exclusive then $\Pr(A\cap B)=0$

Method 1 – Venn diagram

Note: $\frac{1}{3}+\frac{1}{2}$

$=\frac{2}{6}+\frac{3}{6}$

$=\frac{5}{6}$



$\Pr(A\cup B')$ is shaded.

(1 mark)

So $\Pr(A\cup B')=1-\frac{1}{2}=\frac{1}{2}$

(1 mark)

Method 2 – Addition formula

$\Pr(A\cup B')=\Pr(A)+\Pr(B')-\Pr(A\cap B')$ **(1 mark)**

$=\frac{1}{3}+\frac{1}{2}-\frac{1}{3}$

$=\frac{1}{2}$

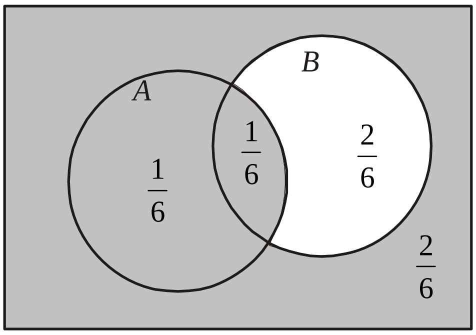
(1 mark)

b. If A and B are independent events then $\Pr(A)\times\Pr(B)=\Pr(A\cap B)$

Method 1 – Venn diagram

So $\Pr(A\cap B)=\frac{1}{3}\times\frac{1}{2}$

$=\frac{1}{6}$ **(1 mark)**



$\Pr(A\cup B')$ is shaded.

So $\Pr(A\cup B')=\frac{1}{6}+\frac{1}{6}+\frac{2}{6}$

$=\frac{2}{3}$

(1 mark)

Method 2 – Addition formula

$\Pr(A\cup B')=\Pr(A)+\Pr(B')-\Pr(A\cap B')$ **(1 mark)**

$=\frac{1}{3}+\frac{1}{2}-\frac{1}{6}$

$=\frac{2}{6}+\frac{3}{6}-\frac{1}{6}$

$=\frac{2}{3}$

(1 mark)

Question 9

Pr(June shops locally exactly once in the coming 3 weeks)

$= \Pr(L, M, M) + \Pr(M, L, M) + \Pr(M, M, L)$ **(1 mark)**

$= 0.4 \times 0.7 \times 0.6 + 0.6 \times 0.4 \times 0.7 + 0.6 \times 0.6 \times 0.4$ **(1 mark)**

$= 0.168 + 0.168 + 0.144$

$= 0.48$

(1 mark)

Question 10

$$y = x^{\frac{3}{5}} + c$$

$$\frac{dy}{dx} = \frac{3}{5}x^{\frac{-2}{5}}$$

At $x = 1$,

$$\frac{dy}{dx} = \frac{3}{5} \times 1$$

$$= \frac{3}{5}$$

The gradient of the tangent at $x = 1$ is $\frac{3}{5}$ so the gradient of the normal at $x = 1$ is $\frac{-5}{3}$.

(1 mark)

The normal passes through the point $(a, 0)$ which is it's x-intercept and through the point of normalcy which occurs at $x = 1$. At $x = 1$, $y = 1^{\frac{3}{5}} + c = 1 + c$.

So the point of normalcy is $(1, 1 + c)$.

The gradient of the normal is therefore given by

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 + c - 0}{1 - a}$$

$$= \frac{1 + c}{1 - a}$$

So $\frac{-5}{3} = \frac{1 + c}{1 - a}$ **(1 mark)**

$$-5(1 - a) = 3(1 + c)$$

$$-5 + 5a = 3 + 3c$$

$$5a = 8 + 3c$$

$$a = \frac{3c + 8}{5}$$

(1 mark)

Question 11

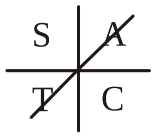
The graphs intersect when $2\sin(x) = 2\cos(x)$

$$\sin(x) = \cos(x)$$

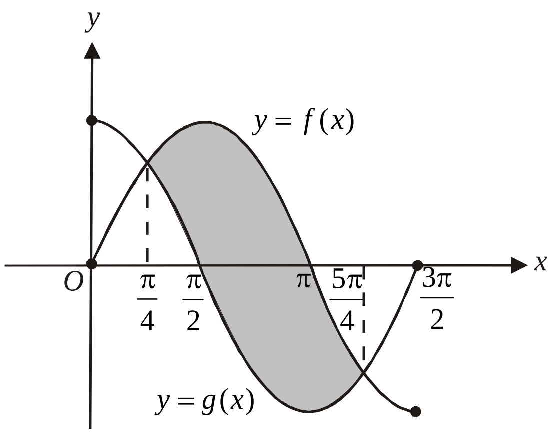
$$\frac{\sin(x)}{\cos(x)} = 1$$

$$\tan(x) = 1$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$



(1 mark)



shaded area $= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (f(x) - g(x))dx$

(1 mark) for correct terminals of integration
(1 mark) for correct integrand

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (2\sin(x) - 2\cos(x))dx$$

(1 mark)

$$= \left[-2\cos(x) - 2\sin(x)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -2\left\{\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\frac{\pi}{4}\right)\right\}$$

$$= -2\left\{\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right\}$$

$$= -2\left(\frac{-2}{\sqrt{2}} - \frac{2}{\sqrt{2}}\right)$$

$$= -2 \times \frac{-4}{\sqrt{2}}$$

$= 4\sqrt{2}$ square units

(1 mark)