The Mathematical Association of Victoria

Trial Exam 2012

MATHEMATICAL METHODS (CAS)

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 21 pages with a detachable sheet of miscellaneous formulas at the back
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your **name** in the space provided above on this page.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The equation of the normal to the curve with equation $y = x^2$ at $x = 2$ is

A. $4x - y = -4$
B. $x - 4y = 18$
C. $x + 4y = 18$
D. $4x + y = 12$
E. $x - 4y = 18$

Question 2

The equation $x^4 + ax^3 + 2x^2 = 0$, where a is a real constant, will have one unique real solution if

A. $a = -2\sqrt{2}$ or $a = 2\sqrt{2}$
B. $-2\sqrt{2} < a < 2\sqrt{2}$
C. $-2\sqrt{2} \le a \le 2\sqrt{2}$
D. $a < -2\sqrt{2}$ or $a > 2\sqrt{2}$
E. $a \le -2\sqrt{2}$ or $a \ge 2\sqrt{2}$

Question 3

If $f(x) = |\cos(x)|$ then

A. $f'(x) = \begin{cases} -\sin(x) \text{ when } \cos(x) > 0 \\ \text{undefined when } \cos(x) = 0 \\ \sin(x) \text{ when } \cos(x) < 0 \end{cases}$

B. $f'(x) = -\sin(x)$ for $x \in R$

C. $f'(x) = \begin{cases} -\sin(x) \text{ when } \cos(x) > 0 \\ 0 \text{ when } \cos(x) = 0 \\ \sin(x) \text{ when } \cos(x) < 0 \end{cases}$

D. $f'(x) = \sin(x)$ for $x \in R$

E. $f'(x) = \begin{cases} -\sin(x) \text{ when } \cos(x) < 0 \\ \text{undefined when } \cos(x) = 0 \\ \sin(x) \text{ when } \cos(x) > 0 \end{cases}$

SECTION 1 - continued

Question 4
For $x \in R$, there are no stationary points on the curve of *f* with equation

A. $f(x) = x^3 - 4x$

B. $f(x) = x^3 - 4x + 2$

C. $f(x) = (x-2)^3 - 4(x-2)$

D. $f(x) = x^4 + 4x$

E. $f(x) = (x-2)^3 + 4(x-2)$

Question 5
Given λ is a parameter, the solutions to $5y = 10x - 3$ and $20x - 10y = 6$ can be described by

A. $\left\{\left(\frac{5\lambda+3}{10}, \lambda\right) : \lambda \in Z\right\}$

B. $\left\{\left(\frac{5\lambda+3}{10}, \lambda\right) : \lambda \in R\right\}$

C. $\left\{\left(\lambda, \frac{5\lambda+3}{10}\right) : \lambda \in Z\right\}$

D. $\left\{\left(\lambda, \frac{5\lambda+3}{10}\right) : \lambda \in R\right\}$

E. $\left\{\left(\lambda, \frac{5\lambda+3}{10}\right) : \lambda \in R^+\right\}$

Question 6
The equation of the image of the curve $y = e^{2x+3}$ under the transformation described by the matrix $\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$ is

A. $y = \log_e\left|\frac{-x}{3}\right| - 3$

B. $y = -\frac{1}{3}\log_e(2x) + 1$

C. $y = -\frac{1}{3}e^{4x+3}$

D. $y = \log_e\left(\frac{-x}{3}\right) - 3,\ x < 0$

E. $y = \log_e\left(\frac{-x}{3}\right) - \frac{3}{2},\ x < 0$

SECTION 1 – continued
TURN OVER

Question 7

The graph of the inverse function of g where $g(x) = 1 + 3\log_e(1 - 2x)$ has

A. an asymptote with equation $x = 1$ and an x-axis intercept at 1

B. an asymptote with equation $y = \frac{1}{2}$ and an x-axis intercept at $\frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$

C. an asymptote with equation $x = \frac{1}{2}$ and an x-axis intercept at $\frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$

D. an asymptote with equation $y = \frac{1}{2}$ and a y-axis intercept at $\frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$

E. an asymptote with equation $y = 1$ and a y-axis intercept at 1

Question 8

If $f(x) = \sqrt{ax + b}$ and $g(x) = \sqrt{b - ax}$, where a and b are positive real constants then the maximal domain of the **derivative** of $f + g$ is

A. R

B. $\left[-\frac{b}{a}, \frac{b}{a}\right]$

C. $\left(-\frac{b}{a}, \frac{b}{a}\right)$

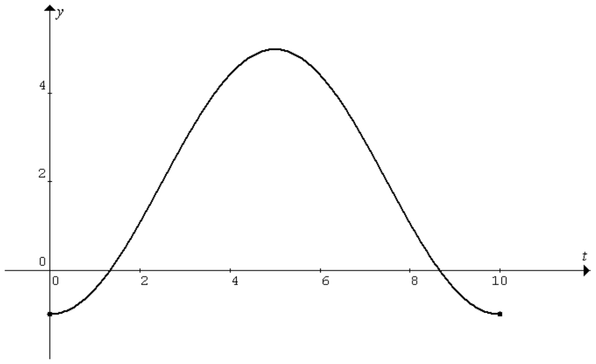
D. $\left[-\frac{a}{b}, \frac{a}{b}\right]$

E. $\left(-\frac{a}{b}, \frac{a}{b}\right)$

SECTION 1 - continued

Question 9

The graph of $a\cos(nt)+b$ is shown.



The values of *a*, *n* and *b* respectively, could be

A. 6, 10 and -1

B. -6, 5π and -1

C. 3, 10π and 2

D. -3, $\frac{\pi}{5}$ and 2

E. -3, 5π and 2

Question 10

Initially a tank contains 3000 litres of water. If water starts to leak out of the tank at a rate of $t^{\frac{2}{3}}$ litres per minute, the tank will be empty in

A. $50\times 10^{\frac{2}{5}}$ minutes

B. 164 317 minutes

C. $50\times 20^{\frac{2}{5}}$ minutes

D. 166 minutes

E. $30\,000\sqrt{30}$ minutes

SECTION 1 – continued

TURN OVER

Question 11

Using the Linear Approximation formula $f(x+h) \approx f(x) + hf'(x)$, where $f(x) = \frac{1}{\sqrt{x}}$, the approximate value of $\frac{1}{\sqrt{120.9}}$ can be found by evaluating

A. $\frac{1}{\sqrt{121}} + \frac{1}{20(121)^{\frac{3}{2}}}$

B. $\frac{1}{\sqrt{121}}$

C. $\frac{1}{\sqrt{121}} - \frac{1}{20(121)^{\frac{3}{2}}}$

D. $\frac{1}{\sqrt{121}} - \frac{1}{20(121)^{\frac{1}{2}}}$

E. $\frac{1}{\sqrt{120}} + \frac{1}{20(121)^{\frac{3}{2}}}$

Question 12

The **average** rate of change of the function with rule $f(t) = 2\tan(t)$ over the interval $\left[0, \frac{\pi}{3}\right]$ is

A. $\frac{6}{\pi}\log_e(2)$

B. $\frac{6\sqrt{3}}{\pi}$

C. $2\log_e(2)$

D. $-\frac{6\sqrt{3}}{\pi}$

E. $\frac{6}{\pi\sqrt{3}}$

SECTION 1 - continued

Question 13

The area enclosed by the graphs of *f* and *g* where $f(x)=\sin(x)$ and $g(x)=\dfrac{1}{x+2}$, over the domain $x\in[0,\pi]$, is best approximated by

A. $\int_{0.425}^{2.938}\left(\sin(x)-\dfrac{1}{x+2}\right)dx$

B. $\int_{0.425}^{2.938}\left(\dfrac{1}{x+2}-\sin(x)\right)dx$

C. $\int_{0}^{3.142}\left(\dfrac{1}{x+2}-\sin(x)\right)dx$

D. $\int_{0}^{3.142}\left(\sin(x)-\dfrac{1}{x+2}\right)dx$

E. $\int_{1.171}^{2.541}\left(\sin(x)-\dfrac{1}{x+2}\right)dx$

Question 14

If $\int_{1}^{5}\left(f(x)\right)dx=6$ then $2\int_{1}^{5}\left(f(x)+3\right)dx$ equals

A. 9
B. 15
C. 18
D. 24
E. 36

Question 15

The random variable, *X*, has the following probability distribution.

x	0	1	2	3	4
$\Pr(X=x)$	$\dfrac{p}{2}$	p	$3p$	p^2	$2p^2$

$E(X)$, the expected value of *X*, is closest to

A. 0.59
B. 1.59
C. 1.80
D. 2.00
E. 3.00

SECTION 1 – continued
TURN OVER

Question 16

The probability density function of the continuous random variable, *X*, is

$$f(x)=\begin{cases}\frac{3}{16}\left(4-x^2\right) & 0\le x\le 2\\ 0 & \text{elsewhere}\end{cases}.$$

If $\Pr(X>a)=0.4$, then the value of *a*, correct to four decimal places is

A. 0.2960
B. 0.4530
C. 0.5470
D. 0.7040
E. 0.8514

Question 17

In a particular population the probability a person has blue eyes is 0.36. A group of 8 people are selected from this population. It is known that less than 5 of the 8 have blue eyes. Correct to four decimal places, the probability that exactly 3 have blue eyes is

A. 0.2890
B. 0.3181
C. 0.4922
D. 0.5069
E. 0.5417

Question 18

The continuous random variable, *X*, has a normal distribution with a mean 10 and standard deviation 2. The value of *a* such that $\Pr(X>a)=0.7$, correct to three decimal places, is

A. 1.000
B. 7.244
C. 8.317
D. 8.951
E. 11.049

Question 19

A continuous random variable, *X*, has a normal distribution with a mean of 40 and standard deviation σ. Given $\Pr(X<55)=0.8413$, the value of σ is closest to

A. 1
B. 15
C. 16.7781
D. 95
E. 95.001

SECTION 1 - continued

Question 20

The life span of a particular laser light bulb is a continuous random variable, X, with a probability distribution function given by

$$f(x)=\begin{cases}\dfrac{1}{150\,000}x & 0\le x\le 500\\[2ex] \dfrac{1}{50}-\dfrac{1}{30\,000}x & 500<x\le 600\\[2ex] 0 & \text{elsewhere}\end{cases}.$$

The expected life span of a laser light bulb can be found by evaluating

A. $\dfrac{\displaystyle\int_0^{500}\left(\frac{x^2}{150\,000}\right)dx+\int_{500}^{600}\left(\frac{x}{50}-\frac{x^2}{30\,000}\right)dx}{2}$

B. $\displaystyle\int_0^{500}\left(\frac{x^2}{150\,000}\right)dx+\int_{500.001}^{600}\left(\frac{x}{50}-\frac{x^2}{30\,000}\right)dx$

C. $\displaystyle\int_0^{500}\left(\frac{x^2}{150\,000}\right)dx+\int_{500}^{600}\left(\frac{x}{50}-\frac{x}{30\,000}\right)dx$

D. $\displaystyle\int_0^{600}\left(f(x)\right)dx$

E. $\displaystyle-\int_{600}^{0}\left(xf(x)\right)dx$

Question 21

For two events, A and B, $\Pr(A\cap B)=p$, $\Pr(A'\cap B)=2p$ and $\Pr(A')=q$. $\Pr(A\cap B')$ is

A. $1-2p$

B. $\dfrac{1-q-p}{1-q}$

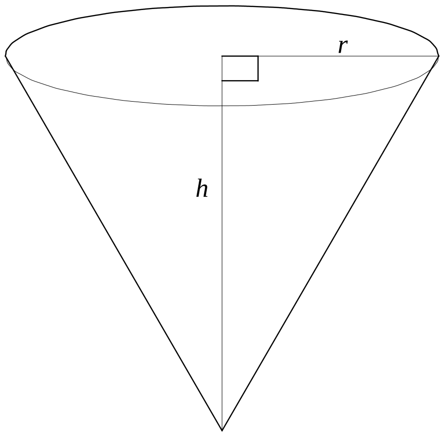
C. $1-p-q$

D. $\dfrac{p}{1-q}$

E. $q-2p$

SECTION 1 – continued
TURN OVER

Question 22
An inverted right circular cone, as shown below, has a radius equal to half its height.



The rate at which the volume changes with respect to the height is given by

A. $4\pi h^3$

B. $\dfrac{\pi h^3}{12}$

C. $\dfrac{3\pi h^2}{4}$

D. $\dfrac{\pi h^2}{4}$

E. $12\pi h^2$

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A channel is to be built as part of an irrigation system to bring water to a large agricultural area.
In the diagram, *PQRS* represents the cross-section of the channel.

PQ and *RS* are inclined at an angle of *x* radians to the base of the channel, *QR*, and $0 < x < \frac{\pi}{2}$.

The sum of the distances *PQ*, *QR* and *RS* is 20 metres. *h* metres is the vertical height of the channel where $0 < h < 10$.

a. **i.** Express the length, *PQ*, in terms of *h* and *x*.

ii. Hence, show $QR = 2\left(10 - \frac{h}{\sin(x)}\right)$.

1 + 1 = 2 marks

SECTION 2 – Question 1 – continued
TURN OVER

b. **i.** Find the length of *PS* in terms of h, $\sin(x)$ and $\tan(x)$.

ii. Hence show that the area, *A*, of the cross section *PQRS* is given by

$$A = 20h - \frac{2h^2}{\sin(x)} + \frac{h^2}{\tan(x)}.$$

iii. If $h = 5$ metres, write the area of the cross section *PQRS* in function notation.

2 + 2 + 1 = 5 marks

c. Show that the maximum area of this cross section occurs when $x = \frac{\pi}{3}$. You are not required to justify the nature of the stationary points.

2 marks

SECTION 2 – Question 1 – continued

d. **i.** Using the formula from **b. ii.**, find the maximum area of the cross section, in terms of *h*.

ii. Find the value of *h*, in metres, that will give the maximum cross sectional area. State the maximum cross sectional area in square metres. Give answers correct to one decimal place.

1 + 3 = 4 marks

Total 13 marks

SECTION 2 - continued
TURN OVER

Question 2

Hannah has found that the number of calls she receives on her mobile phone over a two hour period is a random variable, *X*. The probability distribution of *X* is given by the following formula.

$$\Pr(X = x) = \begin{cases} k & x = 0 \\ 0.3 \times (0.7)^x & x = \{1, 2, 3, 4, 5\} \\ 0 & \text{elsewhere} \end{cases}$$

a. **i.** Show that the value of *k*, correct to three decimal places, is 0.418.
(Use $k = 0.418$ for the remainder of the question.)

ii. Find $\Pr(X \le 3)$, correct to three decimal places.

2 + 2 = 4 marks

b. Find, correct to 2 decimal places
i. $\mathrm{E}(X)$**,** the expected value of *X*.

ii. $\mathrm{SD}(X)$, the standard deviation of *X*.

1 + 2 = 3 marks

SECTION 2 – Question 2 - continued

Hannah determines that the number of minutes, *Y*, she uses her mobile phone in a randomly chosen month is normally distributed with mean of 120 minutes and standard deviation of 7 minutes.

c. **i.** Find the probability that Hannah spends between 90 and 130 minutes using her mobile phone on any month, correct to four decimal places.

ii. The probability of Hannah spending more than *a* minutes on her mobile phone is 0.1 .Find the value of *a* correct to the nearest minute.

iii. Show that the probability Hannah spends more than 126 minutes on her mobile phone in any month, given that she has spent between 90 minutes and 130 minutes during that month is 0.1290 correct to four decimal places.

1 + 1 + 2 = 4 marks

SECTION 2 – Question 2 - continued
TURN OVER

Hannah rents her mobile phone with calls charged at \$1.25 per minute and a fixed charge of \$34 per month.

d. Given *C*, the monthly cost of the mobile phone, is a random variable with a normal distribution, find

i. $\text{E}(C)$, the mean of *C*.

ii. Find $\text{Var}(C)$, the variance of *C* and hence, $\text{SD}(C)$, the standard deviation of *C*.

iii. Calculate the 95% confidence interval for Hannah's monthly phone costs.

iv. Find the probability that in any two consecutive months the cost of using the mobile phone exceeds \$200. Give the answer correct to four decimal places.

1 + 2 + 2 + 2 = 7 marks

Total 18 marks

SECTION 2 - continued

Question 3

Consider $f:(-\infty, 2] \to R$, where $f(x) = (x-2)^2 + 1$.

a. Find f^{-1}.

3 marks

The area bounded by the curves of f and f^{-1} is $\frac{5\sqrt{5}-11}{3}$ units^2.

b. Write down the definite integral which when evaluated will give this area.

2 marks

Working Space

Consider the family of functions $f_k : (-\infty, a] \to R$, where $f_k(x) = (x-2)^2 + 1$ and a is a real constant and $k \in Z$.

c. For any one set of graphs of f_k and f_k^{-1}, how many possible solutions are there to $f_k = f_k^{-1}$?

2 marks

The area bounded by the curves f and f^{-1} between $x = 1$ and $x = \frac{5-\sqrt{5}}{2}$ forms the cross sectional area, in m^2, of a prism of height 2 m.

d. Find the volume of the prism in **m^3**.

1 mark

The prism is filled with water. Water starts to leak out of this prism at a rate of $2\log_e(t+1)$ **cm^3/min**.

e. How long will it take for the prism to empty? Give your answer to the nearest minute.

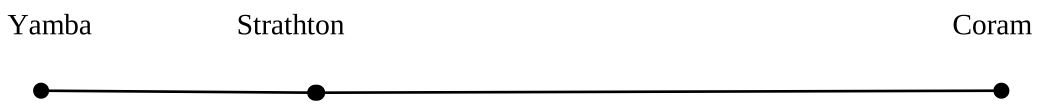
4 marks

Total 12 marks

SECTION 2 - continued

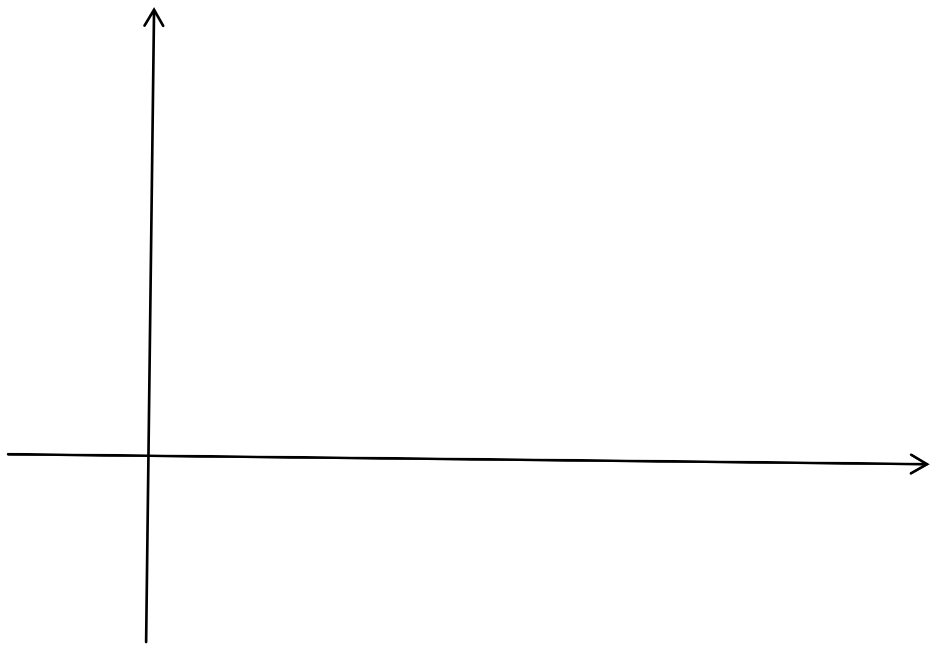
Question 4

To ascertain the suitability for a bike race, Tasmania Jones rode his bike along a straight road that joins the towns of Yamba, Strathton and Coram. He started at Strathton at 9.00 am and travelled towards Yamba but as the road surface was unsafe he turned around and rode back to Strathton and then to Coram and stopped.



The velocity, v km/h of the bike at time t hours is given by $v(t) = -\frac{14}{5}(t-4)^2 + 30$.

a. Sketch the graph of v for $t \in [0, \frac{5}{7}\sqrt{21} + 4]$ on the set of axes below. Label the axial intercepts and turning points with the exact values of their coordinates.



2 marks

b. Find Tasmania's position, x km, from Strathton in terms of t.

2 marks

SECTION 2 – Question 4 - continued
TURN OVER

Hence or otherwise answer the following questions.

c. **i.** How far did Tasmania ride towards Yamba before he turned around? Give your answer to the nearest metre.

ii. At what time did he get back to Strathton? Give your answer to the nearest minute.

iii. How far is Coram from Strathton? Give your answer in kilometres correct to the nearest kilometre.

iv. What was Tasmania's average velocity for when he was travelling directly from Strathton to Coram? Give your answer in km/h correct to one decimal place.

2 + 1 + 2 + 2 = 7 marks

Tasmania decides that the race is going to be from Strathton to Coram and that there should be two checkpoints along the route. The checkpoints will evenly divide the distance between Strathton and Coram.

.

d. At what times did Tasmania pass the location of the checkpoints on his initial ride? Give your answer to the nearest minute.

4 marks

Total 15 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Mathematical Methods (CAS)
Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc\sin A$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$\int \frac{1}{x}\,dx = \log_e|x| + c$

$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2\,p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

END OF FORMULA SHEET

MULTIPLE CHOICE ANSWER SHEET

Student Name: ....

Circle the letter that corresponds to each correct answer

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E