MATHEMATICAL MATHODS (CAS)

Unit 4
Targeted Evaluation Task for School-assessed Coursework 1

2012 Analysis Problems Task on Logs &Exponentials for

Outcomes 1, 2 & 3

SOLUTIONS & RESPONSE GUIDE

The marks given are allocated to the 3 outcomes according to the following:
A – Outcome 1, B – Outcome 2, C – Outcome 3

Question 1

a. $2^{x+1} = \left(2^3\right)^{3x} = 2^{9x}$

A1

Equating indices $\Rightarrow x+1=9x$

$x=\frac{1}{8}$ or $x=0.125$

A1

b. $e^{-5x}=12$

$x=\frac{\ln(12)}{-5}=-0.4970$

C1

Question 2

a. $\log_5 2x = \log_5 4^3 = \log_5 64$

$2x=64$

$x=32$

A1

b. $\text{LHS} = \log_7 x^2 + \log_7 x^{-6} - \log_7 3^4$

$=\log_7\left(\frac{x^2x^{-6}}{3^4}\right)$

A1

$=\log_7\left(\frac{1}{3^4x^4}\right)$

$=\log_7(3x)^{-4}$

$= -4\log_7 3x$ as required

B1

Question 3

a. Let $y=16(1-e^{-0.4x})$

Interchanging x and y

$x=16(1-e^{-0.4y})$

B1

$$\frac{x}{16} = 1 - e^{-0.4y}$$

$$e^{-0.4y} = 1 - \frac{x}{16}$$

$$y = \frac{1}{-0.4}\log\left(1 - \frac{x}{16}\right)$$

$$f^{-1}(x) = -2.5\log\left(1 - \frac{x}{16}\right)$$

A1

b. dom f^{-1} = ran f =[0, 16)
ran f^{-1} = dom f = [0, ∞)

B1

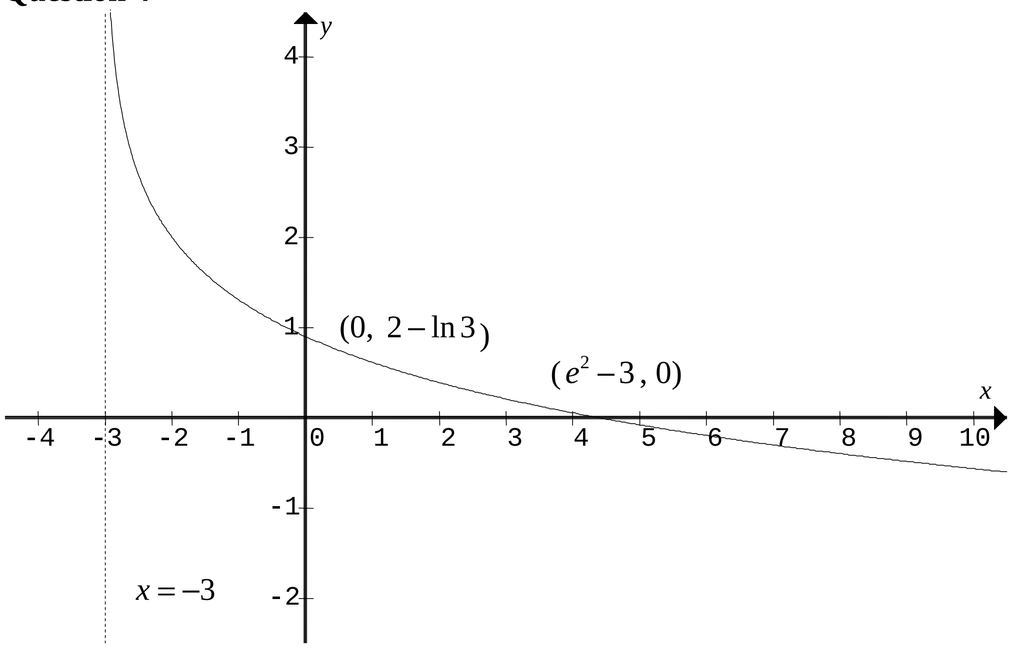
c. Graph both functions on a graphics calculator. Using the intersection function gives the intersection point as (15.973, 15.973)

C1

d. The time elapsed from when the parachutist jumped as a function of the speed of the parachutist.

B1

Question 4



Correct graph shape and asymptote A2
Correct intercepts B2

Question 5

At the stationary points $\frac{dy}{dx} = 0$

$\therefore e^x + 6e^{-x} - 5 = 0$

$e^{2x} + 6 - 5e^x = 0$

$(e^x)^2 - 5e^x + 6 = 0$

A1

Let $u = e^x$

$u^2 - 5u + 6 = 0$

$(u-2)(u-3) = 0$

$u = 2$ or $u = 3$

$e^x = 2$ or $e^x = 3$

$x = \ln 2$ or $x = \ln 3$

A1

When $x = \ln 2$

$y = 2 - \frac{6}{2} - 5\ln 2 = -1 - 5\ln 2$

When $x = \ln 3$

$y = 3 - \frac{6}{3} - 5\ln 3 = 1 - 5\ln 3$

Therefore coordinates of stationary points are
(ln2, -1-5 ln2) and (ln3, 1 -5ln3)

A1

Question 6

a. If half the original amount remains then $m = \frac{A}{2}$ at $t = 20$ as A is the original amount.

$0.5 = e^{-20k}$

$\ln(0.5) = -20k$

$k = -\frac{\ln(0.5)}{20} = 0.03466$

B2

b. Initial amount = $A = \frac{m}{e^{-0.03466t}}$

$A = \frac{9.48}{e^{-0.03466 \times 6.8}} = 11.9996 \approx 12$

Therefore initial amount is 12 g

B1

Question 7

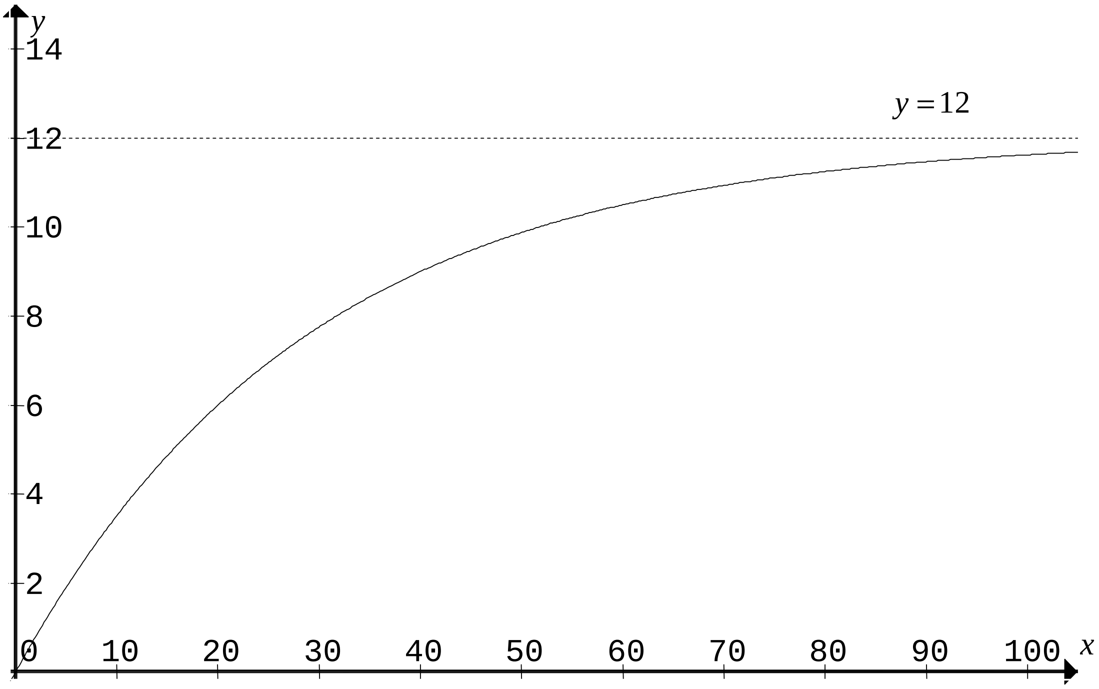
a. The mass of B will be the difference between the original mass of A and the amount of A remaining at any time *t*.

$$m_B = 12 - 12e^{-0.03466t}$$

$$m_B = 12(1 - e^{-0.03466t})$$

B1

b.



Note: Graph should pass through (20, 6)

Correct shape and asymptote A2

c. The graph would start off like the graph in part **b** but would then gradually drop below it while still increasing for some time. The graph would then reach a maximum value and start to decrease. For larger values of *t* the graph would be very close to a typical radioactive decay curve.

B2

Question 8

a. Substituting the given values into the equation gives:

$250 = P_0 e^{-10k}$ (1)

$15 = P_0 e^{-30k}$ (2)

$\frac{(1)}{(2)} \Rightarrow 16.6667 = e^{20k}$

$k = \frac{\ln(16.6667)}{20} = 0.14067 \approx 0.1407$ to four decimal places

B1, C1

b. $P_0 = \frac{250}{e^{-0.1407\times10}} = 1020.92$

$P_0 = 1021$ to the nearest millibar

A1, C1

c. If If $P = P_0 e^{-kh}$ then

$$h = -\frac{\ln\left(\frac{P}{P_0}\right)}{k} = \frac{\ln\left(\frac{65}{1021}\right)}{0.1407} = 19.5746 \text{ km}$$

therefore height is 19.575 km to the nearest metre

B1, C1

Question 9

Draw a graph of $y = 2xe^{-0.08x} + 0.0017x^2$ on a graphics calculator.

a. Use the maximum and minimum functions to find:
Maximum at (13.3218, 9.4797) and minimum at (43.9605, 5.8959)

C2

b. Either use the *x*-Cal function to find the values when *y* = 7.8 or find the intersections of the graph of $y = 2xe^{-0.08x} + 0.0017x^2$ with $y = 7.8$. Correct set of values is:
$(6.4990,\ 25.1578) \cup (64.4235,\ \infty)$.

Correct intersection points C2
Correct range of values C1

Question 10

a. Using the product and chain rules:

$f'(x) = 2(x-a)e^{-bx} - b(x-a)^2 e^{-bx}$

A1

And factorising gives:

$f'(x) = (x-a)[2-b(x-a)]e^{-bx}$

B1

b. Let $(x-a)[2-b(x-a)]e^{-bx} = 0$

$e^{-bx} \neq 0$ therefore

$(x-a) = 0$ or $2 - b(x-a) = 0$

$x = a$ or $x = \frac{2}{b} + a$

B1

Summary of mark allocation by Outcome

Question	Part	Outcome 1	Outcome 2	Outcome 3
1	a	2		
	b			1
2	a	1		
	b	1	1	
3	a	1	1	
	b		1	
	c			1
	d		1	
4		2	2	
5		3		
6	a		2	
	b		1	
7	a		1	
	b	2		
	c		2	
8	a		1	1
	b	1		1
	c		1	1
9	a			2
	b			3
10	a	1	1	
	b		1	
Raw marks		14	16	10
Adjusted marks		7	8	5