itute.com

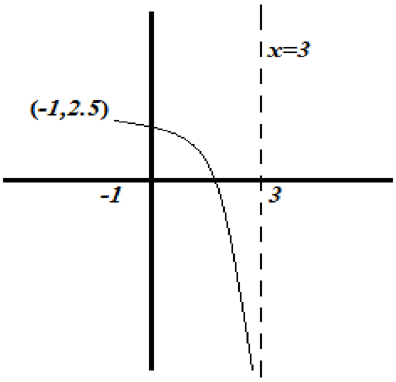
2013 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
A	D	D	C	C	C	D	D	D	A	A

12	13	14	15	16	17	18	19	20	21	22
B	B	E	E	B	E	E	A	E	E	C

Q1 $-1 \le x < 3$, $f(x) \le 2.5$, the only asymptote is $x = 3$. A



Q2 $\log_{\frac{1}{e}}\left(\sqrt[3]{e}\times 3\right) = \log_{\frac{1}{e}}\left(3e^{\frac{1}{3}}\right) = \dfrac{\log_e\left(3e^{\frac{1}{3}}\right)}{\log_e\left(\frac{1}{e}\right)} = \dfrac{\log_e 3 + \frac{1}{3}}{-1}$

$= -\log_e 3 - \dfrac{1}{3}$ D

Q3 $\sqrt{a-3x} + \log_b(3x) = \log_b a$, $\sqrt{a-3x} = \log_b a - \log_b(3x)$,

$\sqrt{a-3x} = \log_b\left(\dfrac{a}{3x}\right)$, both sides equal zero when $3x = a$, i.e.

$x = \dfrac{a}{3}$. D

Q4 *Range of* $f \subseteq$ *domain of* g, $\therefore$ $(0, 1-a] \subseteq (a, 1]$

$\therefore$ $a \le 0$ and $1 - a \le 1$, i.e. $a \le 0$ and $a \ge 0$

$\therefore$ $a = 0$ C

Q5 At $x = 2$, $e^{ax} = \dfrac{\log_e x}{a}$, $\therefore$ $e^{2a} = \dfrac{\log_e 2}{a}$, $ae^{2a} = \log_e 2$.

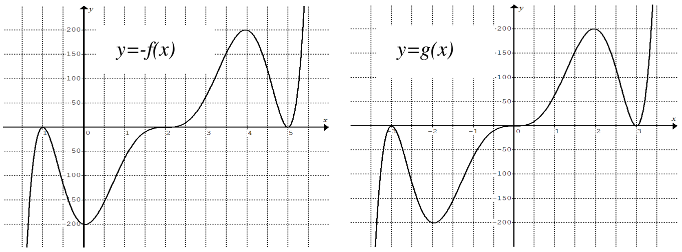
Since e^{2a} and $\log_e 2 \in R^+$, $\therefore$ $a > 0$.

By inspection $a \ne 1$, $a \ne \log_e 2$, $\therefore$ $a = \log_e \sqrt{2}$ C

Check:

$ae^{2a} = \left(\log_e \sqrt{2}\right)e^{2\log_e \sqrt{2}} = \left(\log_e \sqrt{2}\right)\left(e^{\log_e \sqrt{2}}\right)^2 = \left(\log_e \sqrt{2}\right)(2) = \log_e 2$

Q6 The graphs of $y = -f(x)$ and $y = g(x)$ are shown below.



$y = g(x)$ is the translation of $y = -f(x)$ to the left by 2 units, and $g(-x) = -g(x)$. $\therefore$ $a = -2$ C

Q7 Let (p, q) be the point of intersection.

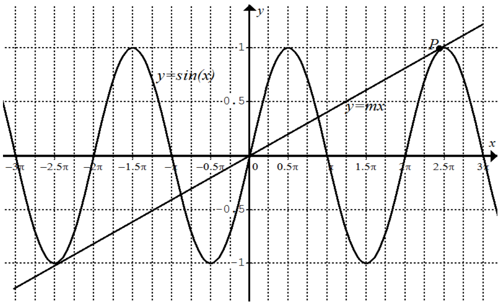
$\therefore$ $4ap + (a+b)q + 4b = 0$ independent of the values of a and b.

Rearrange the equation: $(4p+q)a + (q+4)b = 0$

$\therefore$ $4p + q = 0$ and $q + 4 = 0$

$\therefore$ $q = -4$ and $p = 1$ D

Q8 The following graphs of $y = \sin(x)$ and $y = mx$ intersect at exactly 5 points.



The coordinates of point P is $(x, \sin(x))$.

$\dfrac{dy}{dx} = m = \cos x$ at P.

$\therefore$ $m = \dfrac{\sin(x)}{x} = \cos(x)$. By CAS, $m = 0.12837455$ D

Q9 $y = f(1-x)$

$\rightarrow$ $y = f(1-(x-1))$, i.e. $y = f(2-x)$

$\rightarrow$ $y = f(2-(-x))$, i.e. $y = f(2+x)$ D

Q10 Even degree, positive coefficient of leading term, two x-intercepts, one of them a stationary point of inflection. A

Q11 $f(x) = |4a - 6x| = \sqrt{(4a-6x)^2} = \sqrt{4(2a-3x)^2}$

$= 2\sqrt{(2a-3x)^2}$ A

Q12 B

Q13 $g(x)=b-f(-x)$, $g'(x)=-f'(-x)\times -1=f'(-x)$
.: $g'(-a)=f'(a)=b$ B

Q14 $y=f(|x|)=\begin{cases} f(x), & x\ge 0 \\ f(-x), & x<0 \end{cases}$
$y=f(-x)$, $x<0$ is the reflection in the *y*-axis of $y=f(x)$, $x>0$, .: $y=f(|x|)$ does not have an inflection point. E

Q15 $\int_a^b f(x)dx=\log_e\left|\frac{b}{a}\right|$, .: $f(x)=\frac{1}{x}$ for $x\in R\setminus\{0\}$
.: $f(x)$ is discontinuous at $x=0$
.: $\int_{-1}^{2} f(x)dx$ is undefined. E

Q16 $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} g(x)dx=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\left(|x-\pi|+\sin x-\frac{\pi}{2}\right)dx\approx -2.19$ by CAS

$$\textit{Average value}=\frac{\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} g(x)dx}{\frac{4\pi}{3}-\frac{2\pi}{3}}\approx\frac{-2.19}{\frac{2\pi}{3}}\approx -1.04$$ B

Q17 $f(\theta)=\cos^2\theta+2\sin^3\theta-1=(\cos^2\theta-1)+2\sin\theta\sin^2\theta$
$=(\cos^2\theta-1)-2\sin\theta(\cos^2\theta-1)$
$=(\cos^2\theta-1)(1-2\sin\theta)$
$=(\cos\theta-1)(\cos\theta+1)(1-2\sin\theta)$ E

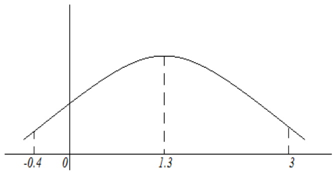
Q18 E

Q19 $\Pr(ABABB)=1\times b\times a\times b\times(1-a)=b^2a(1-a)$ A

Q20 Binomial:
${}^{10}C_5p^5(1-p)^5=0.1$, ${}^{10}C_6p^6(1-p)^4=0.2$
$\frac{{}^{10}C_6p^6(1-p)^4}{{}^{10}C_5p^5(1-p)^5}=\frac{0.2}{0.1}$, $\frac{p}{1-p}=\frac{12}{5}$, $p=\frac{12}{17}\approx 0.7$ E

Q21 $\Pr(A)-\Pr(A\mid B')\Pr(B')=\Pr(A)-\Pr(A\cap B')$
$=\Pr(A\cap B)=\Pr(B)-\Pr(B\cap A')$
$=\Pr(B)-\Pr(B\mid A')\Pr(A')$ E

Q22 -0.4 and 3 are the same distance from 1.3.
Given $\Pr(X<3)=0.85$, .: $\Pr(X>3)=0.15$ and $\Pr(X<-0.4)=0.15$
.: $\Pr(-0.4<X<3)=0.70$ C



SECTION 2

Q1a $(3,0)$ and $(0,p)$ are the *x* and *y* intercepts respectively.
$\frac{x}{3}+\frac{y}{p}=1$, .: $y=-\frac{p}{3}(x-3)$

Q1b $y=(x-3)(x^2+bx+c)$
$(0,-3)$, $-3=(-3)(c)$, .: $c=1$
$(-1,-4)$, $-4=(-4)(1-b+1)$, .: $b=1$

Q1ci Equation of the cubic function is $y=(x-3)(x^2+x+1)$.
Solve the above equation and $y=-\frac{p}{3}(x-3)$ simultaneously.
$(x-3)(x^2+x+1)=-\frac{p}{3}(x-3)$, $(x-3)(x^2+x+1)+\frac{p}{3}(x-3)=0$
$(x-3)\left(x^2+x+1+\frac{p}{3}\right)=0$
$x-3=0$, .: $x=3$ is the *x*-coordinate of one of the intersections.
The other intersection(s) comes from $x^2+x+1+\frac{p}{3}=0$.
One solution only: $b^2-4ac=1^2-4(1)\left(1+\frac{p}{3}\right)=0$, .: $p=-\frac{9}{4}$

Q1cii $x^2+x+1+\frac{1}{3}\times\frac{-9}{4}=0$, $x^2+x+\frac{1}{4}=0$, $\left(x+\frac{1}{2}\right)^2=0$,
.: $x=-\frac{1}{2}$, $y=-\frac{p}{3}(x-3)=-\frac{1}{3}\times\frac{-9}{4}\left(-\frac{1}{2}-3\right)=-\frac{21}{8}$,
$\left(-\frac{1}{2},-\frac{21}{8}\right)$

Q1d $y=(x-3)(x^2+x+1)=x^3-2x^2-2x-3$
$\frac{dy}{dx}=3x^2-4x-2=-\frac{p}{3}$ where $p=-\frac{9}{4}$
.: $3x^2-4x-2=\frac{3}{4}$, $12x^2-16x-11=0$, $(6x-11)(2x+1)=0$
.: $x=\frac{11}{6}$ and $y=-\frac{1561}{216}$, the point is $\left(\frac{11}{6},-\frac{1561}{216}\right)$.

Q1ei $-3x+18=-x^3+4x^2+8x+24$, $x=-1$ or 6 by CAS

Q1eii $-\frac{1}{2}\to -1$ and $3\to 6$, the factor of dilation from the *y*-axis is 2.

Q1fi and ii $\int_{-1}^{6}\left(-x^3+4x^2+8x+24-(-3x+18)\right)dx$
$=\int_{-1}^{6}\left(-x^3+4x^2+11x+6\right)dx=\frac{2401}{12}$

Q1g Original line: *y*-intercept is $\left(0,-\frac{9}{4}\right)$; new line: *y*-intercept is $(0,18)$. The factor of dilation from the *x*-axis is $\frac{18}{\frac{9}{4}}=8$.

Q2a

$a\log_e(3+b)+c=1$.................(1)

$a\log_e(2+b)+c=0$.................(2)

$a\log_e(1.5+b)+c=-1$............(3)

Q2b

(1) – (2): $a\log_e\frac{3+b}{2+b}=1$ …………(4)

(2) – (3): $a\log_e\frac{2+b}{1.5+b}=1$ …………(5)

$\therefore\ \log_e\frac{3+b}{2+b}=\log_e\frac{2+b}{1.5+b}$, $\therefore\ \frac{3+b}{2+b}=\frac{2+b}{1.5+b}$, $\therefore\ b=-1$

Substitute $b=-1$ in (2): $c=0$

Substitute $b=-1$ and $c=0$ in (1): $a=\frac{1}{\log_e 2}$

Q2c Wall A: $y=\frac{1}{\log_e 2}\log_e(x-1)$

Reflection of $y=\frac{1}{\log_e 2}\log_e(x-1)$ in the line $y=x$ is

$x=\frac{1}{\log_e 2}\log_e(y-1)$, $\therefore\ (\log_e 2)x=\log_e(y-1)$, $y-1=\left(e^{\log_e 2}\right)^x$,

$\therefore\ y=2^x+1$

Q2d NE direction: $m=1$

$y=\frac{1}{\log_e 2}\log_e(x-1)$, $\frac{dy}{dx}=\frac{1}{(\log_e 2)(x-1)}=1$, $\therefore\ x=\frac{1}{\log_e 2}+1$

$\therefore\ y=\frac{1}{\log_e 2}\log_e\left(\frac{1}{\log_e 2}\right)=-\frac{\log_e(\log_e 2)}{\log_e 2}$

$\left(\frac{1}{\log_e 2}+1,\ -\frac{\log_e(\log_e 2)}{\log_e 2}\right)$

Q2e The point on Wall B with a NE tangent is

$\left(-\frac{\log_e(\log_e 2)}{\log_e 2},\ \frac{1}{\log_e 2}+1\right)$.

The shortest distance is the distance between

$\left(\frac{1}{\log_e 2}+1,\ -\frac{\log_e(\log_e 2)}{\log_e 2}\right)$ and $\left(-\frac{\log_e(\log_e 2)}{\log_e 2},\ \frac{1}{\log_e 2}+1\right)$.

The shortest distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\approx 2.71\,\text{m}$

Q2f *Area* $=3\times 3-2\int_2^3\frac{1}{\log_e 2}\log_e(x-1)dx$

$\approx 9-2\times 0.5573\approx 7.89\,\text{m}^2$ by CAS

Q3a When the depth is h metres, $V=\pi r^2h=\pi^3h$, $\frac{dV}{dh}=\pi^3$.

Related rates: $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $\therefore\ \frac{dV}{dt}=\pi^3\times\frac{dh}{dt}$,

$\frac{dh}{dt}=\frac{1}{\pi^3}\left(\sin\frac{2t}{\pi}+\cos\frac{2t}{\pi}+\pi\right)$

Q3b $T=\frac{2\pi}{n}=\frac{2\pi}{\frac{2}{\pi}}=\pi^2$ minutes

Q3ci *Average over n periods* $=\frac{\int_0^{n\pi^2}\left(\sin\frac{2t}{\pi}+\cos\frac{2t}{\pi}+\pi\right)dt}{n\pi^2}$

$=\frac{\int_0^{n\pi^2}(\pi)dt}{n\pi^2}=\frac{[\pi t]_0^{n\pi^2}}{n\pi^2}=\frac{n\pi^3}{n\pi^2}=\pi\,\text{m}^3\ \text{min}^{-1}$

Q3cii *Volume of the tank* $=4\pi^3\,\text{m}^3$, *time* $=\frac{4\pi^3}{\pi}=4\pi^2$ min,

number of periods $=\frac{4\pi^2}{\pi^2}=4$

Q3d *Volume to be filled* $=\pi r^2h=\pi(3^2)(4)=36\pi\,\text{m}^3$

Let τ minutes be the time to fill the second tank.

$\frac{dV}{dt}=\sin\frac{2t}{\pi}+\cos\frac{2t}{\pi}+\pi$

$V=\int_0^{\tau}\left(\sin\frac{2t}{\pi}+\cos\frac{2t}{\pi}+\pi\right)dt=36\pi$

$\tau\approx 35.3$ minutes by CAS

Q3e Find the local maxima and minima of

$\frac{dh}{dt}=\frac{1}{\pi^3}\left(\sin\frac{2t}{\pi}+\cos\frac{2t}{\pi}+\pi\right)$, let $\frac{d}{dt}\left(\frac{dh}{dt}\right)=0$

$\therefore\ \frac{2}{\pi^4}\left(\cos\frac{2t}{\pi}-\sin\frac{2t}{\pi}\right)=0$, $\sin\frac{2t}{\pi}=\cos\frac{2t}{\pi}$, $\tan\frac{2t}{\pi}=1$ and

$0<t<35.3$, i.e. $0<\frac{2t}{\pi}<22.4727$

$\therefore\ \frac{2t}{\pi}=\frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \frac{17\pi}{4},\ \frac{21\pi}{4},\ \frac{25\pi}{4}$

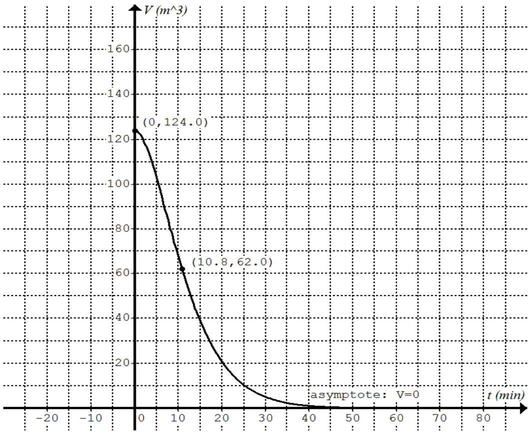
$\therefore\ t=\frac{\pi^2}{8},\ \frac{5\pi^2}{8},\ \frac{9\pi^2}{8},\ \frac{13\pi^2}{8},\ \frac{17\pi^2}{8},\ \frac{21\pi^2}{8},\ \frac{25\pi^2}{8}$

The greatest rate of increase occurs when

$t=\frac{\pi^2}{8},\ \frac{9\pi^2}{8},\ \frac{17\pi^2}{8},\ \frac{25\pi^2}{8}$

Q3fi $9.164(t+10)^2e^{-(0.2t+2)}=\frac{1}{2}\times 4\pi^3$, $t\approx 10.8$ min by CAS

Q3fii



Q4a $\Pr(A>5)$, by CAS $normalcdf(5,e\text{^}99,8.53,5.23)\approx 0.75$

Q4b $\Pr(7.00-5.23<A<7.00+5.23)=\Pr(1.77<A<12.23)$
≈ 0.66

Q4ci $\Pr(X=7)$, by CAS $binompdf(10,0.75,7)\approx 0.25$

Q4cii $E(X)=np=10\times 0.75\approx 8$

Q4d $\Pr(B>5)=0.6615$ and $\Pr(B<6)=0.4013$

.: $\Pr\left(Z>\frac{5-\mu}{\sigma}\right)=0.6615$, i.e. $\Pr\left(Z<\frac{5-\mu}{\sigma}\right)=0.3385$ and

$\Pr\left(Z<\frac{6-\mu}{\sigma}\right)=0.4013$

By CAS, invNorm, $\frac{5-\mu}{\sigma}\approx -0.41656$ and $\frac{6-\mu}{\sigma}\approx -0.24998$

.: $\mu=7.50$ and $\sigma=6.00$ minutes

Q4e $E(B)=\mu=7.50$

Q4f $\Pr(B<6\,|\,B>5)=\frac{\Pr(B<6\cap B>5)}{\Pr(B>5)}=\frac{\Pr(5<B<6)}{\Pr(B>5)}$

$\approx\frac{0.06283}{0.66154}\approx 0.095$

Q4g $A\to A$, $\frac{2}{5}$; $A\to B$, $\frac{3}{5}$; $B\to B$, $\frac{1}{4}$; $B\to A$, $\frac{3}{4}$

.: she shopped at Bestbuy last time.

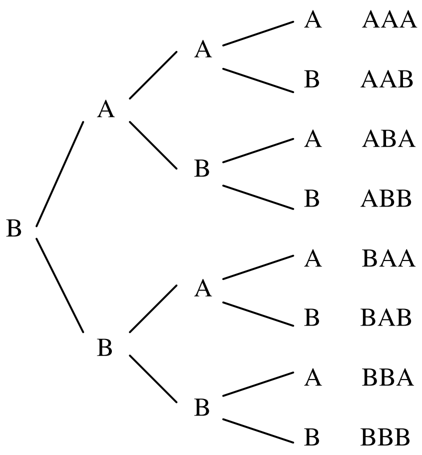
Q4h $\Pr(AB\cup BA)=\Pr(AB)+\Pr(BA)=\frac{3}{4}\times\frac{3}{5}+\frac{1}{4}\times\frac{3}{4}=\frac{51}{80}$

Q4i *Transition matrix* $=\begin{array}{cc} A & B \\ \left[\begin{array}{cc} \frac{2}{5} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{4} \end{array}\right] & \begin{array}{c} A \\ B \end{array} \end{array}$

In the long run, probability of shopping at Bestbuy $=\frac{\frac{3}{5}}{\frac{3}{4}+\frac{3}{5}}=\frac{4}{9}$

Q4j

210th 211^{th} 212^{th} 213^{th}



Let X be the number of times that Sofia goes to Bestbuy in her next three shopping trips.

$\Pr(AAA)=\frac{3}{4}\times\frac{2}{5}\times\frac{2}{5}$, $\Pr(AAB)=\frac{3}{4}\times\frac{2}{5}\times\frac{3}{5}$,
$\Pr(ABA)=\frac{3}{4}\times\frac{3}{5}\times\frac{3}{4}$, $\Pr(ABB)=\frac{3}{4}\times\frac{3}{5}\times\frac{1}{4}$
$\Pr(BAA)=\frac{1}{4}\times\frac{3}{4}\times\frac{2}{5}$, $\Pr(BAB)=\frac{1}{4}\times\frac{3}{4}\times\frac{3}{5}$
$\Pr(BBA)=\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}$, $\Pr(BBB)=\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}$

x	0	1	2	3
$\Pr(X=x)$	0.12	0.5925	0.271875	0.015625

$E(X)=0\times 0.12+1\times 0.5925+2\times 0.271875+3\times 0.015625\approx 1.18$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors