The Mathematical Association of Victoria

SOLUTIONS: Trial Exam 2013
MATHEMATICAL METHODS
Written Examination 2

SECTION 1: Multiple Choice

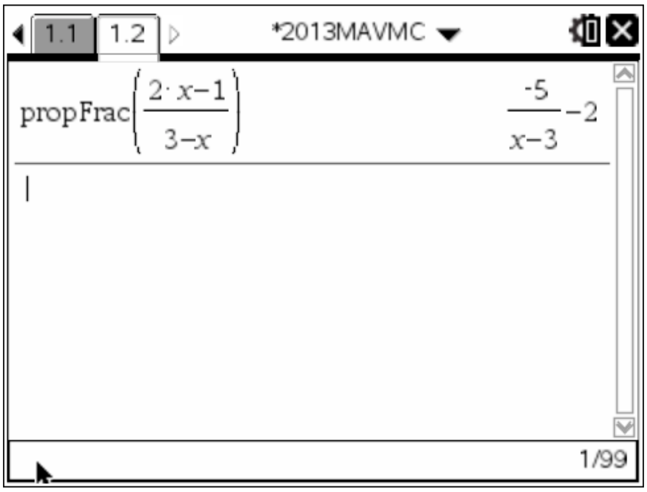
1. D	2. C	3. B	4. B	5. A	6. D	7. C	8. E	9. B	10. A	11. D
12. B	13. D	14. E	15. D	16. E	17. C	18. A	19. E	20. A	21. D	22. C

Question 1

$f(x) = \frac{2x-1}{3-x} = \frac{-5}{x-3} - 2$

The maximal domain is $R \setminus \{3\}$.

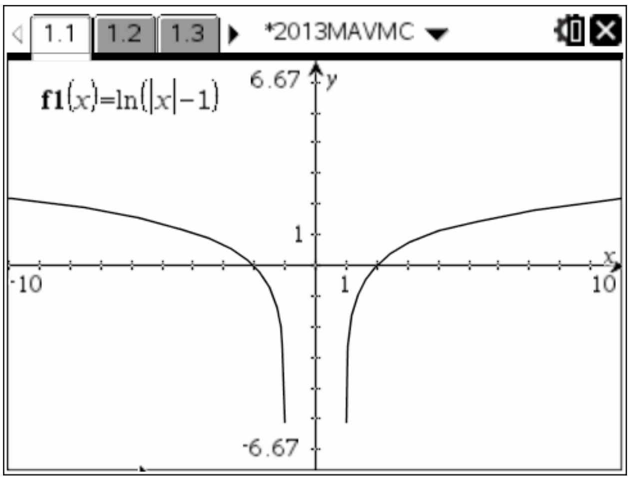
The range is $R \setminus \{-2\}$. **D**



Question 2

The equations of the asymptotes are

$x = -1$ and $x = 1$ **C**



Question 3

$f: R^{+} \rightarrow R, f(x) = \log_e(x)$, and $g: \left(\frac{1}{2}, \infty\right) \rightarrow R,\ g(x) = (2x-1)^2$

$f\left(g(x)\right) = \log_e(2x-1)^2$

$f\left(g(x)\right) = 2\log_e\left(2(x-\frac{1}{2})\right)$

Dilation by a factor of 2 from the x-axis, dilation by a factor of a $\frac{1}{2}$ from the y-axis and a translation of a $\frac{1}{2}$ of a unit to the right. **B**

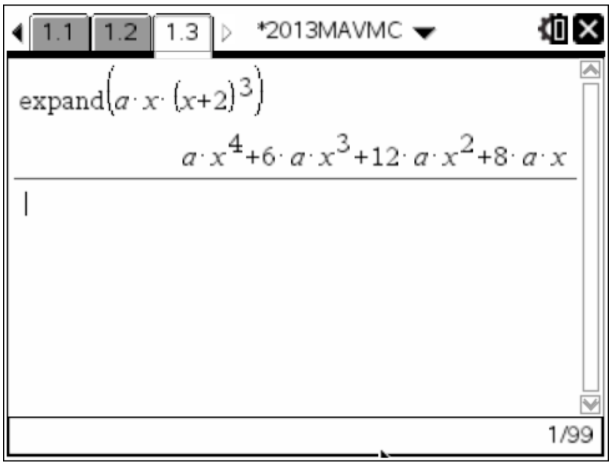
Question 4

$f(x) = Ax(x+2)^3 = Ax^4 + 6Ax^3 + 12Ax^2 + 8Ax$

$f(x) = ax^4 + bx^3 - 24x^2 + cx$

$12A = -24,\ A = -2$

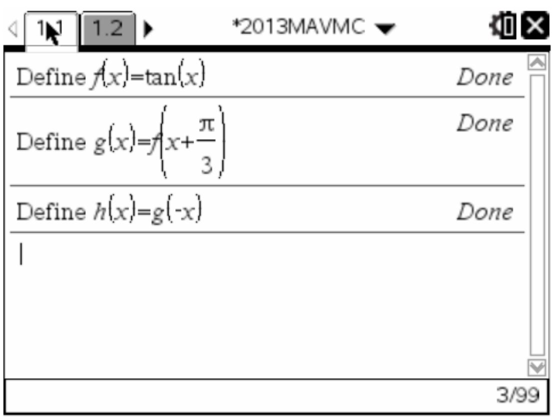
$b = 6A = -12$ **B**

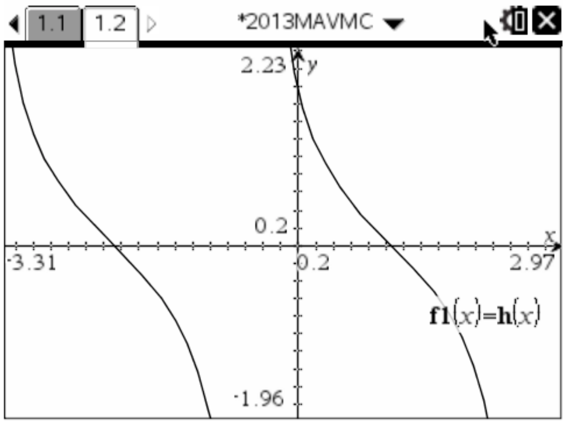


Question 5

Translation of $\frac{\pi}{3}$ in the negative direction of the x axis $\rightarrow y = \tan\left(x + \frac{\pi}{3}\right)$,

Refection in the y axis $\rightarrow y = \tan\left(-x + \frac{\pi}{3}\right)$ **A**



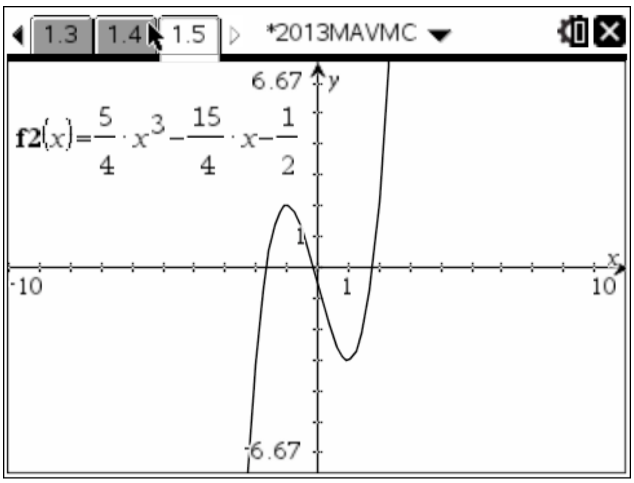


Question 6
Amplitude of 2
Translation of c in the negative direction from the x axis
Range is $[-2-c, 2-c] = [-(2+c), (2-c)]$ **D**

Question 7
$f(x+h) \approx f(x) + hf'(x)$
$f(x+h) - f(x) \approx hf'(x)$
$h = 0.1,\ f(x) = \sqrt[3]{x},\ f'(x) = \dfrac{1}{3x^{\frac{2}{3}}},\ f'(8) = \dfrac{1}{3 \times 8^{\frac{2}{3}}} = \dfrac{1}{12}$
$hf'(x) = 0.1 \times \dfrac{1}{12} = \dfrac{1}{120}$ **C**

Question 8
$f(x) + c = 0$
Sketch a possible graph for f.
For one solution f needs to be translated down more than 2 units or translated up more than 3 units.

$\{c : c < -2\} \cup \{c : c > 3\}$ **E**

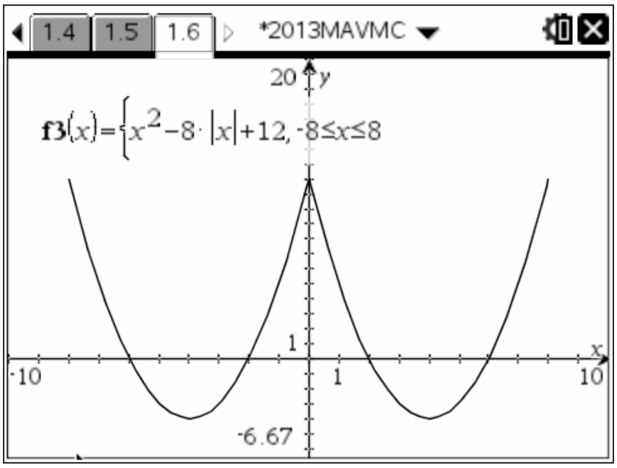


Question 9

$g:[-8,8] \to R$, where $g(x) = x^2 - 8|x| + 12$

The graph is not differentiable at the endpoints or the sharp point.

$x = -8,\ x = 0,\ x = 8$ **B**



Question 10

$g(x) = e^{-x}$

Area = $0.5\big(g(0) + g(0.5) + g(1) + g(1.5)\big)$

$= 0.5\left(1 + \frac{1}{e^{0.5}} + \frac{1}{e} + \frac{1}{e^{1.5}}\right)$ **A**

Question 11

$f:[a,b] \to R$, where $f(x) = x - 1$

The average value will be zero if a and b are equally spaced either side of $x = 1$ as the area above the line $y = 0$ will equal the area below the line $y = 0$.

$[-2,2]$ **D**

Question 12

$\int_1^3 (f(x))dx = 5$

$2\int_1^3 (f(x) - 1)dx = 2\int_1^3 (f(x)) - 2[x]_1^3$

$= 10 - (6 - 2) = 6$ **B**

Question 13

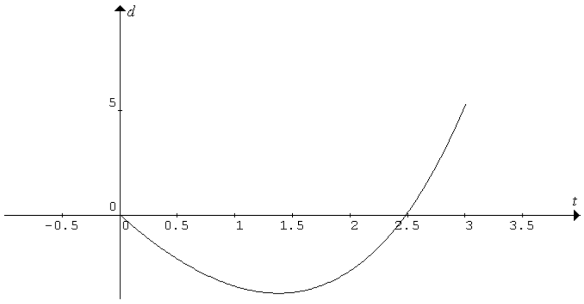
$a = t^2 + 3$

$v = \frac{t^3}{3} + 3t + c$

$-5 = \frac{0}{3} + 3 \times 0 + c$

$v = \frac{t^3}{3} + 3t - 5$

$d = \frac{t^4}{12} + \frac{3t^2}{2} - 5t + c_1$



D

Option A is acceleration against time
Option B is velocity against time
Option C is $y = 2t - 5$
Option E is $y = 2$

Question 14

$$SD(X) = \sqrt{E(X^2) - [E(X)]^2}$$
$$= \sqrt{1.44 - a^2}$$

E

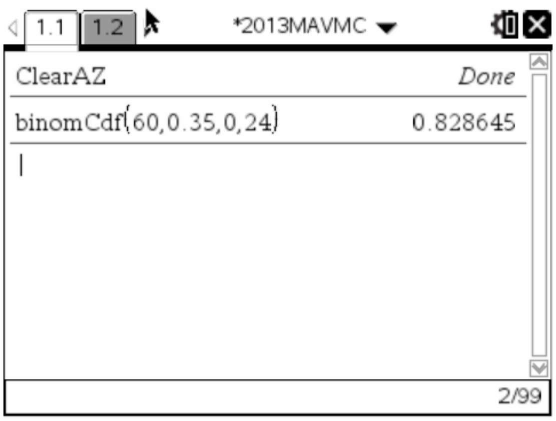
Question 15

Let X be the number who play a musical instrument out of 60

$X \sim \text{Bi}(60, 0.35)$

$\Pr(X < 25) = 0.8286$ correct to four decimal places

D



Question 16

$$\begin{bmatrix} 0.75 & 0.55 \\ 0.25 & 0.45 \end{bmatrix}^2 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.657 \\ 0.343 \end{bmatrix}$$

0.657 **E**

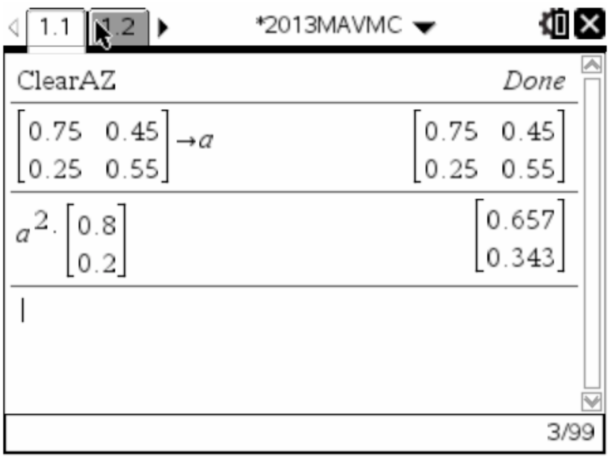
OR

$w \times w \times w + w \times w' \times w + w' \times w \times w + w' \times w' \times w$

$0.8 \times 0.75^2 + 0.8 \times 0.25 \times 0.45 + 0.2 \times 0.45 \times 0.75 + 0.2 \times 0.55 \times 0.45$ **E**

$= 0.45 + 0.09 + 0.0675 + 0.0495$

$= 0.657$



Question 17

$\int_2^6 f(t)dt = -\int_6^2 f(t)dt$ **C**

Question 18

Gradient is $a \rightarrow a = \dfrac{2a}{2}$

$\text{Area} = \dfrac{1}{2} \times 2 \times 2a$

$= 2a$

$2a = 1$

$a = \dfrac{1}{2}$ **A**

OR

$a\int_1^3 (x-1)\,dx = 1$ Solve on the CAS or

$a\left[\dfrac{x^2}{2} - x\right]_1^3 = 1$

$a\left\{\left[\dfrac{9}{2} - 3\right] - \left[\dfrac{1}{2} - 1\right]\right\} = 1$

$a\{4 - 2\} = 1$

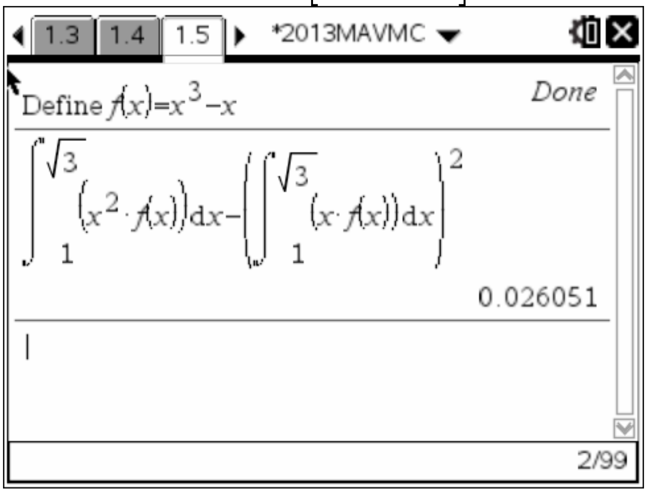
$2a = 1$

$a = \dfrac{1}{2}$ **A**

Question 19

Define $f(x) = x^3 - x$

$$\text{Var}(X) = \int_1^{\sqrt{3}} x^2 f(x)dx - \left[\int_1^{\sqrt{3}} xf(x)dx\right]^2$$

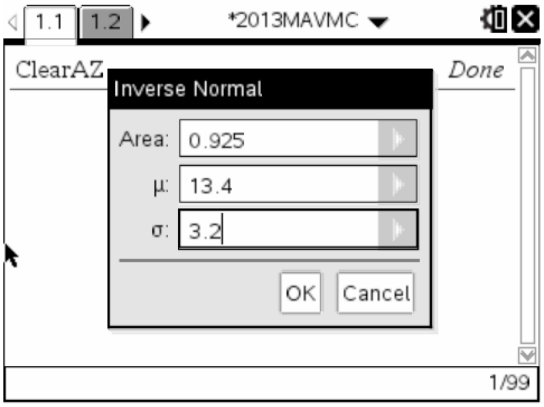


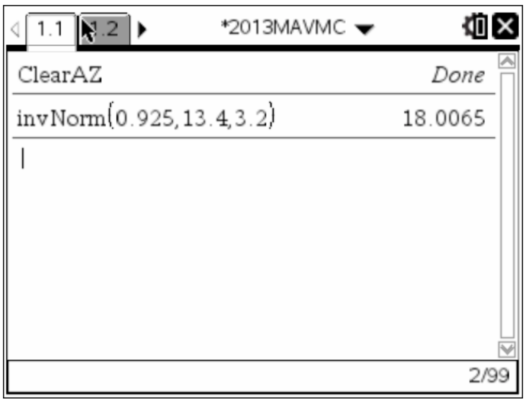
0.0261 correct to four decimal places **D**

Question 20

$\Pr\left(X < a_2\right) = 0.925$

$\Pr\left(Z < \dfrac{a_2 - 13.4}{3.2}\right) = 0.925$

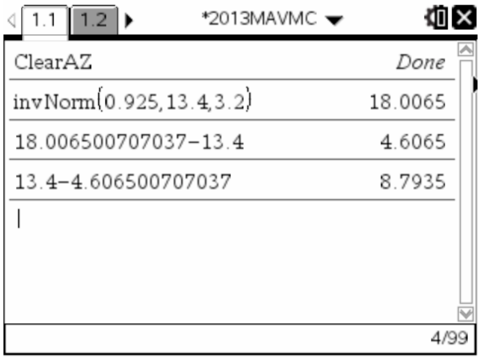




$18.0065 - 13.4 = 4.6065$

$13.4 - 4.6065 = 8.7935$

$a_1 = 8.79; a_2 = 18.01$ **A**



Question 21

$1-\left[\Pr(X=0)+\Pr(X=1)\right]$

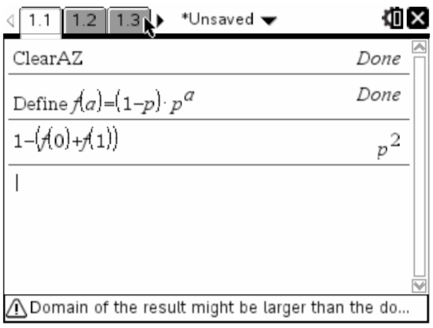
$1-\left[(1-p)\times 1+(1-p)\times p\right]$

$1-\left[(1-p)(1+p)\right]$

$1-\left[1-p^2\right]$

p^2 **D**

OR



Question 22

$$\frac{dV}{dt} = -750\text{cm}^3 / \text{min}$$

$$h = 3r$$

$$r = \frac{h}{3}$$

$$V = \frac{1}{3}\pi r^3 h$$

$$= \frac{1}{3}\pi \frac{h^3}{9}$$

$$= \frac{\pi h^3}{27}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{9}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{9}{\pi h^2} \times -750$$

C

$$= -\frac{6750}{\pi h^2}$$

END OF SECTION 1 SOLUTIONS

SECTION 2: Extended Answer Solutions

Question 1

a. TSA = $\pi rs + \pi r^2$

Curved surface area = πrs = 100

$s = \sqrt{r^2 + h^2}$ **1M**

$\pi r\sqrt{r^2 + h^2} = 100$ **1M**

$\sqrt{r^2 + h^2} = \frac{100}{\pi r}$

$r^2 + h^2 = \frac{10\ 000}{\pi^2 r^2}$

$h^2 = \frac{10\ 000}{\pi^2 r^2} - r^2$

$h^2 = \frac{10\ 000 - \pi^2 r^4}{\pi^2 r^2}$

$h = \frac{\sqrt{10\ 000 - \pi^2 r^4}}{\pi r}$ as required **1M** *Show that*

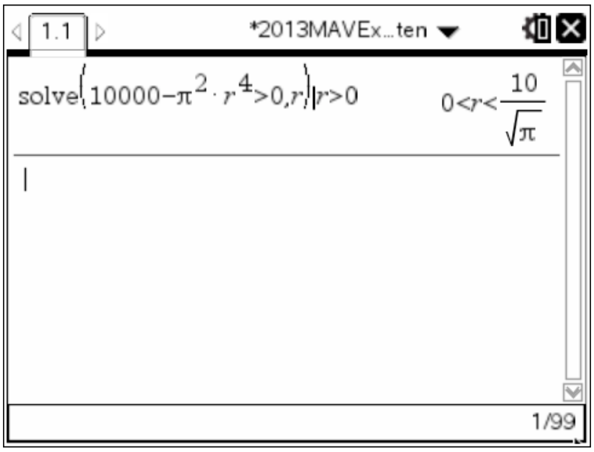
b. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2 \frac{\sqrt{10\ 000 - \pi^2 r^4}}{\pi r}$

$V = \frac{r\sqrt{10\ 000 - \pi^2 r^4}}{3}$ as required **1M** *Show that*

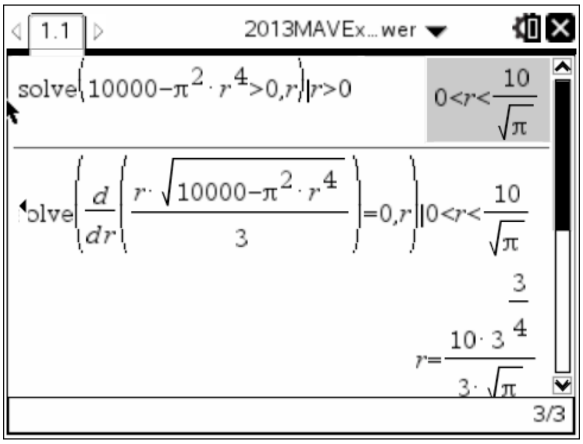
c. $10\ 000 - \pi^2 r^4 > 0$ **1M**

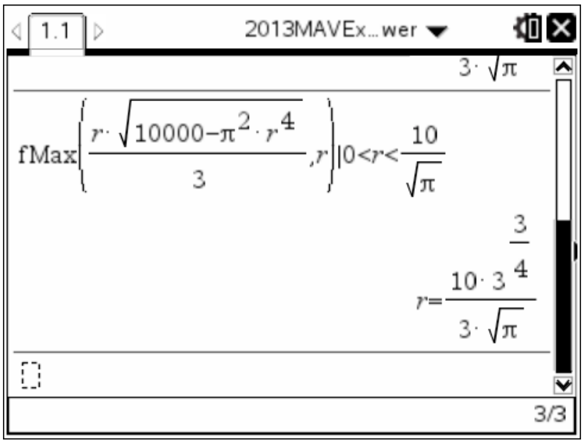
$0 < r < \frac{10}{\sqrt{\pi}}$ **1A**



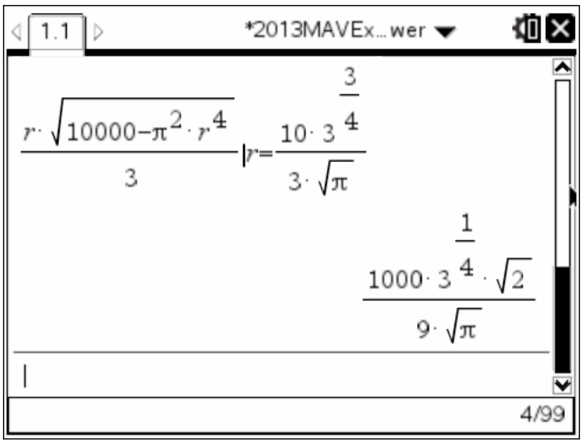
d. Solve $V'(r) = 0$ or find the maximum value

$r = \dfrac{10}{\sqrt[4]{3}\sqrt{\pi}}$ **1A**

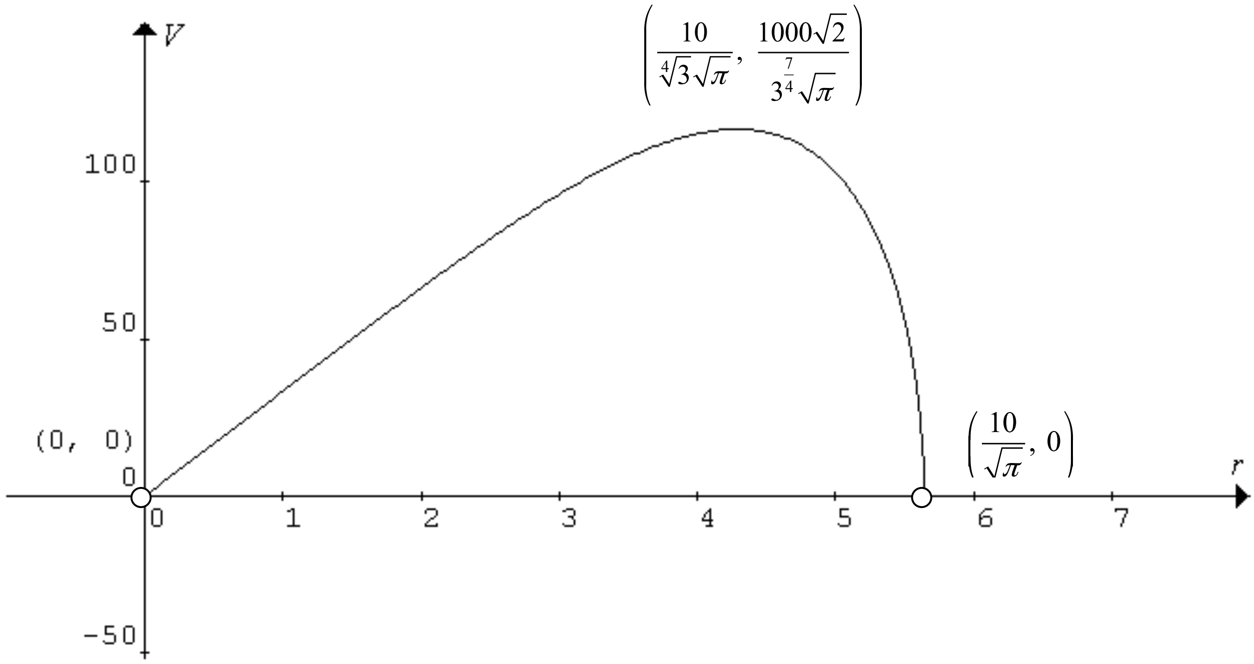




$V(r_{\max}) = \dfrac{1000\sqrt{2}}{3^{\frac{7}{4}}\sqrt{\pi}}$ **1A**



e.



Shape 1A

Coordinates 1A

Drawn to scale ½ A

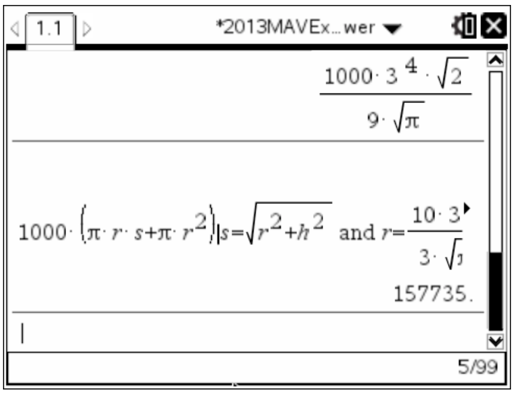
Open circles ½ A

Round down

f. TSA for 1000 cones = $1000(\pi rs + \pi r^2)$ **1M**

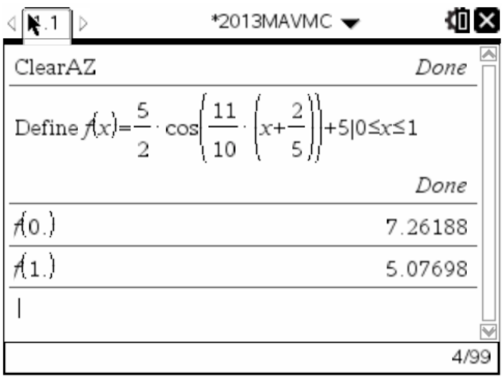
Substitute $\pi rs = 100$ and $r = \dfrac{10}{\sqrt[4]{3}\sqrt{\pi}}$ **1M**

TSA for 1000 cones = 157 735 cm^2 **1A**

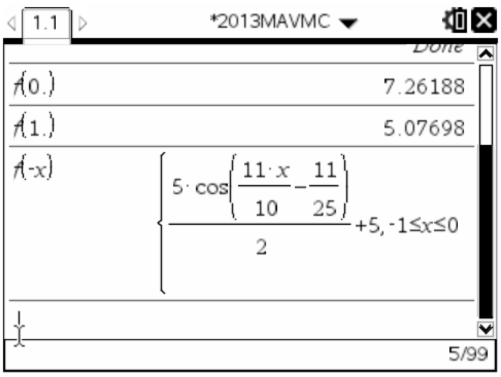


Question 2

a. i. $A=(0, 7.262), B=(1, 5.077)$ **2x1A**



ii. $g(x)=f(-x)$



$$g(x)=\frac{5}{2}\cos\left(\frac{11}{10}x-\frac{11}{25}\right)+5$$ **1A**

or

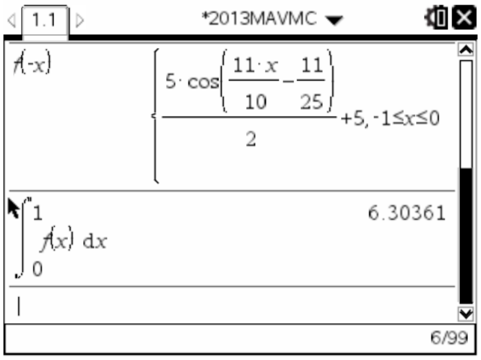
$$g(x)=\frac{5}{2}\cos\left(-\frac{11}{10}x+\frac{11}{25}\right)+5$$ **1A**

iii. domain: $-1 \le x \le 0$ **1A**

b. i. Area $=\int_0^1 f(x)\,dx$

$= 6.3036...$

$= 6.304 \text{ m}^2$ correct to three decimal places **1A**

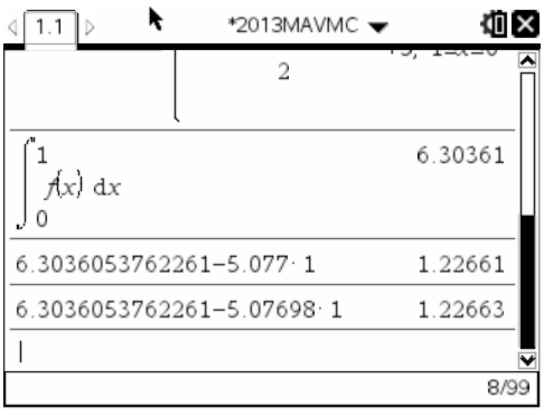


ii. By symmetry, $\text{Total area} = 2\times 6.3036... = 12.607 \text{ m}^2$ correct to three decimal places **1A**

iii. $a = 5.0769...$ **1M**

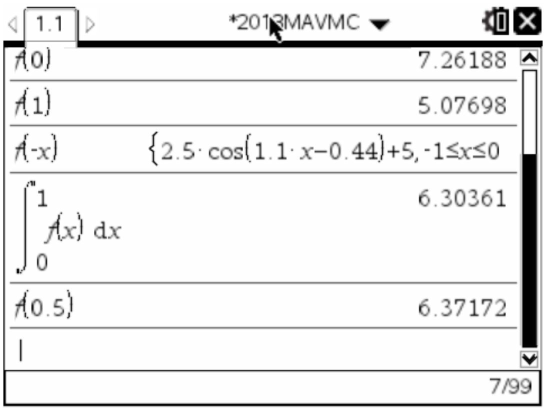
Area: $\int_0^1 f(x)dx - (5.0769...\times 1) = 1.2266...$

Area: 1.227 m^2 correct to three decimal places **1A**



c. i. $C = (0.500, 6.372)$ **1A**

By symmetry, $F = (-0.500, 6.372)$ **1A**

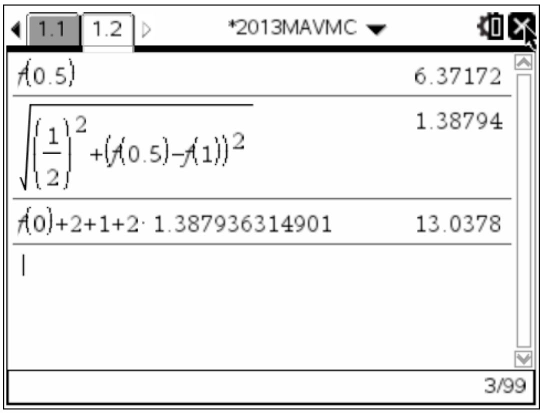


ii. $AO = 7.261...$, $BD = 2$, $FC = 1$, **1M**

$$CE = \sqrt{\left(\frac{1}{2} - 0\right)^2 + (6.371... - 5.076...)^2} = 1.387... = FE$$ **1M**

Total length $= 7.216... + 2 + 1 + 2 \times 1.387...$

$= 13.04$ m (nearest cm) **1A**

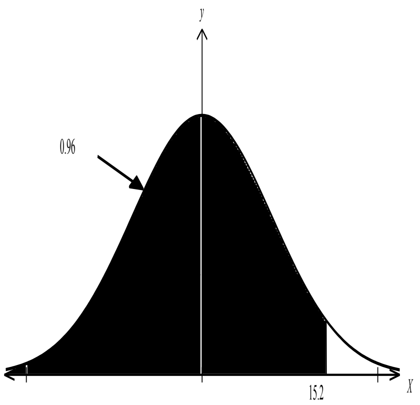


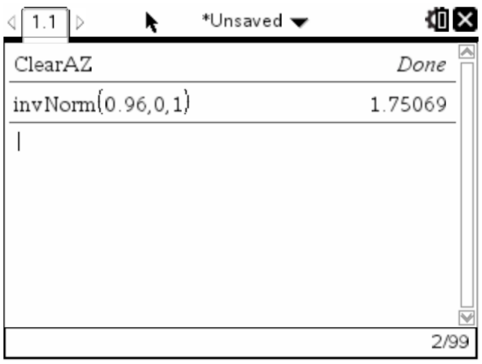
Question 3

a. i. % non-defective: $100-(4+6)=90\%$ **1A**

ii. Let X mm be the diameter of a cylinder.

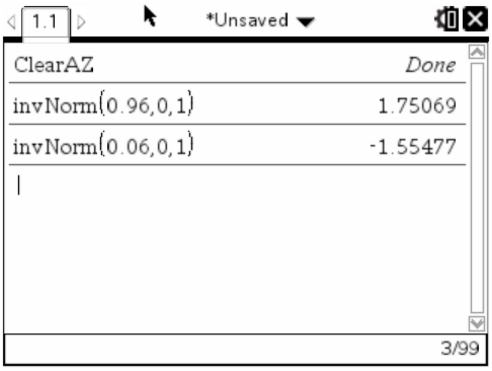
$$\Pr(X<15.2)=0.96 \rightarrow \Pr\left(Z<z=\frac{15.2-\mu}{\sigma}\right)=0.96$$





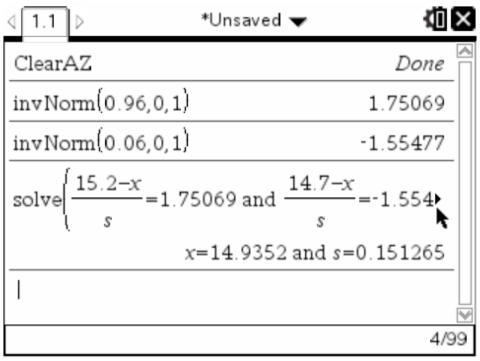
$$\frac{15.2-\mu}{\sigma}=1.75069... \quad \textbf{eq 1}$$ **1A**

$$\Pr(X<14.7)=1-0.94 \rightarrow \Pr\left(Z<z=\frac{14.7-\mu}{\sigma}\right)=0.06$$



$\frac{14.7-\mu}{\sigma}=-1.55477...$ **eq 2** **1A**

Solving the simultaneous for μ and σ :

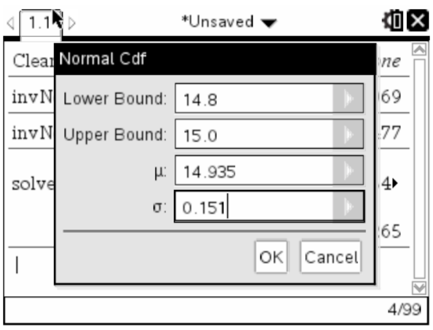


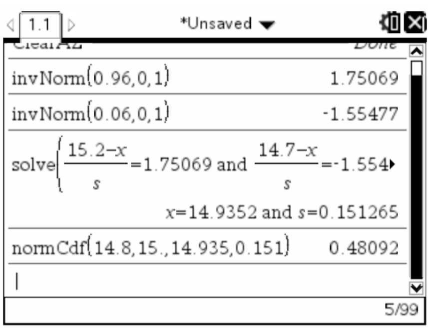
$\mu=14.935, \sigma=0.151$ **1M** *show that*

95% of cylinders: $14.935\pm2\times0.151=14.633$ mm;15.237 mm **1A**

iii. $\Pr(14.8<X<15.0)=0.48092...$

$=0.48$ correct to two decimal places **1A**



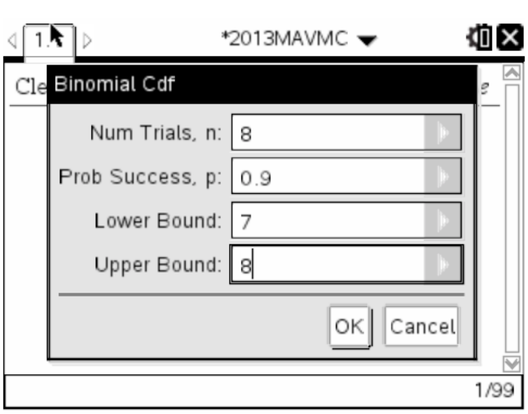


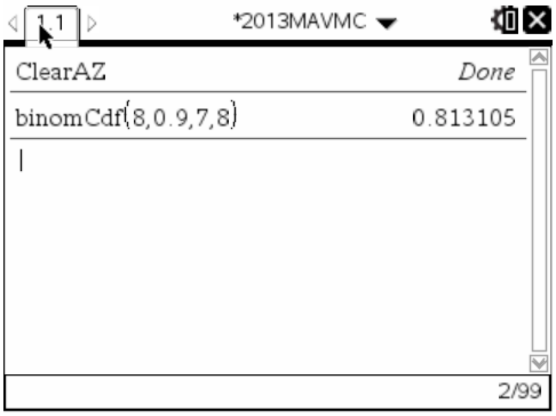
b. i. Y is number of non-defective cylinders out of 8: Binomial $n=8,\ p=0.9$ **1M**

$\Pr(Y\geq7)=\Pr(Y=7)+\Pr(Y=8)$

$$= \binom{8}{7} \times 0.9^7 \times 0.1 + \binom{8}{8} \times 0.9^8 \times 0.1^0$$

$= 0.8131$ correct to four decimal places **1A**





OR

Binomial $n = 8, p = 0.1$ **1M**

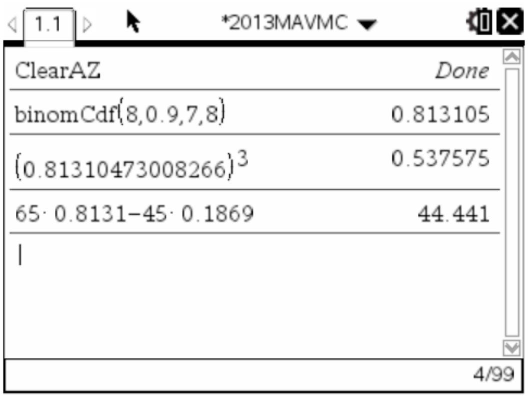
$\Pr(Y \le 1) = \Pr(Y = 0) + \Pr(Y = 1)$

$= 0.8131$ correct to four decimal places **1A**

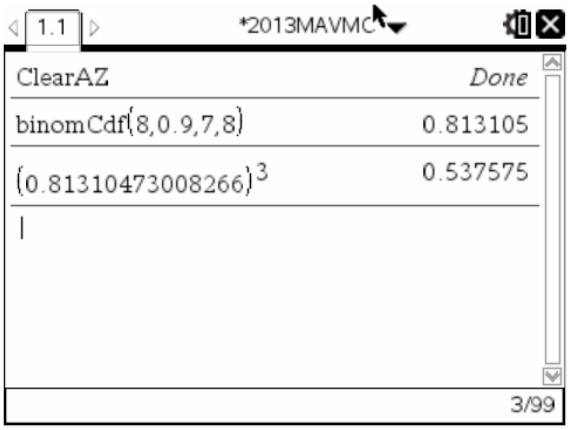
ii. Let $\$P$ be the profit per box.

$P = \{65, -45\}$ **1M**

$$\begin{aligned} E(P) &= 65 \times 0.8131 + -45 \times 0.1869 \\ &= 52.8515 - 8.4105 \\ &= \$44.44 \end{aligned}$$ **1A**



iii. $0.813105^3 = 0.5376$ correct to four decimal places **1A**

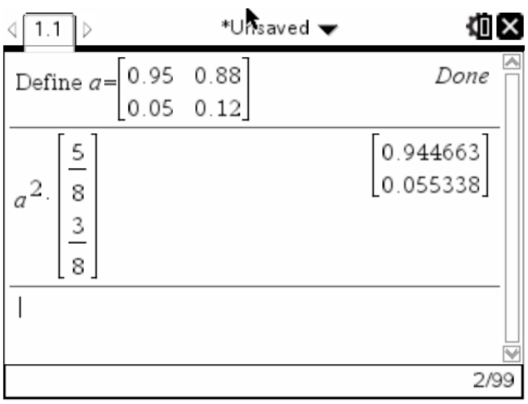


c. i. Let A be a non-defective cylinder.

$\Pr(A \mid M) = 0.95$ $\Pr(A \mid N) = 0.88$

$\Pr(A' \mid M) = 0.05$ $\Pr(A' \mid N) = 0.12$ Transition matrix $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}$ **1M**

Two manufacturing runs: $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^2 \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.944663 \\ 0.055338 \end{bmatrix}$



94% non-defective after two runs **1A**

ii. Long run: $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^{1000} \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.946237 \\ 0.053763 \end{bmatrix}$

Overall percentage of non-defective cylinders: 95% **1A**

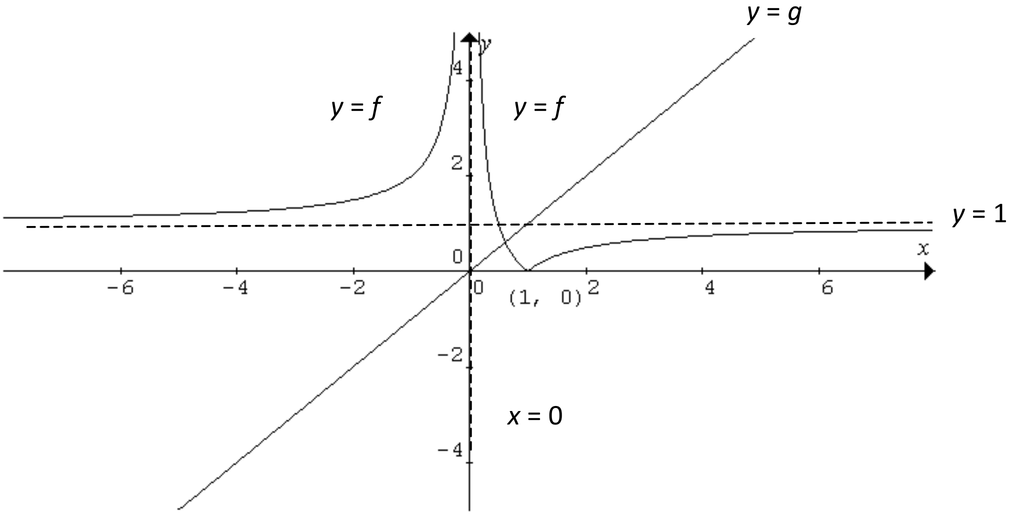
iii. The new factory produces 5% more non-defective cylinders than the original factory.

1A

Question 4

a. Solve $\left|1-\frac{1}{x}\right| = x$ for x

$\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$

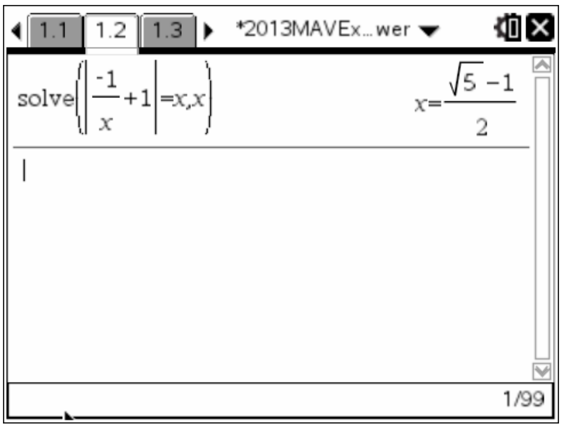


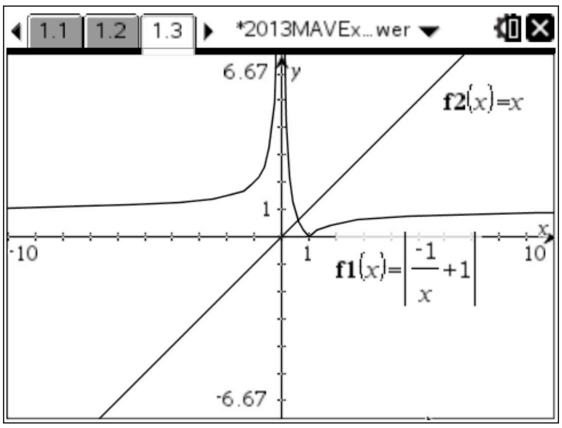
$y = g$ with $\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$ and (0, 0) 1A

Shape for $y = \left|1-\frac{1}{x}\right|$ 1A

Sharp point at (1, 0) 1A

Asymptotes 1A



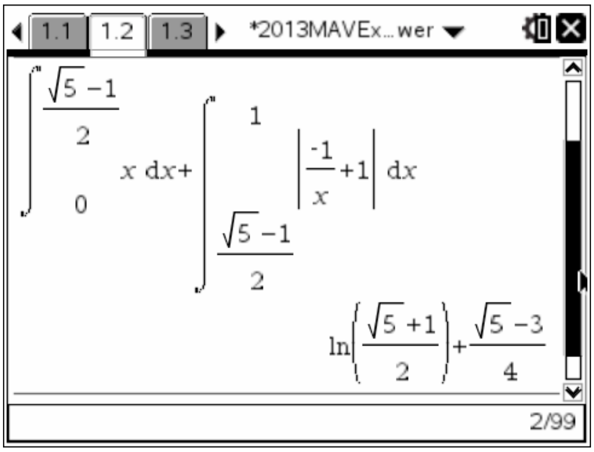


b.

$$f(x)=\begin{cases}1-\dfrac{1}{x}, x<0\ \cup x\geq 1\\ -1+\dfrac{1}{x}\ ,\ 0<x<1\end{cases}$$ **4 x ½ = 2A Round down**

c. $\displaystyle\int_0^{\frac{\sqrt{5}-1}{2}}(x)dx+\int_{\frac{\sqrt{5}-1}{2}}^{1}\left(\frac{1}{x}-1\right)dx$ **2 x 1 = 2A**

$=\log_e\left(\dfrac{\sqrt{5}+1}{2}\right)+\dfrac{\sqrt{5}-3}{4}$ **1A**



d. The area will be same when $y = x + k$ crosses the right hand branch of the hyperbola when $x > 1$.
x-intercept, for $y = x + k$ is $x=-k$. **1A**

x coordinate of the point of intersection with $y=1-\dfrac{1}{x}$ and $x > 1$ and $k < 0$ is

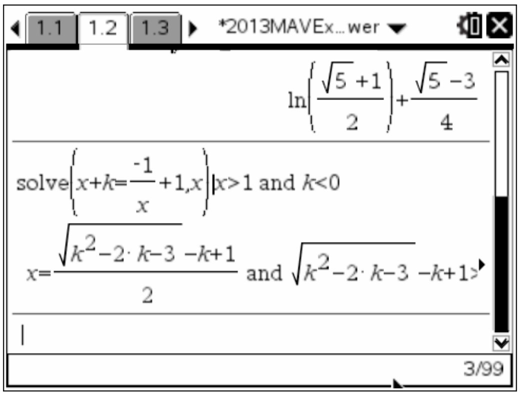
$x=\dfrac{\sqrt{k^2-2k-3}-k+1}{2}$ **1A**

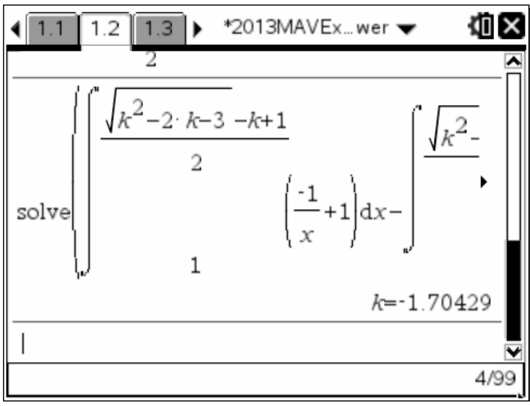
Solve $\displaystyle\int_1^{\frac{\sqrt{k^2-2k-3}-k+1}{2}}\left(1-\frac{1}{x}\right)dx-\int_{-k}^{\frac{\sqrt{k^2-2k-3}-k+1}{2}}(x+k)dx=\log_e\left(\frac{\sqrt{5}+1}{2}\right)+\frac{\sqrt{5}-3}{4}$ for k. **1A**

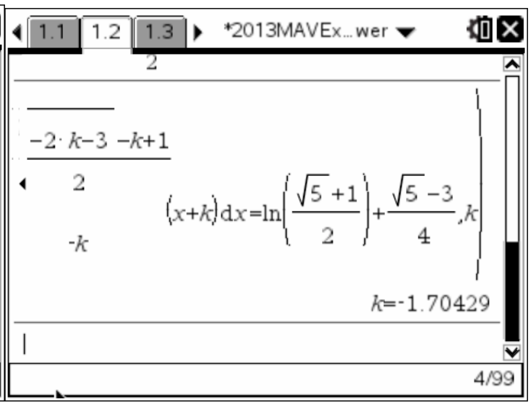
or

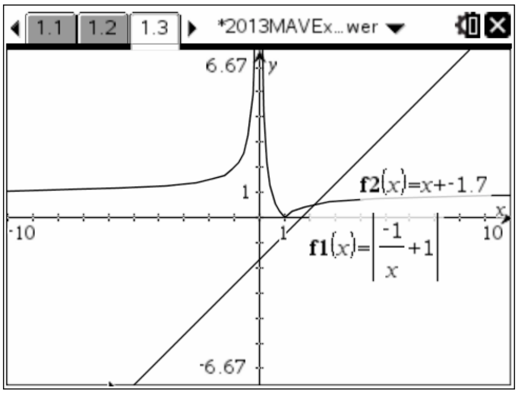
Solve $\displaystyle\int_1^{\frac{\sqrt{k^2-2k-3}-k+1}{2}}\left(1-\frac{1}{x}\right)dx-\int_{-k}^{\frac{\sqrt{k^2-2k-3}-k+1}{2}}(x+k)dx=0.2902$ for k. **1A**

$=-1.7$ correct to one decimal place **1A**









e. There will be three solutions when $y = x + k$ crosses the left hand branch of the hyperbola twice.

x coordinates of the point of intersection with $y = 1 - \frac{1}{x}$ and $x < 0$ and $k > 0$ are

$x = \frac{\pm\sqrt{k^2 - 2k - 3} - k + 1}{2}$ **1A from part d.**

Solve $k^2 - 2k - 3 > 0$ for $k > 0$. **1M**

$k > 3$ **1A**

END OF SECTION 2 SOLUTIONS