

The Mathematical Association of Victoria

SOLUTIONS: Trial Exam 2013

MATHEMATICAL METHODS

Written Examination 2

SECTION 1: Multiple Choice

1. D 2. C 3. B 4. B 5. A 6. D 7. C 8. E 9. B 10. A 11. D
 12. B 13. D 14. E 15. D 16. E 17. C 18. A 19. E 20. A 21. D 22. C

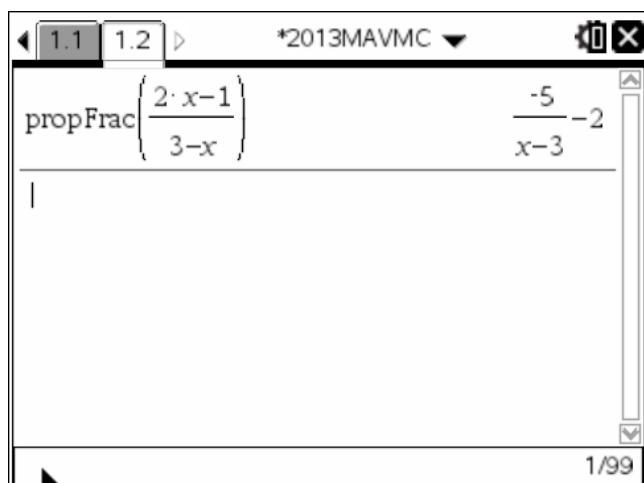
Question 1

$$f(x) = \frac{2x-1}{3-x} = \frac{-5}{x-3} - 2$$

The maximal domain is $\mathbb{R} \setminus \{3\}$.

The range is $\mathbb{R} \setminus \{-2\}$.

D

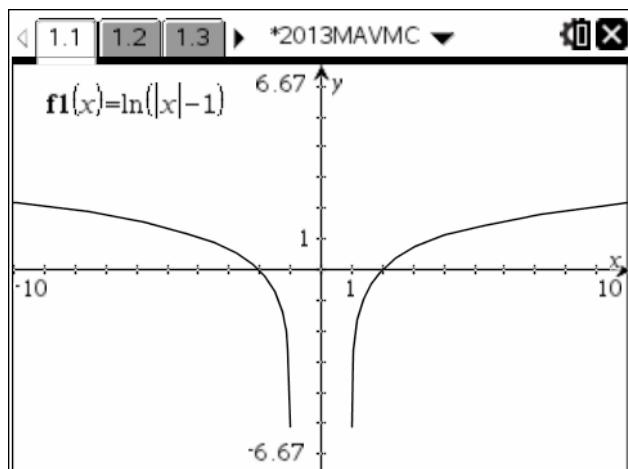


Question 2

The equations of the asymptotes are

$x = -1$ and $x = 1$

C



Question 3

$$f: R^+ \rightarrow R, f(x) = \log_e(x), \text{ and } g: \left(\frac{1}{2}, \infty\right) \rightarrow R, g(x) = (2x - 1)^2$$

$$f(g(x)) = \log_e(2x - 1)^2$$

$$f(g(x)) = 2 \log_e\left(2(x - \frac{1}{2})\right)$$

Dilation by a factor of 2 from the x -axis, dilation by a factor of a $\frac{1}{2}$ from the y -axis and a translation

of a $\frac{1}{2}$ of a unit to the right. **B**

Question 4

$$f(x) = Ax(x+2)^3 = Ax^4 + 6Ax^3 + 12Ax^2 + 8Ax$$

$$f(x) = ax^4 + bx^3 - 24x^2 + cx$$

$$12A = -24, A = -2$$

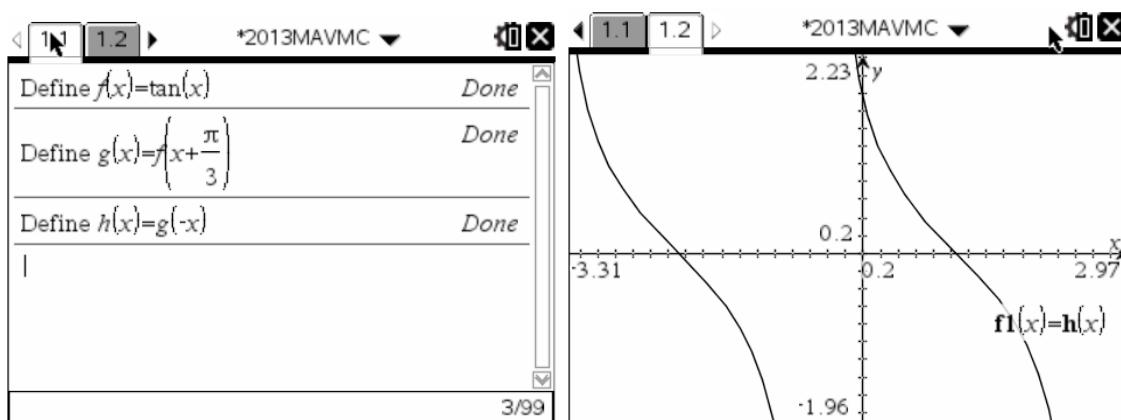
$$b = 6A = -12$$

B

Question 5

Translation of $\frac{\pi}{3}$ in the negative direction of the x axis $\rightarrow y = \tan\left(x + \frac{\pi}{3}\right)$,

Reflection in the y axis $\rightarrow y = \tan\left(-x + \frac{\pi}{3}\right)$ **A**

**Question 6**

Amplitude of 2

Translation of c in the negative direction from the x axis

$$\text{Range is } [-2-c, 2-c] = [-(2+c), (2-c)] \quad \mathbf{D}$$

Question 7

$$f(x+h) \approx f(x) + hf'(x)$$

$$f(x+h) - f(x) \approx hf'(x)$$

$$h = 0.1, f(x) = \sqrt[3]{x}, f'(x) = \frac{1}{3x^{\frac{2}{3}}}, f'(8) = \frac{1}{3 \times 8^{\frac{2}{3}}} = \frac{1}{12}$$

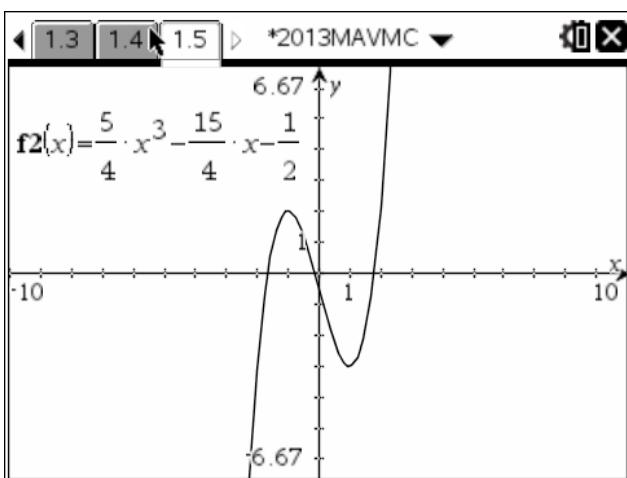
$$hf'(x) = 0.1 \times \frac{1}{12} = \frac{1}{120} \quad \mathbf{C}$$

Question 8

$$f(x) + c = 0$$

Sketch a possible graph for f .For one solution f needs to be translated down more than 2 units or translated up more than 3 units.

$$\{c : c < -2\} \cup \{c : c > 3\} \quad \mathbf{E}$$



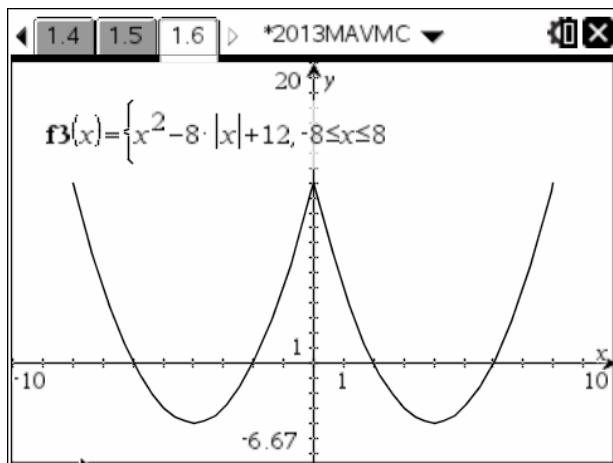
Question 9

$g : [-8, 8] \rightarrow R$, where $g(x) = x^2 - 8|x| + 12$

The graph is not differentiable at the endpoints or the sharp point.

$$x = -8, x = 0, x = 8$$

B

**Question 10**

$$g(x) = e^{-x}$$

$$\text{Area} = 0.5(g(0) + g(0.5) + g(1) + g(1.5))$$

$$= 0.5\left(1 + \frac{1}{e^{0.5}} + \frac{1}{e} + \frac{1}{e^{1.5}}\right)$$

A

Question 11

$$f : [a, b] \rightarrow R, \text{ where } f(x) = x - 1$$

The average value will be zero if a and b are equally spaced either side of $x = 1$ as the area above the line $y = 0$ will equal the area below the line $y = 0$.

$$[-2, 2]$$

D

Question 12

$$\int_1^3 (f(x)) dx = 5$$

$$\begin{aligned} 2 \int_1^3 (f(x) - 1) dx &= 2 \int_1^3 (f(x)) dx - 2 \int_1^3 1 dx \\ &= 10 - (6 - 2) = 6 \end{aligned}$$

B

Question 13

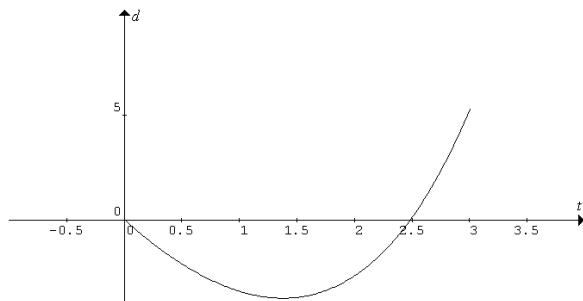
$$a = t^2 + 3$$

$$v = \frac{t^3}{3} + 3t + c$$

$$-5 = \frac{0}{3} + 3 \times 0 + c$$

$$v = \frac{t^3}{3} + 3t - 5$$

$$d = \frac{t^4}{12} + \frac{3t^2}{2} - 5t + c_1$$

**D****Option A** is acceleration against time**Option B** is velocity against time**Option C** is $y = 2t - 5$ **Option E** is $y = 2$ **Question 14**

$$\begin{aligned} SD(X) &= \sqrt{E(X^2) - [E(X)]^2} \\ &= \sqrt{1.44 - a^2} \end{aligned} \quad \text{E}$$

Question 15Let X be the number who play a musical instrument out of 60 $X \sim Bi(60, 0.35)$ $\Pr(X < 25) = 0.8286$ correct to four decimal places**D**
Question 16

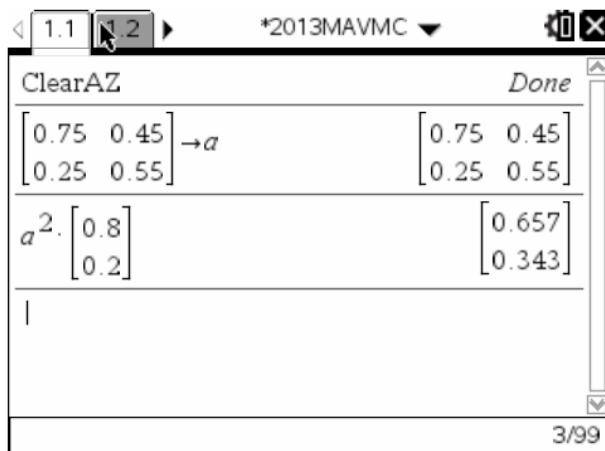
$$\begin{bmatrix} 0.75 & 0.55 \\ 0.25 & 0.45 \end{bmatrix}^2 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.657 \\ 0.343 \end{bmatrix}$$

0.657 **E**

OR

$$\begin{aligned}
 & w \times w \times w + w \times w' \times w + w' \times w \times w + w' \times w' \times w \\
 & 0.8 \times 0.75^2 + 0.8 \times 0.25 \times 0.45 + 0.2 \times 0.45 \times 0.75 + 0.2 \times 0.55 \times 0.45 \\
 & = 0.45 + 0.09 + 0.0675 + 0.0495 \\
 & = 0.657
 \end{aligned}$$

E

**Question 17**

$$\int_2^6 f(t)dt = -\int_6^2 f(t)dt \quad \text{C}$$

Question 18

Gradient is $a \rightarrow a = \frac{2a}{2}$

$$\text{Area} = \frac{1}{2} \times 2 \times 2a$$

$$= 2a$$

$$2a = 1$$

$$a = \frac{1}{2} \quad \text{A}$$

OR

$$a \int_1^3 (x-1) dx = 1 \quad \text{Solve on the CAS or}$$

$$a \left[\frac{x^2}{2} - x \right]_1^3 = 1$$

$$a \left\{ \left[\frac{9}{2} - 3 \right] - \left[\frac{1}{2} - 1 \right] \right\} = 1$$

$$a \{4 - 2\} = 1$$

$$2a = 1$$

$$a = \frac{1}{2} \quad \text{A}$$

Question 19

Define $f(x) = x^3 - x$

$$\text{Var}(X) = \int_1^{\sqrt{3}} x^2 f(x) dx - \left[\int_1^{\sqrt{3}} x f(x) dx \right]^2$$

The calculator screen displays the following:

Define $f(x) = x^3 - x$

$$\int_1^{\sqrt{3}} (x^2 \cdot f(x)) dx - \left(\int_1^{\sqrt{3}} (x \cdot f(x)) dx \right)^2$$

0.026051

2/99

0.0261 correct to four decimal places

D

Question 20

$$\Pr(X < a_2) = 0.925$$

$$\Pr\left(Z < \frac{a_2 - 13.4}{3.2}\right) = 0.925$$

ClearAZ

Inverse Normal

Area: 0.925

μ : 13.4

σ : 3.2

OK Cancel

Done

invNorm(0.925, 13.4, 3.2)

18.0065

1/99

2/99

$$18.0065 - 13.4 = 4.6065$$

$$13.4 - 4.6065 = 8.7935$$

$$a_1 = 8.79; a_2 = 18.01$$

A

The calculator screen shows the following steps:

- ClearAZ
- $\text{invNorm}[0.925, 13.4, 3.2]$ Done 18.0065
- $18.0065 - 13.4$ Done 4.6065
- $13.4 - 4.6065$ Done 8.7935
- 1

Bottom right corner: 4/99

Question 21

$$1 - [\Pr(X = 0) + \Pr(X = 1)]$$

$$1 - [(1-p) \times 1 + (1-p) \times p]$$

$$1 - [(1-p)(1+p)]$$

$$1 - [1 - p^2]$$

$$p^2$$

D**OR**

The calculator screen shows the following steps:

- ClearAZ
- Define $f(a) = (1-p) \cdot p^a$ Done
- $1 - (f(0) + f(1))$ Done p^2
- 1

Bottom right corner: 4/99

Message at bottom: Δ Domain of the result might be larger than the do...

Question 22

$$\frac{dV}{dt} = -750 \text{ cm}^3 / \text{min}$$

$$h = 3r$$

$$r = \frac{h}{3}$$

$$V = \frac{1}{3}\pi r^3 h$$

$$= \frac{1}{3}\pi \frac{h^3}{9}$$

$$= \frac{\pi h^3}{27}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{9}$$

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{9}{\pi h^2} \times -750 \\ &= -\frac{6750}{\pi h^2}\end{aligned}$$

C**END OF SECTION 1 SOLUTIONS**

SECTION 2: Extended Answer Solutions**Question 1**

a. $\text{TSA} = \pi r s + \pi r^2$

Curved surface area $= \pi r s = 100$

$$s = \sqrt{r^2 + h^2} \quad \mathbf{1M}$$

$$\pi r \sqrt{r^2 + h^2} = 100 \quad \mathbf{1M}$$

$$\sqrt{r^2 + h^2} = \frac{100}{\pi r}$$

$$r^2 + h^2 = \frac{10\ 000}{\pi^2 r^2}$$

$$h^2 = \frac{10\ 000}{\pi^2 r^2} - r^2$$

$$h^2 = \frac{10\ 000 - \pi^2 r^4}{\pi^2 r^2}$$

$$h = \frac{\sqrt{10\ 000 - \pi^2 r^4}}{\pi r} \quad \text{as required} \quad \mathbf{1M} \text{ Show that}$$

b. $V = \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} \pi r^2 \frac{\sqrt{10\ 000 - \pi^2 r^4}}{\pi r}$$

$$V = \frac{r \sqrt{10\ 000 - \pi^2 r^4}}{3} \quad \text{as required} \quad \mathbf{1M} \text{ Show that}$$

c. $10\ 000 - \pi^2 r^4 > 0 \quad \mathbf{1M}$

$$0 < r < \frac{10}{\sqrt{\pi}} \quad \mathbf{1A}$$

1.1 *2013MAVE...ten

solve $\left(10000 - \pi^2 \cdot r^4 > 0, r\right) | r > 0$

$0 < r < \frac{10}{\sqrt{\pi}}$

1/99

d. Solve $V'(r) = 0$ or find the maximum value

$$r = \frac{10}{\sqrt[4]{3\sqrt{\pi}}} \quad \text{1A}$$

1.1 2013MAVE...wer

solve $\left(10000 - \pi^2 \cdot r^4 > 0, r\right) | r > 0$

$0 < r < \frac{10}{\sqrt{\pi}}$

solve $\left(\frac{d}{dr}\left(\frac{r \cdot \sqrt{10000 - \pi^2 \cdot r^4}}{3}\right) = 0, r\right) | 0 < r < \frac{10}{\sqrt{\pi}}$

$r = \frac{10 \cdot 3^{\frac{4}{3}}}{3 \cdot \sqrt{\pi}}$

3/3

1.1 2013MAVE...wer

$fMax\left(\frac{r \cdot \sqrt{10000 - \pi^2 \cdot r^4}}{3}, r\right) | 0 < r < \frac{10}{\sqrt{\pi}}$

$r = \frac{10 \cdot 3^{\frac{4}{3}}}{3 \cdot \sqrt{\pi}}$

3/3

$$V(r_{\max}) = \frac{1000\sqrt{2}}{3^{\frac{7}{4}}\sqrt{\pi}} \quad \text{1A}$$

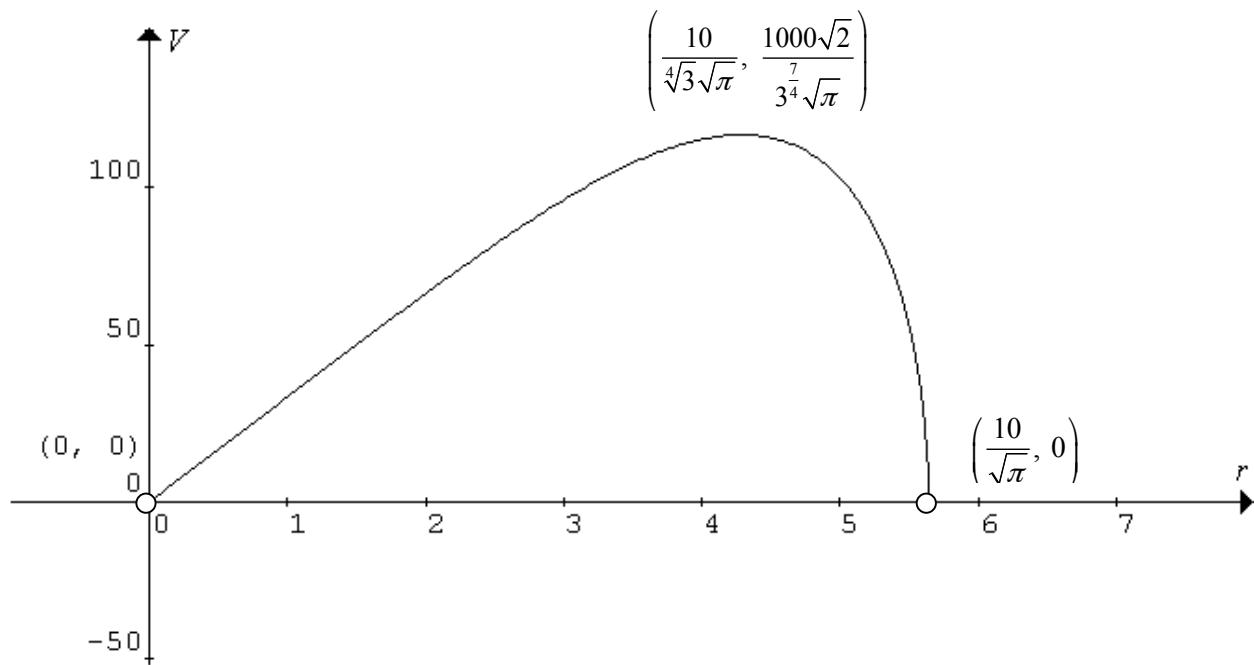
1.1 *2013MAVE...wer

$\frac{r \cdot \sqrt{10000 - \pi^2 \cdot r^4}}{3} | r = \frac{10 \cdot 3^{\frac{4}{3}}}{3 \cdot \sqrt{\pi}}$

$\frac{1}{9 \cdot \sqrt{\pi}} \cdot \frac{1000 \cdot 3^{\frac{4}{3}} \cdot \sqrt{2}}{3}$

4/99

e.

**Shape 1A****Coordinates 1A****Drawn to scale $\frac{1}{2}$ A****Open circles $\frac{1}{2}$ A****Round down**

f. TSA for 1000 cones = $1000(\pi rs + \pi r^2)$ 1M

Substitute $\pi rs = 100$ and $r = \frac{10}{4\sqrt{3}\sqrt{\pi}}$ 1M

TSA for 1000 cones = $157\ 735 \text{ cm}^2$ 1A

A screenshot of a CAS calculator interface. The top part shows the input and output of a complex fraction: $\frac{1000 \cdot 3^{\frac{4}{3}} \cdot \sqrt{2}}{9 \cdot \sqrt{\pi}}$. The bottom part shows the working for the total surface area of 1000 cones. It uses the formula $1000 \cdot (\pi \cdot r \cdot s + \pi \cdot r^2)$ with $s = \sqrt{r^2 + h^2}$ and $r = \frac{10 \cdot 3}{3 \cdot \sqrt{3}}$. The final result is given as 157735.

Question 2

a. i. $A = (0, 7.262), B = (1, 5.077)$

2x1A

The calculator screen shows the following steps:

- Define $f(x) = \frac{5}{2} \cdot \cos\left(\frac{11}{10} \cdot \left(x + \frac{2}{5}\right)\right) + 5 \mid 0 \leq x \leq 1$
- $f(0)$ is calculated as 7.26188
- $f(1)$ is calculated as 5.07698
- The current cursor position is at the bottom of the screen.

ii. $g(x) = f(-x)$

The calculator screen shows the following steps:

- $f(0)$ is shown as 7.26188
- $f(1)$ is shown as 5.07698
- Define $f(-x) = \begin{cases} 5 \cdot \cos\left(\frac{11 \cdot x - 11}{10} - \frac{25}{25}\right) + 5, & -1 \leq x \leq 0 \end{cases}$
- The current cursor position is at the bottom of the screen.

$$g(x) = \frac{5}{2} \cos\left(\frac{11}{10}x - \frac{11}{25}\right) + 5 \quad \textbf{1A}$$

or

$$g(x) = \frac{5}{2} \cos\left(-\frac{11}{10}x + \frac{11}{25}\right) + 5 \quad \textbf{1A}$$

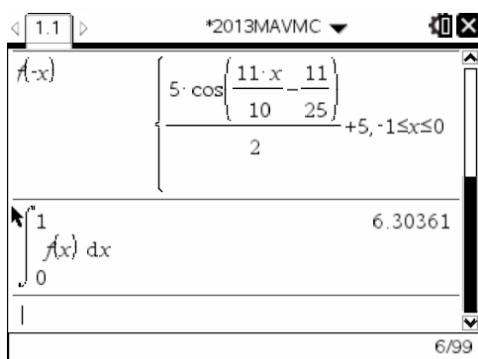
iii. domain: $-1 \leq x \leq 0$ **1A**

b. i. Area = $\int_0^1 f(x) dx$

$$= 6.3036\dots$$

$$= 6.304 \text{ m}^2 \text{ correct to three decimal places}$$

1A

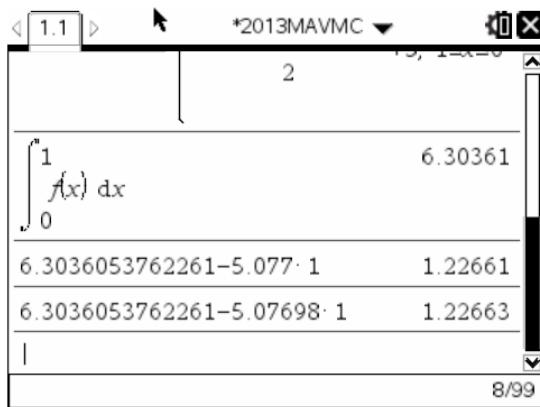


ii. By symmetry, Total area = $2 \times 6.3036\dots = 12.607 \text{ m}^2$ correct to three decimal places **1A**

iii. $a = 5.0769\dots$ **1M**

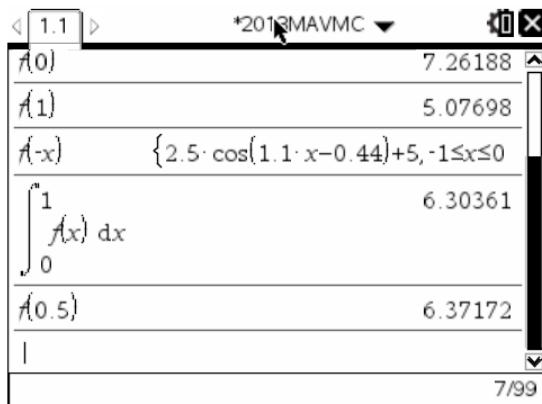
$$\text{Area: } \int_0^1 f(x) dx - (5.0769\dots \times 1) = 1.2266\dots$$

Area: 1.227 m^2 correct to three decimal places **1A**



c. i. $C = (0.500, 6.372)$ **1A**

By symmetry, $F = (-0.500, 6.372)$ **1A**

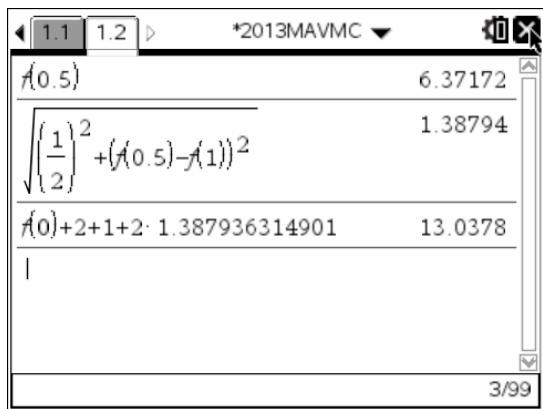


ii. $AO = 7.261\dots$, $BD = 2$, $FC = 1$, 1M

$$CE = \sqrt{\left(\frac{1}{2} - 0\right)^2 + (6.371\dots - 5.076\dots)^2} = 1.387\dots = FE \quad \text{1M}$$

Total length = $7.216\dots + 2 + 1 + 2 \times 1.387\dots$

$$= 13.04 \text{ m (nearest cm)} \quad \text{1A}$$

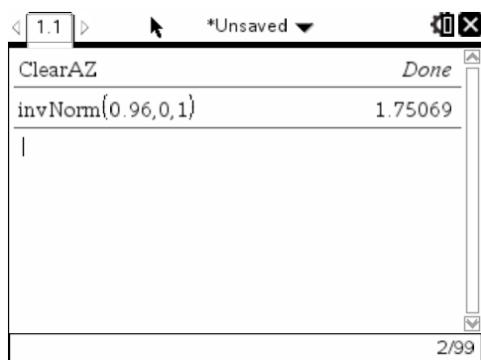
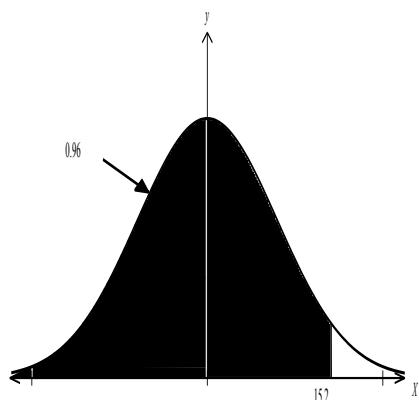


Question 3

a. i. % non-defective: $100 - (4 + 6) = 90\%$ 1A

ii. Let X mm be the diameter of a cylinder.

$$\Pr(X < 15.2) = 0.96 \rightarrow \Pr\left(Z < z = \frac{15.2 - \mu}{\sigma}\right) = 0.96$$



$$\frac{15.2 - \mu}{\sigma} = 1.75069\dots \quad \text{eq 1} \quad \text{1A}$$

$$\Pr(X < 14.7) = 1 - 0.94 \rightarrow \Pr\left(Z < z = \frac{14.7 - \mu}{\sigma}\right) = 0.06$$

1.1 > *Unsaved ▾

ClearAZ

Done

invNorm(0.96,0,1)	1.75069
invNorm(0.06,0,1)	-1.55477

$$\frac{14.7 - \mu}{\sigma} = -1.55477\dots \text{ eq 2} \quad 1A$$

Solving the simultaneous for μ and σ :

The screenshot shows the TI-Nspire CX handheld calculator's TI-Nspire CX CAS software interface. The top bar indicates "1.1" and "Unsaved". The main area displays the following calculations:

- ClearAZ
- Done
- $\text{invNorm}(0.96, 0, 1) = 1.75069$
- $\text{invNorm}(0.06, 0, 1) = -1.55477$
- Solve command: $\left\{ \frac{15.2-x}{s} = 1.75069 \text{ and } \frac{14.7-x}{s} = -1.55477 \right\}$
- Result: $x = 14.9352$ and $s = 0.151265$

$$\mu = 14.935, \sigma = 0.151$$

1M show that

95% of cylinders: $14.935 \pm 2 \times 0.151 = 14.633 \text{ mm}; 15.237 \text{ mm}$

1A

$$\text{iii. } \Pr(14.8 < X < 15.0) = 0.48092\dots$$

= 0.48 correct to two decimal places

The TI-Nspire CX CAS calculator is displaying a normal distribution calculation. The top menu bar shows "1.1 > *Unsaved". The left sidebar lists "Clear", "Normal Cdf", "invNorm", "invN", and "solve". The main workspace shows the following input and output:

Input:

- Lower Bound: 14.8
- Upper Bound: 15.0
- μ : 14.935
- σ : 0.151

Output:

- $\text{invNorm}(0.96, 0, 1)$ = 1.75069
- $\text{invNorm}(0.06, 0, 1)$ = -1.55477
- $\text{solve} \left\{ \frac{15.2-x}{s} = 1.75069 \text{ and } \frac{14.7-x}{s} = -1.55477 \right\}$
- $x = 14.9352$ and $s = 0.151265$
- $\text{normCdf}[14.8, 15., 14.935, 0.151]$ = 0.48092

The bottom status bar shows "4/99" and "5/99".

b. i. Y is number of non-defective cylinders out of 8: Binomial $n = 8, p = 0.9$

1M

$$\Pr(Y \geq 7) = \Pr(Y = 7) + \Pr(Y = 8)$$

$$= \binom{8}{7} \times 0.9^7 \times 0.1 + \binom{8}{8} \times 0.9^8 \times 0.1^0$$

= 0.8131 correct to four decimal places

1A

Binomial Cdf

Num Trials, n: 8
Prob Success, p: 0.9
Lower Bound: 7
Upper Bound: 8

binomCdf(8,0.9,7,8) 0.813105

OR

Binomial $n = 8, p = 0.1$

1M

$$\Pr(Y \leq 1) = \Pr(Y = 0) + \Pr(Y = 1)$$

= 0.8131 correct to four decimal places

1A

ii. Let $\$P$ be the profit per box.

$$P = \{65, -45\}$$

1M

$$E(P) = 65 \times 0.8131 + -45 \times 0.1869$$

$$= 52.8515 - 8.4105$$

1A

$$= \$44.44$$

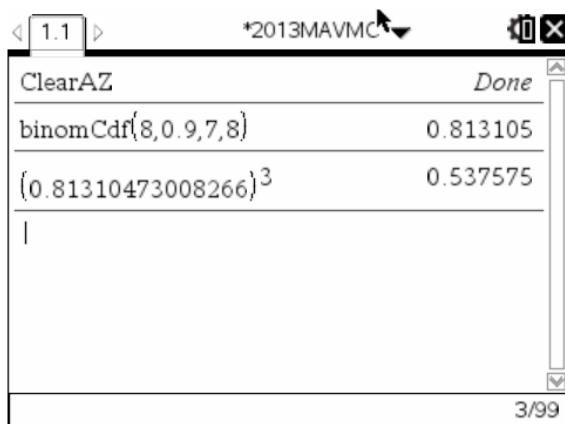
binomCdf(8,0.9,7,8) 0.813105

$(0.81310473008266)^3$ 0.537575

$65 \cdot 0.8131 - 45 \cdot 0.1869$ 44.441

iii. $0.813105^3 = 0.5376$ correct to four decimal places

1A



c. i. Let A be a non-defective cylinder.

$$\begin{aligned} \Pr(A|M) &= 0.95 & \Pr(A|N) &= 0.88 \\ \Pr(A'|M) &= 0.05 & \Pr(A'|N) &= 0.12 \end{aligned} \quad \text{Transition matrix } \begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix} \quad \mathbf{1M}$$

Two manufacturing runs: $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^2 \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.944663 \\ 0.055338 \end{bmatrix}$



94% non-defective after two runs **1A**

ii. Long run: $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^{1000} \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.946237 \\ 0.053763 \end{bmatrix}$

Overall percentage of non-defective cylinders: 95% **1A**

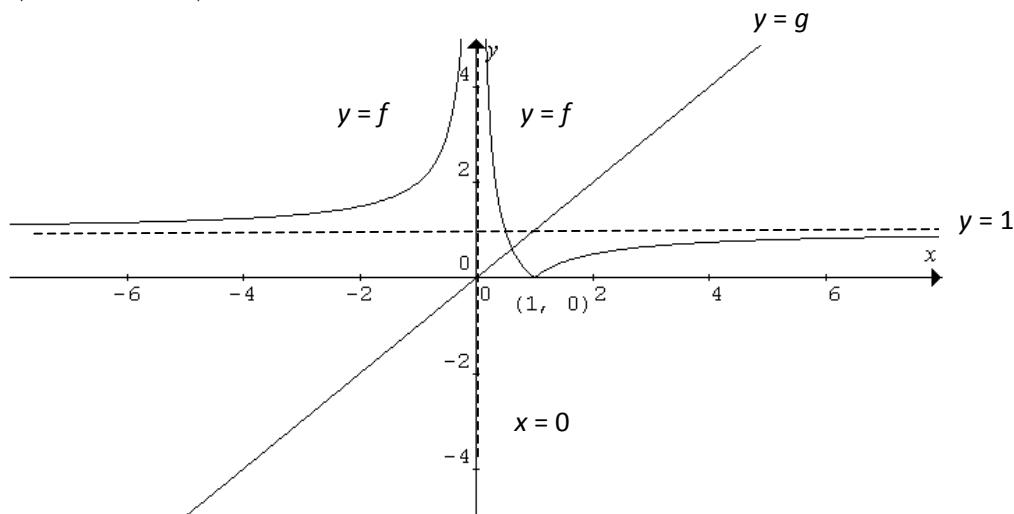
iii. The new factory produces 5% more non-defective cylinders than the original factory.

1A

Question 4

a. Solve $\left|1 - \frac{1}{x}\right| = x$ for x

$$\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$$

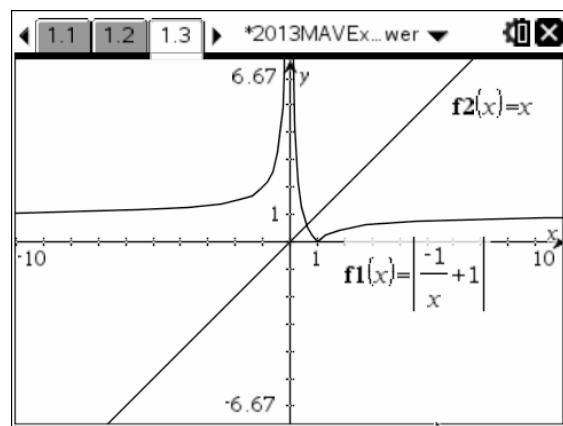
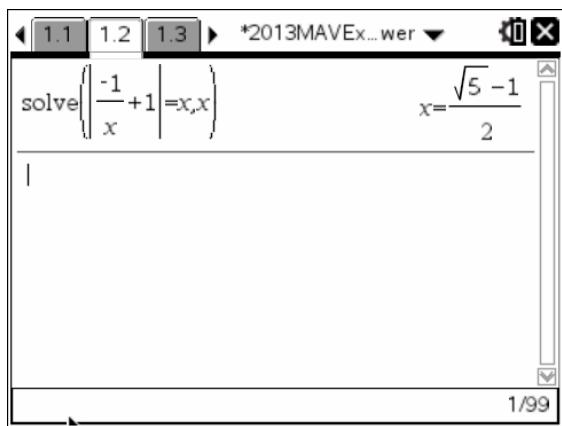


$y = g$ with $\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$ and $(0, 0)$ 1A

Shape for $y = \left|1 - \frac{1}{x}\right|$ 1A

Sharp point at $(1, 0)$ 1A

Asymptotes 1A



b.

$$f(x) = \begin{cases} 1 - \frac{1}{x}, & x < 0 \cup x \geq 1 \\ -1 + \frac{1}{x}, & 0 < x < 1 \end{cases}$$

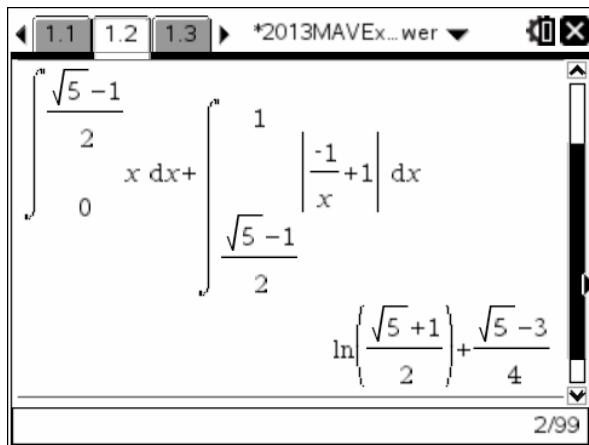
4 x ½ = 2A Round down

c. $\int_0^{\frac{\sqrt{5}-1}{2}} (x) dx + \int_{\frac{\sqrt{5}-1}{2}}^1 \left(\frac{1}{x} - 1 \right) dx$

2 x 1 = 2A

$$= \ln_e \left(\frac{\sqrt{5}+1}{2} \right) + \frac{\sqrt{5}-3}{4}$$

1A



d. The area will be same when $y = x + k$ crosses the right hand branch of the hyperbola when $x > 1$.
 x -intercept, for $y = x + k$ is $x = -k$. **1A**

x coordinate of the point of intersection with $y = 1 - \frac{1}{x}$ and $x > 1$ and $k < 0$ is

$$x = \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}$$

1A

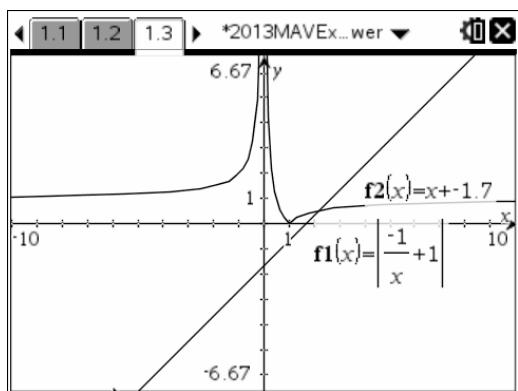
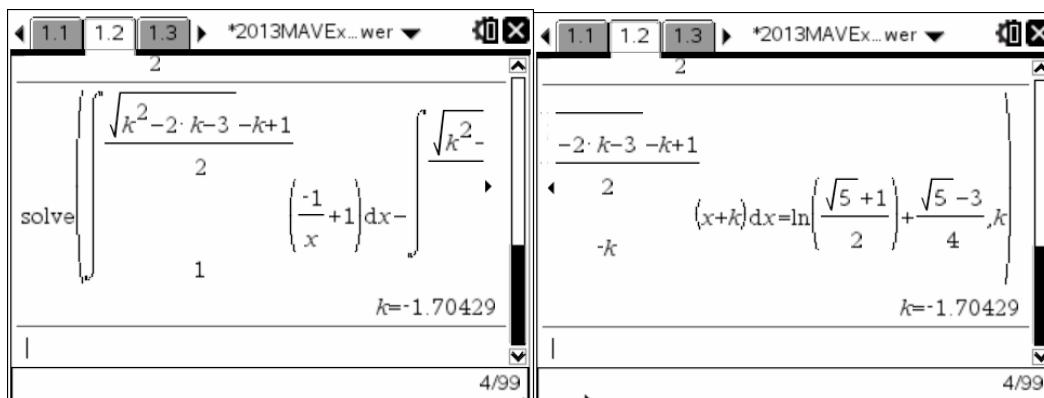
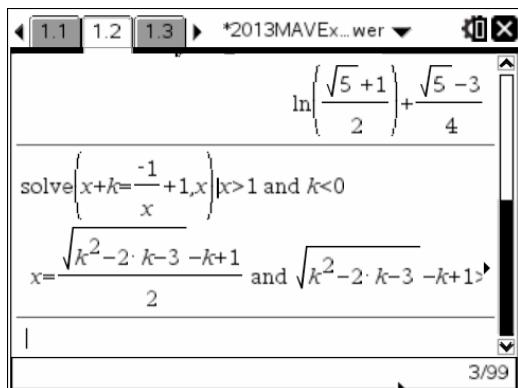
Solve $\int_1^{\frac{\sqrt{k^2-2k-3}-k+1}{2}} \left(1 - \frac{1}{x} \right) dx - \int_{-k}^{\frac{\sqrt{k^2-2k-3}-k+1}{2}} (x+k) dx = \ln_e \left(\frac{\sqrt{5}+1}{2} \right) + \frac{\sqrt{5}-3}{4}$ for k . **1A**

or

Solve $\int_1^{\frac{\sqrt{k^2-2k-3}-k+1}{2}} \left(1 - \frac{1}{x} \right) dx - \int_{-k}^{\frac{\sqrt{k^2-2k-3}-k+1}{2}} (x+k) dx = 0.2902$ for k . **1A**

$$= -1.7 \text{ correct to one decimal place}$$

1A



- e. There will be three solutions when $y = x + k$ crosses the left hand branch of the hyperbola twice.

x coordinates of the point of intersection with $y = 1 - \frac{1}{x}$ and $x < 0$ and $k > 0$ are

$$x = \frac{\pm\sqrt{k^2 - 2k - 3} - k + 1}{2}$$

1A from part d.

Solve $k^2 - 2k - 3 > 0$ for $k > 0$.

1M

$k > 3$

1A

END OF SECTION 2 SOLUTIONS