

Trial Examination 2013

# VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 1

**Suggested Solutions** 

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## Question 1 (3 marks)

**a.** 
$$y = \frac{2}{1-5x} = 2(1-5x)^{-1} \Rightarrow \frac{dy}{dx} = -2(1-5x)^{-2} \times -5$$
  
 $\frac{dy}{dx} = \frac{10}{(1-5x)^2}$  A1

$$f(x) = x\cos^{2}\left(\frac{\pi x}{2}\right)$$

$$= x \cdot 2 \times \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) \left[-\sin\left(\frac{\pi x}{2}\right)\right] + \cos^{2}\left(\frac{\pi x}{2}\right)$$

$$= -\pi x \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right) + \cos^{2}\left(\frac{\pi x}{2}\right)$$

$$f'\left(\frac{1}{2}\right) = -\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) + \cos^{2}\left(\frac{\pi}{4}\right)$$

$$= -\frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^{2}$$

$$= \frac{1}{2} - \frac{\pi}{4}$$
A1

## Question 2 (4 marks)

As 
$$\frac{dy}{dx} = \frac{6}{\sqrt{x^3}}$$
, we have  $y = \int \frac{6}{\sqrt{x^3}} dx = \int 6x^{-\frac{3}{2}} dx = -\frac{12}{\sqrt{x}} + c$  M1  
 $4 = \frac{-12}{\sqrt{\frac{1}{4}}} + c \Rightarrow c = 28$ 

Equation of the curve is 
$$y = -\frac{12}{\sqrt{x}} + 28$$
 A1

#### Question 3 (2 marks)

$$f(g(x)) = f\left(1 + e^{\frac{x}{a}}\right) = a\log_e\left(1 + e^{\frac{x}{a}} - 1\right) = a\log_e\left(e^{\frac{x}{a}}\right) = a \times \frac{x}{a} = x$$
 M1

As  $f(g(x)) = x = f(f^{-1}(x))$ , and both f and g are one to one functions, they are inverses. A1

# Question 4 (5 marks)

- a. Define the random variable X as "the number of passengers who show up for the flight". Pr(everyone gets a seat) =  $Pr(X \le 40) = 0.1 + 0.45 + 0.35 = 0.9$  A1
- **b.** Define *Y* as the number of passengers who do not show up

Then 
$$Y = 43 - X$$
 so that  
 $E(Y) = 43 - E(X)$  M1  
 $E(Y) = 43 - (38 \times 0.1 + 39 \times 0.45 + 40 \times 0.35 + 41 \times 0.05 + 42 \times 0.04 + 43 \times 0.01)$   
 $= 43 - (3.8 + 17.55 + 14 + 2.05 + 1.68 + 0.43)$   
 $= 43 - 39.51$   
 $= 3.49$  A1

c. Pr(X = 38 | not all seats filled) =

$$Pr(X = 38 | X < 40) = \frac{Pr(X = 38)}{Pr(X < 40)}$$

$$= \frac{0.1}{10} = \frac{10}{2} = \frac{2}{10}$$
A1

$$\overline{0.1+0.45} = \overline{55} = \overline{11}$$

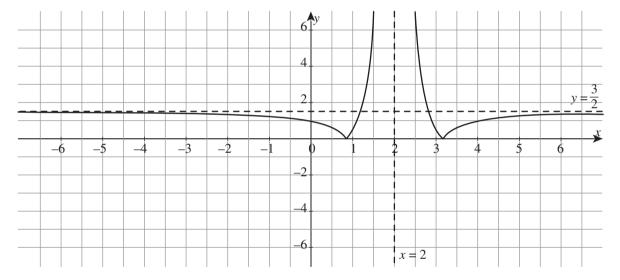
## Question 5 (7 marks)

**a.** A vertical asymptote with equation x = 2 means b = -2. A1

$$f(x) = \frac{a}{(x-2)^2}$$
 and  $f(0) = -\frac{1}{2} \Rightarrow \frac{a}{4} = -\frac{1}{2} \Rightarrow a = -2$  A1

**b.** x-intercepts by solving  $\frac{3}{2} - \frac{2}{(x-2)^2} = 0$  $(x-2)^2 = \frac{4}{3}$ 

$$x = 2 \pm \frac{2}{\sqrt{3}} \approx 2 \pm 2(0.6) \approx 0.8$$
 and 3.2



*Vertical asymptote* x = 2 *and horizontal asymptote*  $y = \frac{3}{2}$  A1

- *x*-intercepts correct and shown on graph A1
  - correct shape A1

c. Starting with 
$$f(x) = \frac{-2}{(x-2)^2}$$
, under a dilation of scale factor 2 from the y-axis we have

$$x \to \frac{x}{2}$$
, so  $y = \frac{-2}{\left(\frac{x}{2} - 2\right)^2}$  M1

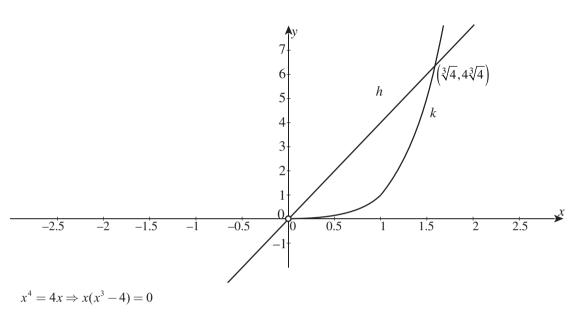
A translation of 3 units in the negative direction of the x-axis,  $x \rightarrow x+3$ 

So 
$$y = \frac{-2}{\left(\frac{x+3}{2}-2\right)^2} = \frac{-2}{\left(\frac{x-1}{2}\right)^2} = \frac{-8}{(x-1)^2}$$
 A1

## Question 6 (4 marks)

b.

**a.** 
$$h(x) = f(g(x)) = f(e^{2x}) = \log_e(e^{2x})^2 = \log_e(e^{4x}) = 4x$$
 A1  
 $k(x) = g(f(x)) = g(\log_e(x^2)) = e^{2\log_e(x^2)}$   
 $k(x) = e^{\log_e(x^4)} = x^4$  A1



 $x = \sqrt[3]{4}$  is the only solution in the domain

graphs of h and k on correct domain with (0,0) excluded A1

 $\left(\sqrt[3]{4}, 4\sqrt[3]{4}\right)$  as intersection point A1

#### Question 7 (3 marks)

Given  $2\log_3(x+4) - \log_3(-x) = 2$  we have x+4 > 0 and -x > 0, which together means -4 < x < 0.  $\log_3(x+4)^2 - \log_3(-x) = 2$ 

$$\log_3\left(\frac{(x+4)^2}{-x}\right) = 2 = \log_3(3^2)$$
 M1

$(x+4)^2 = -9x$	
$x^2 + 8x + 16 = -9x$	
$x^2 + 17x + 16 = 0$	
(x+16)(x+1) = 0	
x = -16, -1	A1
Since $-4 < x < 0$ , $x = -1$ is the only allowable solution.	A1

## Question 8 (5 marks)

**a. i.** As *X* is a probability density function,

$$\int_{0}^{2} kx(4-x^{2})dx = 1$$

$$\left[2x^{2} - \frac{x^{4}}{4}\right]_{0}^{2} = \frac{1}{k}$$

$$4 = \frac{1}{k} \Rightarrow k = \frac{1}{4}$$
A1

ii. The median, *m*, is such that 
$$\int_0^m \frac{1}{4} x(4-x^2) dx = \frac{1}{2}$$
 M1

$$\left[2x^{2} - \frac{x^{4}}{4}\right]_{0}^{m} = 2$$
$$2m^{2} - \frac{m^{4}}{4} = 2$$

$$\begin{array}{c}
4 \\
m^4 - 8m^2 + 8 = 0 \\
Y = aX + b
\end{array}$$
A1

Thus 
$$E(Y) = aE(X) + b$$
 and  $\sigma_Y = a\sigma_X$   
 $65 = 72a + b$  and  $5 = 8a$   
M1

So 
$$a = \frac{5}{8}$$
 and  $b = 20$  A1

## Question 9 (3 marks)

b.

We have 
$$\frac{dN}{dt} = \frac{10\ 000}{(t+3)^3}$$
 so that  $N = \int_0^\infty \frac{10\ 000}{(t+3)^3} dt$  A1

$$N = \int_{0}^{\infty} 10\ 000(t+3)^{-3} dt = \left[ -\frac{5000}{(t+3)^{2}} \right]_{0}^{\infty}$$
M1

$$N = 0 + \frac{5000}{9} = 556$$
 A1

## Question 10 (7 marks)

a.  $f(x) = a\cos(ax)$ 

The *x*-intercepts are given by  $a\cos(ax) = 0$ 

So the closest x-intercepts to the y-axis are  $x = \pm \frac{\pi}{2a}$ . (0, a)  $(\frac{-\pi}{2a}, 0)$ (0, a)  $(\frac{\pi}{2a}, 0)$ Cosine graph between  $\left[-\frac{\pi}{2a}, \frac{\pi}{2a}\right]$  A1

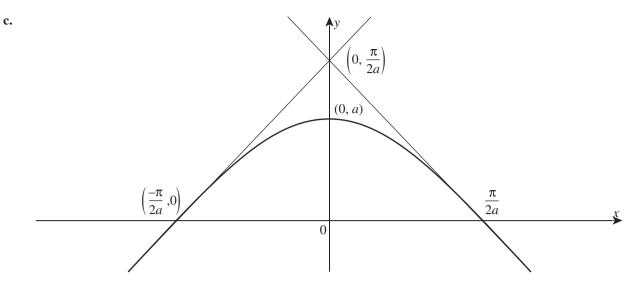
y-intercept at (0, a) A1

**b.** 
$$f'(x) = -a^{2} \sin(ax)$$

$$f'\left(-\frac{\pi}{2a}\right) = -a^{2} \sin\left(-\frac{\pi}{2}\right) = -a^{2}(-1) = a^{2}$$
Equation of tangent at  $\left(-\frac{\pi}{2a}, 0\right)$ :
$$y - 0 = a^{2}\left(x + \frac{\pi}{2a}\right)$$

$$y = \frac{a\pi}{2} + a^{2}x$$
A1
$$f'\left(\frac{\pi}{2a}\right) = -a^{2} \sin\left(\frac{\pi}{2}\right) = -a^{2}$$
Equation of tangent at  $\left(\frac{\pi}{2a}, 0\right)$ :
$$y - 0 = -a^{2}\left(x - \frac{\pi}{2a}\right)$$

$$y = \frac{a\pi}{2} - a^{2}x$$
A1



Area required = area of triangle formed by tangents and *x*-axis – area under curve

$$A = \frac{1}{2} \times \frac{\pi}{a} \times \frac{a\pi}{2} - 2 \int_0^{\frac{\pi}{2a}} a \cos(ax) dx$$

$$M = \frac{\pi^2}{4} - 2 [\sin(ax)]_0^{\frac{\pi}{2a}}$$

$$A = \frac{\pi^2}{4} - 2 \sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} - 2$$
, which is independent of  $a$ . A1

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