Neap:

Trial Examination 2013

# VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 1

## Suggested Solutions

**Question 1 (3 marks)**

**a.** $y = \frac{2}{1-5x} = 2(1-5x)^{-1} \Rightarrow \frac{dy}{dx} = -2(1-5x)^{-2} \times -5$

$\frac{dy}{dx} = \frac{10}{(1-5x)^2}$ A1

**b.** $f(x) = x\cos^2\left(\frac{\pi x}{2}\right)$

$= x \cdot 2 \times \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\left[-\sin\left(\frac{\pi x}{2}\right)\right] + \cos^2\left(\frac{\pi x}{2}\right)$ M1

$= -\pi x \cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) + \cos^2\left(\frac{\pi x}{2}\right)$

$f'\left(\frac{1}{2}\right) = -\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right)$

$= -\frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2} - \frac{\pi}{4}$ A1

**Question 2 (4 marks)**

As $\frac{dy}{dx} = \frac{6}{\sqrt{x^3}}$, we have $y = \int \frac{6}{\sqrt{x^3}}dx = \int 6x^{-\frac{3}{2}}dx = -\frac{12}{\sqrt{x}} + c$ M1

$4 = \frac{-12}{\sqrt{\frac{1}{4}}} + c \Rightarrow c = 28$

Equation of the curve is $y = -\frac{12}{\sqrt{x}} + 28$ A1

**Question 3 (2 marks)**

$f(g(x)) = f\left(1 + e^{\frac{x}{a}}\right) = a\log_e\left(1 + e^{\frac{x}{a}} - 1\right) = a\log_e\left(e^{\frac{x}{a}}\right) = a \times \frac{x}{a} = x$ M1

As $f(g(x)) = x = f(f^{-1}(x))$, and both $f$ and $g$ are one to one functions, they are inverses. A1

**Question 4 (5 marks)**

**a.** Define the random variable $X$ as "the number of passengers who show up for the flight".

Pr(everyone gets a seat) = $\Pr(X \le 40) = 0.1 + 0.45 + 0.35 = 0.9$ A1

**b.** Define $Y$ as the number of passengers who do not show up

Then $Y = 43 - X$ so that

$E(Y) = 43 - E(X)$ M1

$E(Y) = 43 - (38\times0.1 + 39\times0.45 + 40\times0.35 + 41\times0.05 + 42\times0.04 + 43\times0.01)$

$= 43 - (3.8 + 17.55 + 14 + 2.05 + 1.68 + 0.43)$

$= 43 - 39.51$

$= 3.49$ A1

**c.** Pr($X$ = 38 | not all seats filled) =

$$\Pr(X=38\,|\,X<40) = \frac{\Pr(X=38)}{\Pr(X<40)}$$ M1

$$= \frac{0.1}{0.1+0.45} = \frac{10}{55} = \frac{2}{11}$$ A1

**Question 5 (7 marks)**

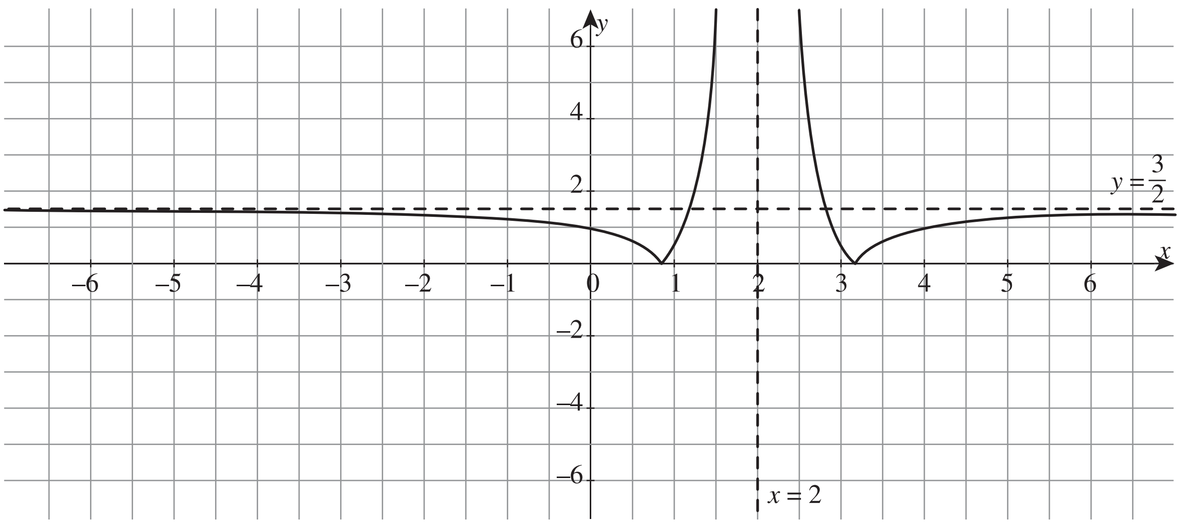
**a.** A vertical asymptote with equation $x = 2$ means $b = -2$. A1

$f(x) = \dfrac{a}{(x-2)^2}$ and $f(0) = -\dfrac{1}{2} \Rightarrow \dfrac{a}{4} = -\dfrac{1}{2} \Rightarrow a = -2$ A1

**b.** $x$-intercepts by solving $\dfrac{3}{2} - \dfrac{2}{(x-2)^2} = 0$

$(x-2)^2 = \dfrac{4}{3}$

$x = 2 \pm \dfrac{2}{\sqrt{3}} \approx 2 \pm 2(0.6) \approx 0.8$ and $3.2$




*Vertical asymptote $x$ = 2 and horizontal asymptote $y = \frac{3}{2}$* A1

*x-intercepts correct and shown on graph* A1

*correct shape* A1

**c.** Starting with $f(x) = \dfrac{-2}{(x-2)^2}$, under a dilation of scale factor 2 from the $y$-axis we have

$x \to \dfrac{x}{2}$, so $y = \dfrac{-2}{\left(\dfrac{x}{2}-2\right)^2}$ M1

A translation of 3 units in the negative direction of the $x$-axis, $x \to x+3$

So $y = \dfrac{-2}{\left(\dfrac{x+3}{2}-2\right)^2} = \dfrac{-2}{\left(\dfrac{x-1}{2}\right)^2} = \dfrac{-8}{(x-1)^2}$ A1

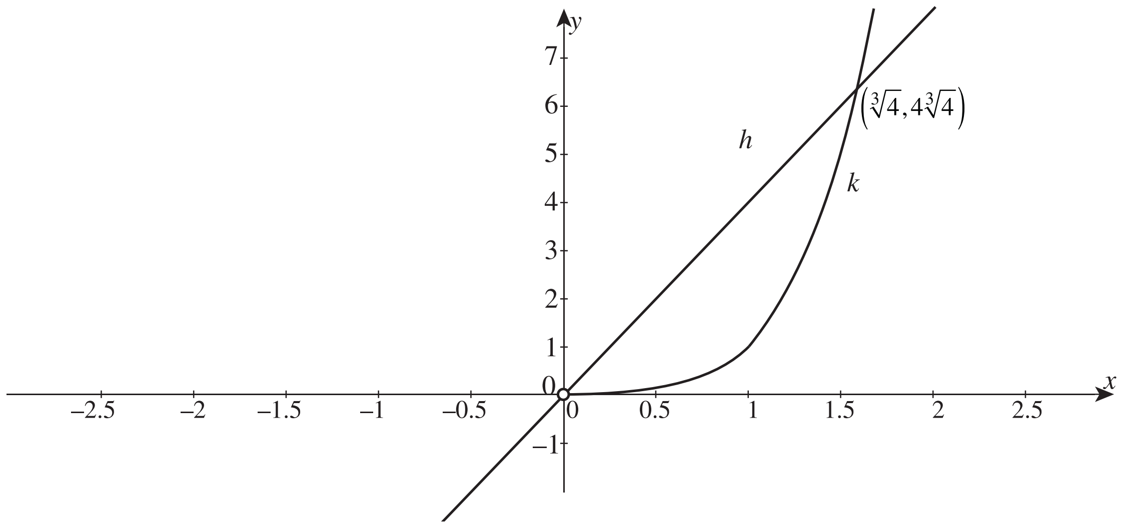
**Question 6 (4 marks)**

**a.** $h(x) = f(g(x)) = f\left(e^{2x}\right) = \log_e\left(e^{2x}\right)^2 = \log_e\left(e^{4x}\right) = 4x$ A1

$k(x) = g(f(x)) = g(\log_e(x^2)) = e^{2\log_e(x^2)}$

$k(x) = e^{\log_e(x^4)} = x^4$ A1

**b.**




$x^4 = 4x \Rightarrow x(x^3 - 4) = 0$

$x = \sqrt[3]{4}$ is the only solution in the domain

*graphs of h and k on correct domain with (0,0) excluded* A1

$\left(\sqrt[3]{4}, 4\sqrt[3]{4}\right)$ *as intersection point* A1

**Question 7 (3 marks)**

Given $2\log_3(x+4) - \log_3(-x) = 2$ we have $x + 4 > 0$ and $-x > 0$, which together means $-4 < x < 0$.

$\log_3(x+4)^2 - \log_3(-x) = 2$

$$\log_3\left(\frac{(x+4)^2}{-x}\right) = 2 = \log_3(3^2)$$ M1

$(x+4)^2 = -9x$

$x^2 + 8x + 16 = -9x$

$x^2 + 17x + 16 = 0$

$(x+16)(x+1) = 0$

$x = -16, \ -1$ A1

Since $-4 < x < 0$, $x = -1$ is the only allowable solution. A1

**Question 8 (5 marks)**

**a.** **i.** As $X$ is a probability density function,

$$\int_0^2 kx(4-x^2)dx = 1$$

$$\left[2x^2 - \frac{x^4}{4}\right]_0^2 = \frac{1}{k}$$

$$4 = \frac{1}{k} \Rightarrow k = \frac{1}{4}$$ A1

**ii.** The median, $m$, is such that $\int_0^m \frac{1}{4}x(4-x^2)dx = \frac{1}{2}$ M1

$$\left[2x^2 - \frac{x^4}{4}\right]_0^m = 2$$

$$2m^2 - \frac{m^4}{4} = 2$$

$$m^4 - 8m^2 + 8 = 0$$ A1

**b.** $Y = aX + b$

Thus $E(Y) = aE(X) + b$ and $\sigma_Y = a\sigma_X$ M1

$65 = 72a + b$ and $5 = 8a$

So $a = \frac{5}{8}$ and $b = 20$ A1

**Question 9 (3 marks)**

We have $\frac{dN}{dt} = \frac{10\ 000}{(t+3)^3}$ so that $N = \int_0^\infty \frac{10\ 000}{(t+3)^3}dt$ A1

$$N = \int_0^\infty 10\ 000(t+3)^{-3}dt = \left[-\frac{5000}{(t+3)^2}\right]_0^\infty$$ M1

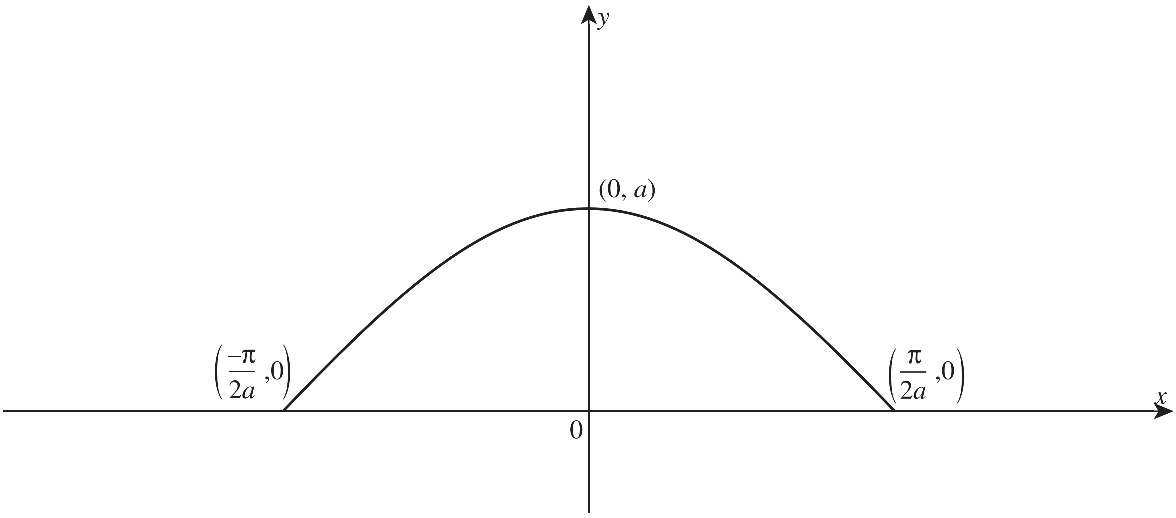
$$N = 0 + \frac{5000}{9} = 556$$ A1

**Question 10 (7 marks)**

**a.** $f(x) = a\cos(ax)$

The $x$-intercepts are given by $a\cos(ax) = 0$

So the closest $x$-intercepts to the $y$-axis are $x = \pm\frac{\pi}{2a}$.




*Cosine graph between* $\left[-\frac{\pi}{2a}, \frac{\pi}{2a}\right]$ A1

y-*intercept at (0, a)* A1

**b.** $f'(x) = -a^2 \sin(ax)$

$f'\left(-\frac{\pi}{2a}\right) = -a^2 \sin\left(-\frac{\pi}{2}\right) = -a^2(-1) = a^2$ M1

Equation of tangent at $\left(-\frac{\pi}{2a}, 0\right)$:

$y - 0 = a^2\left(x + \frac{\pi}{2a}\right)$

$y = \frac{a\pi}{2} + a^2 x$ A1

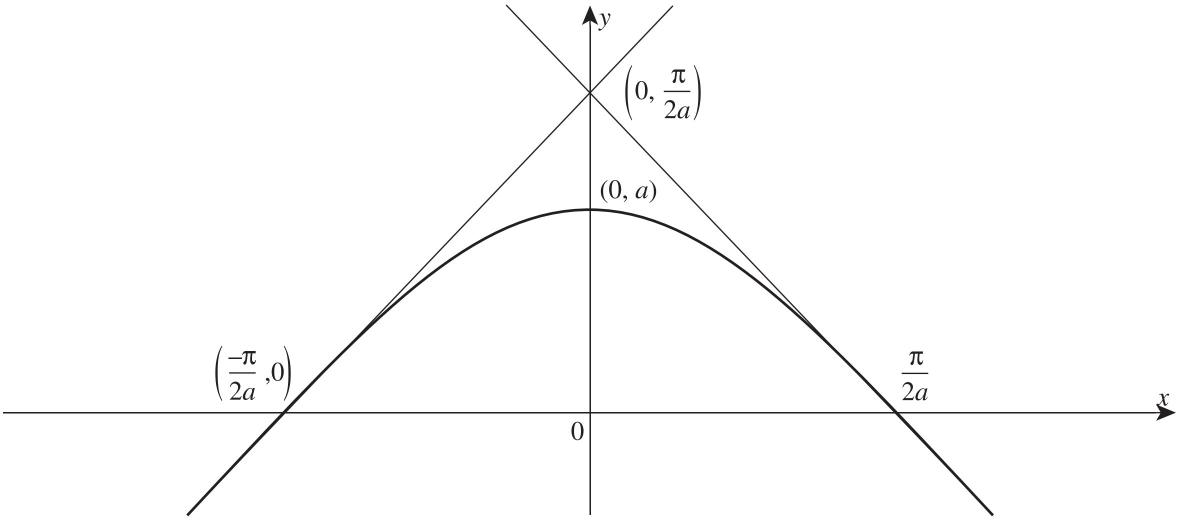
$f'\left(\frac{\pi}{2a}\right) = -a^2 \sin\left(\frac{\pi}{2}\right) = -a^2$

Equation of tangent at $\left(\frac{\pi}{2a}, 0\right)$:

$y - 0 = -a^2\left(x - \frac{\pi}{2a}\right)$

$y = \frac{a\pi}{2} - a^2 x$ A1

**c.**




Area required = area of triangle formed by tangents and $x$-axis – area under curve

$$A = \frac{1}{2} \times \frac{\pi}{a} \times \frac{a\pi}{2} - 2\int_0^{\frac{\pi}{2a}} a\cos(ax)dx$$ M1

$$A = \frac{\pi^2}{4} - 2\left[\sin(ax)\right]_0^{\frac{\pi}{2a}}$$

$$A = \frac{\pi^2}{4} - 2\sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} - 2$$ , which is independent of $a$. A1