QATs
Quality Assessment Tasks

UNIT 4 ■ OUTCOMES 1, 2 & 3

VCE Mathematics Methods (CAS)

SCHOOL-ASSESSED COURSEWORK

Introduction

Outcome 1
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

Outcome 2
Apply mathematical processes in non-routine contexts, and analyse and discuss these applications of mathematics.

Outcome 3
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modeling or investigative techniques or approaches.

Task

Analysis Task- range of problems

The task has been designed to allow achievement up to and including the highest level in the performance Descriptors. It covers a broad range of **key knowledge** and **key skills** over the three outcomes for Unit 4.

It will contribute 20 out of the total (40) marks allocated for SAC in Unit 4.

This task will be marked out of 80 and then will be converted to a proportion of the contribution of this task to SAC in this unit.

The marks for each part are indicated in brackets. Answer in space provided or as directed.

You have 120 minutes over no more than two days. Your work in progress will be collected.

You can access your logbook and an approved CAS calculator.

T Indicates where use of the technology is specifically required in order to answer the question.

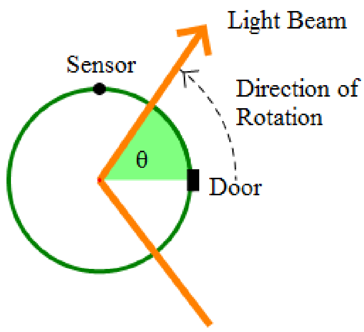
Your teacher will advise you of any variation to these conditions.

NAME:

Task

PART 1

Question 1 (9 marks)



A lighthouse is being built on the top of a cliff. The Lantern Room at the top of the light house is to have a radius of 1.5 m.

The lantern itself is to rotate once every 12 seconds.

The function for the intensity of the reflected light in the room is of the form:

$$V(t) = A\sin(n\,t) + k$$

a) Determine the value of n, for the function. (1 mark)

A quarter of the way around the room from the door is a light sensor that measures the amount of reflected light in the room on a constant basis. The sensor produces the strongest signal when the lamp is shining directly at the it and the weakest signal when it is pointing the opposite way.

The sensor registers signal strength in volts with the highest value being 6.5 volts and the lowest value being 0.5 volts.

b) Write down the amplitude, A, for the function describing the light intensity measured by the sensor. (1 mark)

c) What is the median voltage output of the sensor? (1 mark)

d) Write down the complete equation for output of the sensor in the form $V = Asin(nt) + k$. (1 mark)

Task

e) Using the equation you have developed, show algebraically that when V is 2 volts, then t = 11 seconds. (3 marks) **T**

f) There is a second time during the 12 second rotation where the output will be 2 volts. Determine this second value. (2 marks)

Task

Question 2 (10 marks)

A second beam of light is to follow the first with a 4 second delay.

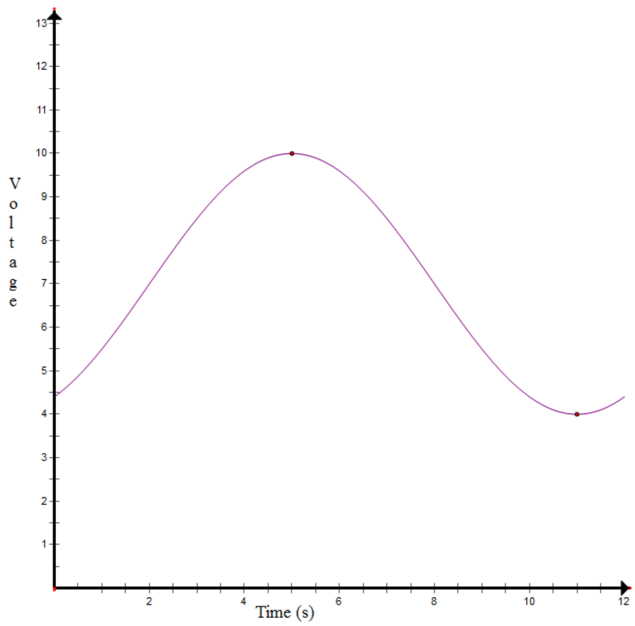
a) What is the smaller angle between the two beams? (1 mark)

b) If the equation for the second beam is:

$$V_2(t) = 3\sin\left(\frac{\pi}{6}(t-4)\right) + 3.5$$

explain how the value found in a) is arrived at.(1 mark)

The voltage curve resulting from the two lamps rotating together is shown below.



c) Explain how this curve can be produced mathematically. Give the name for this technique in your answer. (2 marks)

Task

d) Show, using calculus, that the derivative of the combined function is

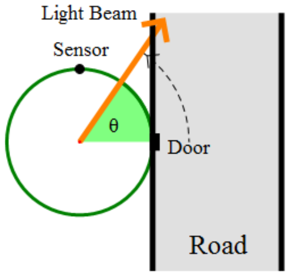
$$\frac{dV_T}{dt} = \frac{\pi}{2}\left(cos\left(\frac{\pi\, t}{6}\right) + \ cos\left(\frac{\pi}{6}(t-4)\right)\right)$$ (3 marks)

e) Determine the maximum and minimum voltages for this curve. (3 marks) **T**

Task

Question 3 (21 marks)

A road runs North-South past the right hand side of the lighthouse. The base of the lighthouse has a diameter of 8 metres.

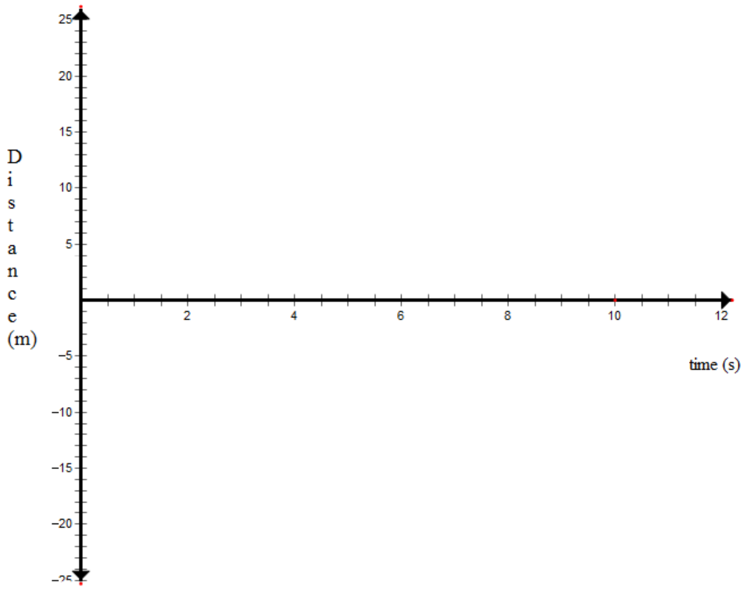


a) Write down the equation of the point of contact between the first beam and the left edge of the road in the form $D(t) = A\tan(nt)$. (2 marks)

b) What is the domain for one full rotation? (1 mark)

c) For what section(s) of the domain is the function D(x) defined? (2 marks)

d) Graph the distance vs time graph of the first beam for one rotation. (3 marks)



Task

e) When t = 2 seconds, how far along the road is the beam? Quote your answer to two decimal places. (1 mark) **T**

f) Remember that the second beam is 4 seconds behind. For what times during one rotation is the second beam lighting up the road, relative to the first beam? Take t = 0 seconds to be when the first beam is at right angles to the road. (2 marks)

g) Now express the equation of the position of the second beam on the road in the form
$D(x) = A\,tan\big(n(t-h)\big)$. (2 marks)

h) Graph, using that provided in question **d)**, the curve of this second beam on the same set of axes for the same domain. (3 marks)

i) For how long during a single rotation of the lamp are both beams shining on the road? (2 marks)

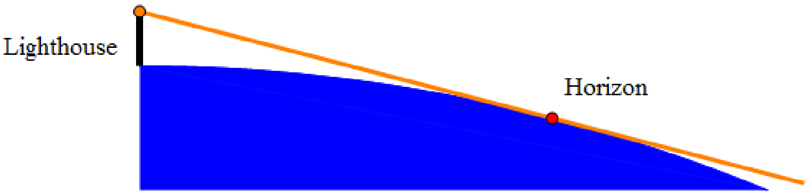
j) For what times during a rotation is none of the road lit by either beam? (3 marks)

Task

PART 2

Question 4 (16 marks)

Light travels in straight lines. Since the Earth is curved, the straight line distance from an observer to the visible horizon increases with height.

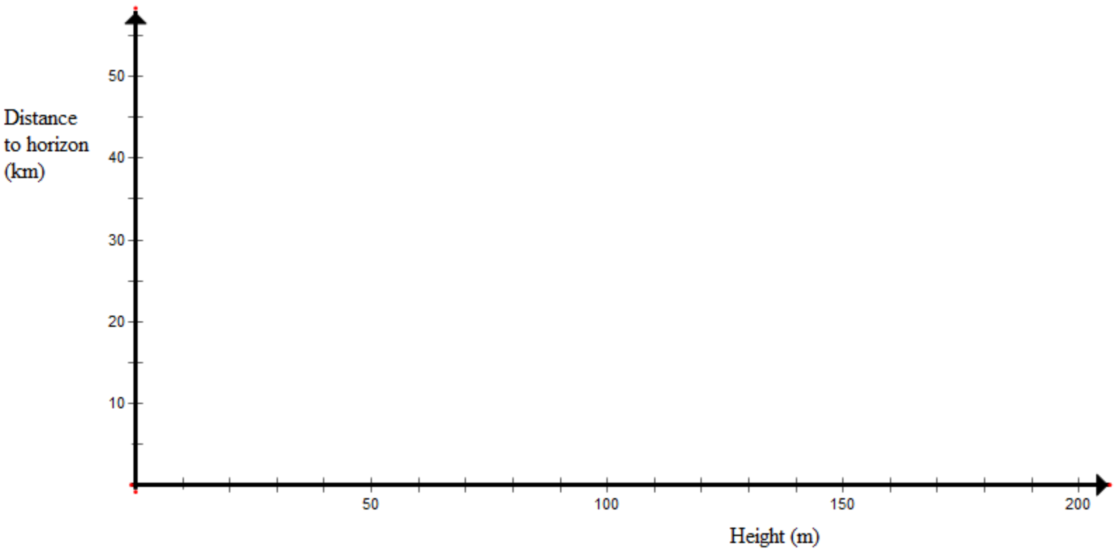


The relationship between the height of an observer, or lighthouse, is modelled by the equation:

$$d = 3.57\sqrt{h}$$

Where d is distance in kilometres and h is height above sea level in metres.

a) Sketch this function on the axes provided. (2 marks)



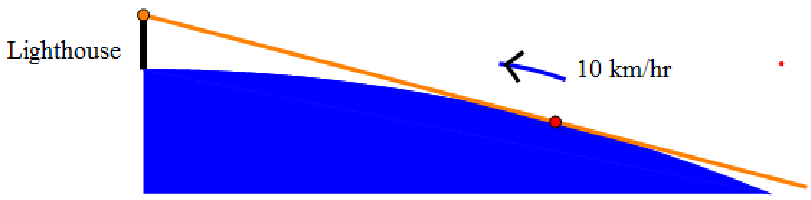
b) From a height of 10 metres above sea level, what is the straight line distance to the horizon to the nearest metre? (2 marks) **T**

Task

c) At what height above sea level, to 2 decimal places, does the lamp need to be if it is to be seen from a distance of 50 km? (2 marks) **T**

d) Differentiate d in terms of h. (1 mark)

A ship is travelling directly toward the lighthouse at a rate of 10 km per hour.



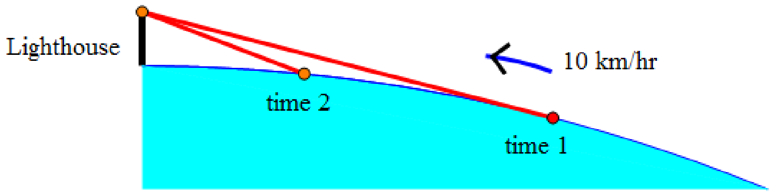
e) Write an expression for this rate in terms of distance, d, and time, t. (2 marks)

f) Using related rates and the chain rule, write down an expression for $\frac{dh}{dt}$. (3 marks)

Task

g) Explain what this rate is measuring. (2 marks)

As the ship approaches the lighthouse, more and more of the tower can be seen.

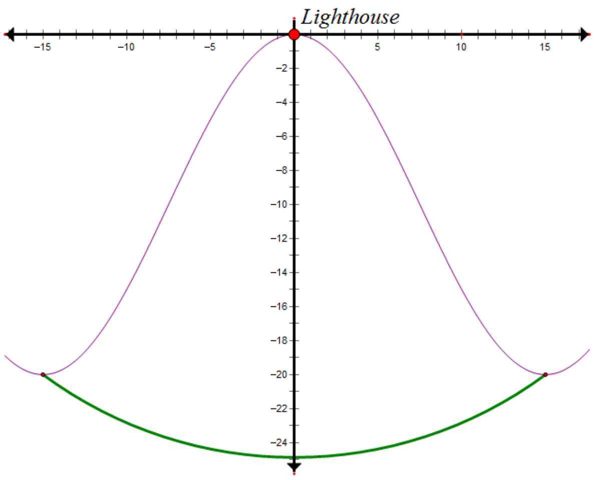


h) When the ship is at a distance of 100 km, at what rate does the tower appear to be coming revealing itself? Include units in your answer. (2 marks) **T**

Task

Question 5 (10 marks)

The lighthouse is situated at the top of a large bay where the coastline can be modelled by the function $k(x) = 10\,sin\left(\frac{\pi(x+7.5)}{15}\right) - 10$. The light from the two lamps extends out and can just be seen from the two furthest points of land.



a) Determine the coordinates of these two points. (2 marks) **T**

b) The light beams sweep out a circular arc across the water between these two points. Determine the radius of this arc. (2 marks) **T**

c) Write down a rule for this arc, A(x), including the domain. (3 marks)

Task

The surface area of the water in the bay that is touched by the light is given by

$$Area = 702.188 - \left|\int_a^b k(x)dx\right|$$

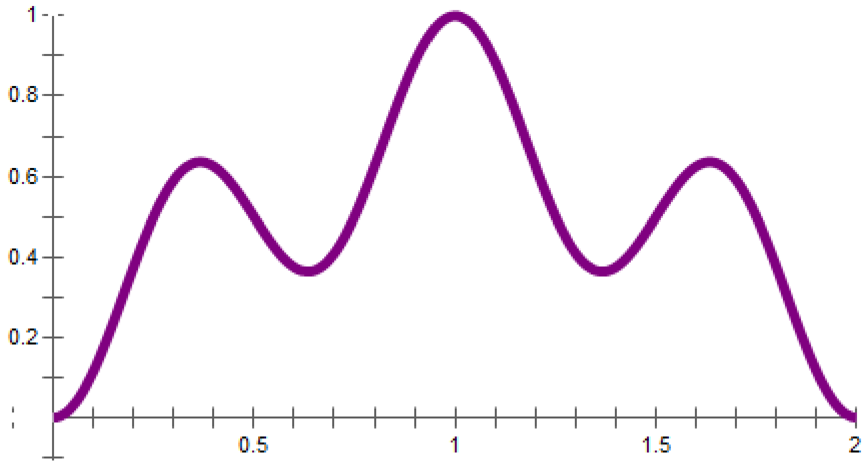
d) Use integral calculus to set up an expression for determining the area of water touched by the lights. *Do not attempt to evaluate the integral*. (1 mark)

e) Integrate this expression and find the area correct to two decimal places. (2 marks) **T**

Task

Question 6 (14 marks)

The lighthouse is situated on the top of a cliff with a cross section as shown. The scale is 1 unit = 100 metres:



This can be modelled using the function $f(x) = \dfrac{-\cos(3\pi x)}{4} - \dfrac{\cos(\pi x)}{4} + \dfrac{1}{2}$

The cliff face has become fragile over time and needs to be stabilised. To do this engineers are thinking about spraying concrete on the cliff. This means that the area of the cliff face needs to be calculated.

a) Find an indefinite integral for *f(x)*. (3 marks)

b) Using the integral from **a)** find the area under the curve for the given domain. (3 marks) **T**

Task

c) The engineering company considers that a layer of concrete 10 cm thick should be enough to keep the cliff from deteriorating any more. What volume of concrete is needed? (1 mark) **T**

To make sure that the concrete does not slide off the cliff, the engineers erect a wire mesh first. If this were not done the concrete would settle into the shape of a rectangular prism 10 cm thick and 200 metres wide.

d) Find the average value of the function $f(x)$ over the given domain of 200 metres. (2 marks) **T**

The entire amount of concrete is to be pumped out of cylindrical storage tank with a radius of 2 metres.

e) To what height up the cylinder would the concrete reach? Quote your answer to the nearest centimetre. (2 marks) **T**

f) If the concrete is to be pumped out at a rate of 2 m^3 per minute, at what rate will the height of the concrete in the tank drop? Quote your answer in exact form. (2 marks) **T**

g) At what rate would the concrete need to be pumped if the desired change in height was to be 20cm/minute? (1 mark) **T**

Teacher Advice

Assessment planning

This is the Analysis Task, as suggested to be undertaken in weeks 7 and 8 in the sample teaching sequence on page 142 of the VCAA Study Design.

This task covers a broad range of **key knowledge** and **key skills** across Outcomes 1, 2 and 3 as per the VCAA VCE Study Design 2006-14.

This task covers assessment in:

- domain, maximal domain, range and asymptotic behaviour of functions;
- features which link the graph of a function to the graph of the corresponding gradient function or its numerical values, the tangent and normal to a curve at a given point and how the sign and magnitude of the derivative of a function can be used to describe key features of the function and its derivative function;
- chain, product and quotient rules for differentiation;
- related rates of change;
- properties of anti-derivatives and definite integrals.

This task contributes 20 of the 40 SAC marks in Unit 4.
The coursework scores for this task are:

Outcome 1	7 marks	35%
Outcome 2	8 marks	40%
Outcome 3	5 marks	25%
TOTAL	20 marks	

This weighting can be used in the conversion of their mark out of 50.
For example, a score of 40 results in:

OUTCOME 1	OUTCOME 2	OUTCOME 3
40/80*20*0.35	40/80*20*0.4	40/80*20*0.25
= 3.5	= 4	= 2.5
= 4 (rounded up)		

The above can be established in an Excel file.

Alternatively marks can be allocated according to the table on the next page.

	Outcome 1 (7)	**Outcome 2 (8)**	**Outcome 3 (5)**
Question			
Part 1			
1 a)	1		
b)	1		
c)	1		
d)	1		

Teacher Advice

e)	1	1	1
f)		2	
2 a)		1	
b)		2	
c)		2	
d)	2	1	
e)	1	1	1
3 a)		2	
b)		1	
c)		2	
d)	2	1	
e)			1
f)		2	
g)	1	1	
h)	2	1	
i)		2	
j)	2	1	
Part 2			
4 a)	2		
b)			2
c)			2
d)	1		
e)		1	
f)	2	1	
g)		2	
h)		1	1
5 a)		1	1
b)		1	1
c)	2	1	
d)		1	
e)			2
6 a)	2		
b)	2		1
c)			1
d)			2
e)			2
f)	1		1
g)	1		1

Teacher Advice

This QAT has been designed to meet the highest level in the Performance Descriptors provided by

VCAA for each outcome in Unit 3 in the Assessment Handbook.

Notes on the two parts of the task

Both PARTS 1 and 2 each contain 3 extended response questions focussing on the properties of sine, cosine and tangent functions and their graphs, their derivatives and integrals. Other functions from the course are also covered to some degree, specifically square roots and semicircles.

Solution Pathway

PART 1

Question 1 (9 marks)

a)

$Period = \frac{2\pi}{n}$

So $n = \frac{2\pi}{Period} = \frac{2\pi}{12} = \frac{\pi}{6}$

(1 mark)

b)

$A = \frac{6.5-0.5}{2} = 3$ volts **(1 mark)**

c) *Halfway between the maximum and minimum outputs, so 3.5 volts.* **(1 mark)**

d)

$V_1(t) = 3\,sin\left(\frac{\pi}{6}t\right) + 3.5$ **(1 mark)**

e) *Substituting V = 2 gives* $2 = 3\sin\left(\frac{\pi}{6}\,t\right) + 3.5$

$\sin\left(\frac{\pi}{6}\,\theta\right) = \frac{-1.5}{3}$ **(1 mark)**

$\frac{\pi\,t}{6} = sin^{-1}\left(\frac{-1}{2}\right)$

$\frac{\pi\,t}{6} = \frac{-\pi}{6}$

t = -1 seconds **(1 mark)**

However, the domain is between 0 and 12 seconds so t = 12 – 1 = 11 seconds, as required. **(1 mark)**

f) *The median value of 3.5 volts occurs three times in 12 seconds, at t = 0, t = 6 and t = 12. For the first half of the cycle the voltage is above this and for the second half it is below. Using symmetry the other value of t is 6 + 1 = 7 seconds.*

Method/Reasoning: 1 mark Answer: 1 mark. **(2 marks)**

Alternatively, if both values have been found in solving e) give the marks for working reasoning based on that solution.

Solution Pathway

Question 2 (10 marks)

a) *A 4 second delay means* $\frac{1}{3}$ *of a rotation or* $\frac{2\pi}{3}$ *radians for the smaller angle.*

(1 mark)

b) *In transformation form the equation is* $V_2(t) = A\sin\left(n(t-h)\right) + k$

In this function the value of $n = \frac{\pi}{6}$ *and the value of* $h = 4$ *. Expanding the inside brackets means multiplying these values together giving* $\frac{2\pi}{3}$ **(1 mark)**

c) *The functions are added together.* (1 mark)

Specifically the y values are added together. This is known as Addition of ordinates. (1 mark)

(2 marks)

d)

$$\frac{dV_T}{dt} = \frac{\pi}{2}\left(cos\left(\frac{\pi\,t}{6}\right) - cos\left(\frac{\pi}{(t6} - 4\right)\right)$$

Start with $V_T = 3\,sin\left(\frac{\pi}{6}t\right) + 3.5 + 3\sin\left(\frac{\pi}{6}(t - 4\right)) + 3.5$ **(1 mark)**

Differentiate: $\frac{dV_T}{dt} = \frac{\pi}{2}cos\left(\frac{\pi\,t}{6}\right) + \frac{\pi}{2}cos\left(\frac{\pi}{6}(t - 4\right))$ **(1 mark)**

Factorise to give: $\frac{dV_T}{dt} = \frac{\pi}{2}\left(cos\left(\frac{\pi\,t}{6}\right) + cos\left(\frac{\pi}{6}(t - 4\right)\right)$ **(1 mark)**

(3 marks)

e)

Let $\frac{dV_T}{Dt} = 0 = \frac{\pi}{2}\left(cos\left(\frac{\pi\,t}{6}\right) - cos\left(\frac{\pi}{6}(t - 4\right)\right)$

Divide by $\frac{\pi}{2}$: $0 = cos\left(\frac{\pi\,t}{6}\right) - cos\left(\frac{\pi}{6}(t - 4)\right)$ (1 mark)

$$cos\left(\frac{\pi\,t}{6}\right) = cos\left(\frac{\pi}{6}(t - 4\right))$$

From here or earlier students could use the calculator to determine that t = 5 and 11 and then find the y values of the coordinates.

Solution Pathway

(5, 10) and (11, 4) (2 marks)

(3 marks)

Alternatively, by using knowledge of the two parent curves and addition of ordinates, students can determine that the x-value of the maximum of the combined function is midway between the maximums of the two parent functions. The same is true for the minimums. Then they can find the corresponding y-values. (2 marks)

Question 3 (21 marks)

a) A = 4 (1 mark) n = $\frac{\pi}{6}$ (1 mark) **(2 marks)**

b) $0 \leq t \leq 12$ **(1 mark)**

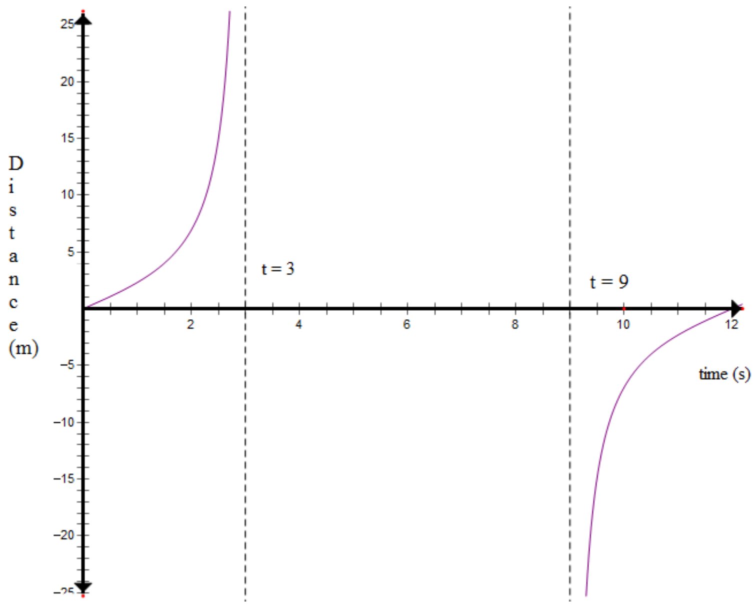
c) For $0 \leq t < 3 \cup 9 < t \leq 12$ seconds (1 mark for each) **(2 marks)**

d)

Shape: 1 mark

Position: 1 mark

Undefined region correct and clear: 1 mark



(3 marks)

e) $f(2) = 4\tan\left(\frac{2\pi}{6}\right) = 6.93$ metres **(1 mark)**

Solution Pathway

f) *The total rotation period is 12 seconds. A quarter of a rotation is 3 seconds long. The second beam does not begin to touch the road until after the third second. It is then in contact for another 6 seconds (half the rotation time) so 7 seconds.*

$$1 < t < 7 \text{ seconds}$$

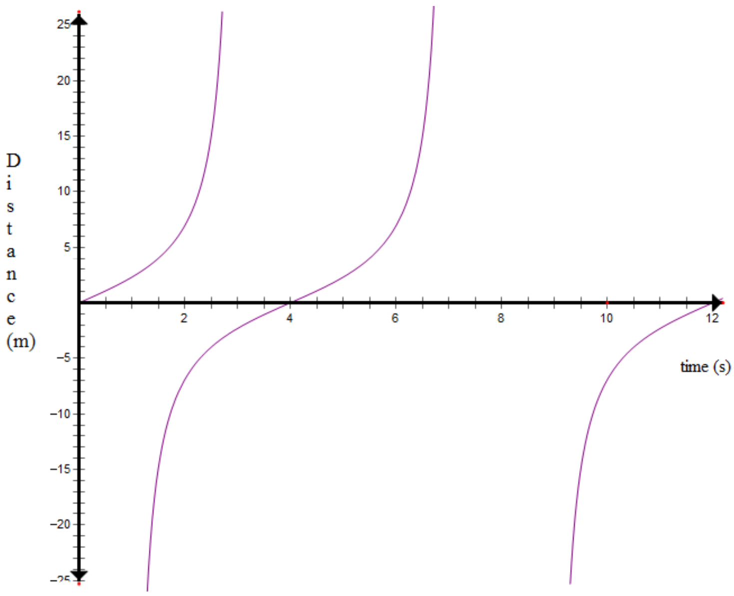
(2 marks)

g) $D(x) = 4\,tan(\frac{\pi}{6}(t-4))$

All of A, n and h correct – 2 marks

Only 2 correct – 1 mark **(2 marks)**

h)



(3 marks)

i)

$1 < t < 3$ *seconds* (1 mark for each value) **(2 marks)**

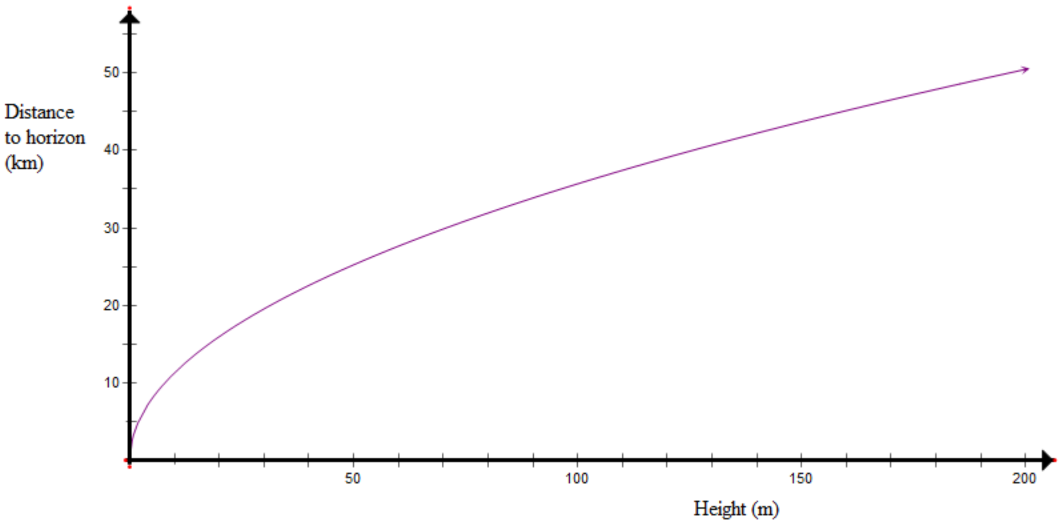
j) $7 \le t \le 9$ *seconds*. 1 mark for each value and a mark for realising $\le$ not $<$. **(3 marks)**

Solution Pathway

PART B

Question 4 (16 marks)

a)



Shape: 1 mark Position: 1 mark **(2 marks)**

b) *Let h = 10, therefore d = 11.289 km.*
Calculation 1 mark. 3 decimal places or conversion to metres, 1 mark. **(2 marks)**

c) $Let\ d = 50 = 3.57\sqrt{h}$

$Therefore\ h = \left(\frac{50}{3.57}\right)^2$ 1 mark

$h = 196.16\ m$ 1 mark

(2 marks)

d)

$$\frac{dd}{dh} = \frac{3.57}{2\sqrt{h}}$$

(1 mark)

Solution Pathway

e) *This is a speed so* $\frac{dd}{dt} = 10$

1 mark for recognising the derivative notation is to be used and 1 mark for setting it equal to 10.

(2 marks)

f)

$\frac{dd}{dh} = \frac{3.57}{2\sqrt{h}}$ and $\frac{dd}{dt} = 10$

To get $\frac{dh}{dt} = \frac{dh}{dd} \times \frac{dd}{dt}$ (1 mark)

So $\frac{dh}{dt} = \frac{2\sqrt{h}}{3.57} \times 10$ (1 mark)

$\frac{dh}{dt} = \frac{20\sqrt{h}}{3.57} = \frac{2000\sqrt{h}}{357}$ (1 mark)

(3 marks)

g) *This rate is the speed at which the light house is coming into view from the top down*.

1 mark for recognising that it had to do with the amount of the light house that was visible, and 1 mark for realising it was from the top down. **(2 marks)**

h)

$\frac{dh}{dt} = \frac{200\sqrt{100}}{357} = 5.60$ metres per hour. (Calculation: 1 mark. Units: 1 mark) **(2 marks)**

Solution Pathway

Question 5 (10 marks)

a) *The period of k(x) is 30. The function is symmetrical about a maximum at (0, 0) to the x values of these two points of land are -15 and 15 km respectively.* (1 mark)

The amplitude of k(x) is 10, so the y value of both points is -20 km. (1 mark)

The coordinates of both points are (-15, -20) and (15, -20). **(2 marks)**

b) *Use Pythagoras and the points (0, 0) and (-15, -20)* (1 mark)

$Radius = \sqrt{(15)^2 + (20)^2}$ = 25 km (1 mark)

(2 marks)

c) *This is the equation of a semicircle with radius 25.*

Given the graph, it is the lower half of the semicircle, so is negative. (1 mark)

$A(x) = -\sqrt{625 - x^2}$ (1 mark)

Domain: $-15 \leq x \leq 15$ (1 mark)

(3 marks)

d) *Area* = $702.188 - \left|\int_{-15}^{15} 10\, sin\left(\frac{\pi(x+7.5)}{15}\right) - 10\, dx\right|$ **(1 mark)**

e) *Area* = $702.188 - \left|\left[-\frac{150}{\pi} cos\left(\frac{\pi(x+7.5)}{15}\right) - 10x\right]_{-15}^{15}\right|$ (1 mark)

Substituting in terminal values

Area = $702.188 - 300 = 402.188\ km^2$ (1 mark)

(2 marks)

Solution Pathway

Question 6 (14 marks)

a) Find an indefinite integral for *f(x)*.

$$F(x) = \frac{-\sin(3\pi x)}{12\pi} - \frac{\sin(\pi x)}{4\pi} + \frac{x}{2} + C$$

1 mark for each component.

Note: the +C is optional since the question asked for "an indefinite integral". **(3 marks)**

b)

$Area = \left[\frac{-\sin(3\pi x)}{12\pi} - \frac{\sin(\pi x)}{4\pi} + \frac{x}{2}\right]_0^2$ (1 mark)

$= \left(\frac{-\sin(6\pi)}{12\pi} - \frac{\sin(2\pi)}{4\pi} + \frac{2}{2}\right) - 0$ (1 mark)

= 1 unit^2*. Now 1 unit = 100 m so Area =* $100^2\ m^2 = 10\ 000\ m^2$ (1 mark)

(3 marks)

c) 10 000 m^2 x 0.01 m = 100 m^3 of concrete. **(1 mark)**

d) *In general average value* $= \frac{1}{b-a}\int_a^b f(x)dx$

So for this function average value $= \frac{1}{2}\int_0^2 \frac{-cos(3\pi x)}{4} - \frac{cos(\pi x)}{4} + \frac{1}{2}dx$

$= \frac{1}{2} \times 1$ (1 mark)

But 1 unit = 100 metres so ½ unit = 50 m. (1 mark)

(2 marks)

Solution Pathway

e)

$V = \pi r^2 h$

So $h = \frac{V}{\pi r^2}$ (1 mark)

$h = \frac{100}{4\pi} = 7.96$ metres (1 mark) **(2 marks)**

f)

$\frac{dV}{dt} = 2$ and $\frac{dV}{dh} = 4\pi$ (1 mark)

So, $\frac{dh}{dt} = 2 \times \frac{1}{4\pi} = \frac{1}{2\pi}$ metres per minute (1 mark)

(2 marks)

g)

$0.2 = \frac{dh}{dt} \times \frac{1}{4\pi}$ so $\frac{dh}{dt} = 2 \times 4\pi = 8\pi = 25.13$ m^3/min **(1 mark)**