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MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2014

Section 1 – Multiple-choice answers

1. C	9. B	17. D
2. C	10. D	18. A
3. A	11. C	19. C
4. E	12. D	20. C
5. E	13. E	21. B
6. E	14. A	22. E
7. D	15. B	
8. B	16. C	

Section 1 – Multiple-choice solutions

Question 1

Since $a \in R^+$, immediately eliminate option **A** which is – 2.
Let $p(x) = 5x^4 + ax^3 - 16x^2$
Since $x + a$ is a factor, $p(-a) = 0$
So $5 \times (-a)^4 + a \times (-a)^3 - 16(-a)^2 = 0$

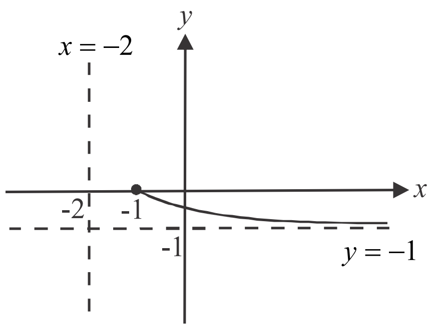
$$5a^4 - a^4 - 16a^2 = 0$$
$$4a^4 - 16a^2 = 0$$
$$4a^2(a^2 - 4) = 0$$
$$4a^2(a-2)(a+2) = 0$$

$a = 0$ or $a = \pm 2$

but $a \in R^+$ so $a = 2$
The answer is C.

Question 2

Sketch $y = f(x)$ on your CAS.



The domain of *f* is $x \in [-1, \infty)$.
Therefore $r_f = (-1, 0]$.

The answer is C.

Question 3

$f(x)=x^4+3x^2-1$

average rate of change $=\dfrac{f(2)-f(0)}{2-0}$

$=\dfrac{(16+12-1)-(-1)}{2}$

$=14$

The answer is A.

Question 4

$f:[1,\infty)\to R, f(x)=\sqrt{x-1}$ and $g(x)=x+2$

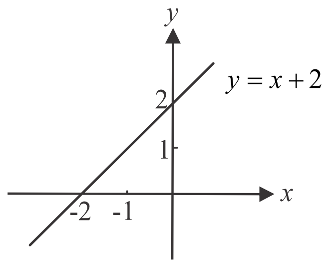
$f(g(x))$ is a composite function which exists iff $r_g \subseteq d_f$.

Since $d_f=[1,\infty)$, we require $r_g \subseteq [1,\infty)$.

Do a quick sketch of $y=x+2$.

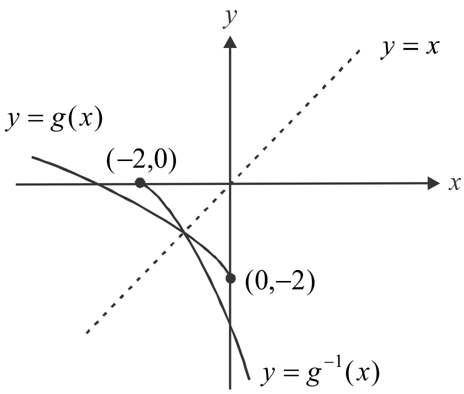
If $r_g=[1,\infty)$, then $d_g=[-1,\infty)$.

The answer is E.



Question 5

The graph of $y=g^{-1}(x)$ is a reflection of the graph of $y=g(x)$ in the line $y=x$.



The point $(0,-2)$ lies on the graph of $y=g(x)$, therefore the point $(-2,0)$ lies on the graph of $y=g^{-1}(x)$. Note that the graphs of the two functions should intersect on the line $y=x$.This latter point rules out option B.

The answer is E.

Question 6

X is normally distributed with $\mu=12$ and $\sigma=0.4$.

$\Pr(X<13)=\text{normCdf}(-\infty,13,12,0.4)$

$=0.99379...$

The closest answer is 0.9938.

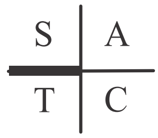
The answer is E.

Question 7

$\cos(3x) = -1$

$$3x = \pi + 2n\pi,\ n \in Z$$

$$x = \frac{\pi}{3} + \frac{2n\pi}{3}$$



The answer is D.

Note, if doing this question using CAS, the solution is $x = \frac{(2n-1)\pi}{3},\ n \in Z.$

Now $2n-1$ always gives an odd number, so the general solution is the set of odd numbered multiples of $\frac{\pi}{3}$. By hand, our solution is $x = \frac{\pi}{3} + \frac{2n\pi}{3} = \frac{\pi(2n+1)}{3}$. Similarly, $2n+1$ always gives an odd number, so the general solution is the set of odd numbered multiples of $\frac{\pi}{3}$.

Question 8

To find the rule for f^{-1}:

Method 1 – using CAS

Let $y = e^{1-2x}$

Swap x and y for inverse.

$$x = e^{1-2y}$$

Solve for y.

$$y = \frac{-(\log_e(x) - 1)}{2}, x > 0$$

$$= -\frac{1}{2}\log_e(x) + \frac{1}{2}$$

$$= \frac{1}{2} - \log_e\left(\sqrt{x}\right)$$

Method 2 – by hand

Let $y = e^{1-2x}$

Swap x and y for inverse.

$$x = e^{1-2y}$$

Rearrange:

$$\log_e(x) = 1 - 2y$$

$$2y = 1 - \log_e(x)$$

$$y = \frac{1}{2} - \frac{1}{2}\log_e(x)$$

$$y = \frac{1}{2} - \log_e\left(x^{\frac{1}{2}}\right)$$

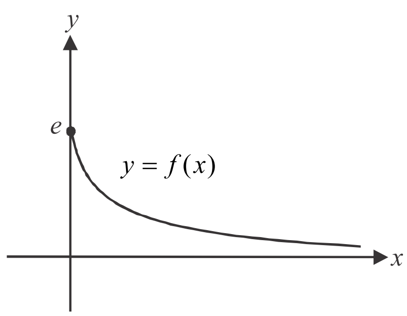
$$y = \frac{1}{2} - \log_e\left(\sqrt{x}\right)$$

To find the domain do a quick sketch of $y = f(x)$.

$$d_f = [0, \infty)$$

$$r_f = (0, e]$$

So $d_{f^{-1}} = (0, e]$

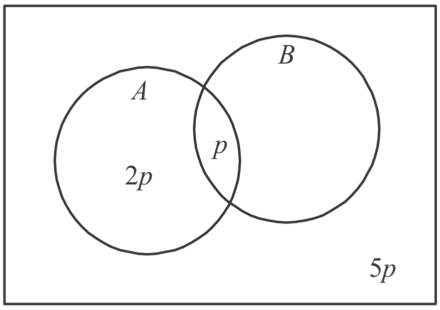


So $f^{-1} : (0, e] \to R, f^{-1}(x) = \frac{1}{2} - \log_e\left(\sqrt{x}\right)$

The answer is B.

Question 9

Draw a Venn diagram and label the probabilities that we have been given.



$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ (Addition formula - formula sheet)

So, using the diagram,

$$1 - 5p = 3p + \Pr(B) - p$$

$$\Pr(B) = -7p + 1 \qquad -(1)$$

Since A and B are independent,

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

$$p = 3p \times \Pr(B)$$

$$\Pr(B) = \frac{p}{3p}$$

$$= \frac{1}{3}$$

In (1) $\frac{1}{3} = -7p + 1$

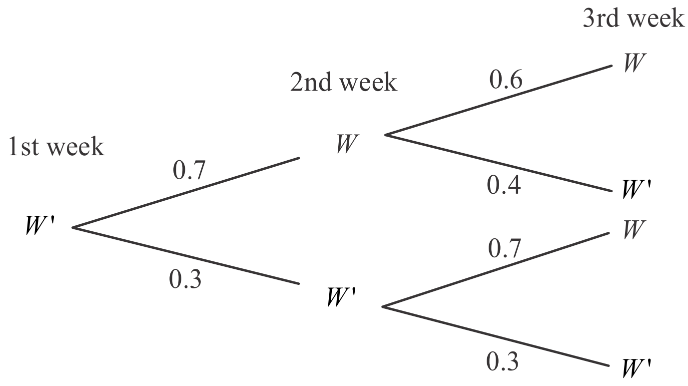
$$7p = \frac{2}{3}$$

$$p = \frac{2}{21}$$

The answer is B.

Question 10

Method 1 – using a tree diagram



$$
\begin{aligned}
&\Pr(W \text{ in } 3^{\text{rd}} \text{ week})\\
&= \Pr(W', W, W) + \Pr(W', W', W)\\
&= 1 \times 0.7 \times 0.6 + 1 \times 0.3 \times 0.7\\
&= 0.42 + 0.21\\
&= 0.63
\end{aligned}
$$

The answer is D.

Method 2 – using a transition matrix

$$
\begin{array}{c} \text{one week} \\ \begin{array}{cc} W & W' \end{array} \end{array}
$$

$$
T = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{matrix} W \\ W' \end{matrix} \text{ next week} \qquad \text{and } S_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} W \\ W' \end{matrix}
$$

$$
\begin{aligned}
S_3 &= T^2 S_1\\
&= \begin{bmatrix} 0.63 \\ 0.37 \end{bmatrix} \begin{matrix} W \\ W' \end{matrix}
\end{aligned}
$$

The probability she drives well in the third week is 0.63.
The answer is D.

Question 11

$$
f(x) = \begin{cases} 2e^{-x} & \text{if } 0 < x < \log_e(2) \\ 0 & \text{elsewhere} \end{cases}
$$

$$
\begin{aligned}
\Pr\left(X < \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} 2e^{-x}\,dx\\
&= 0.78693...
\end{aligned}
$$

The closest answer is 0.7869.
The answer is C.

Question 12

<u>Method 1</u> – using CAS

Define $f(x)=\log_e\left(\frac{x}{2}\right)$

Key in the functional equation and hit ENTER. $f(2(x+y))=f(2x)+f(2y)$

You don't get TRUE, which means it is only sometimes true.

Define $f(x)=x-2$.

Copy and paste the functional equation and hit ENTER.

Again you don't get TRUE.

Define $f(x)=e^{\frac{x}{2}}$.

Copy and paste the functional equation and hit ENTER.

Again you don't get TRUE.

Define $f(x)=\frac{x}{2}$.

Copy and paste the functional equation and hit ENTER.

This time you get TRUE. So this function satisfies the functional equation for all values of x and y.

The answer is D.

<u>Method 2</u> – by hand

$$\begin{aligned}
\text{If } f(x)&=\log_e\left(\frac{x}{2}\right)\\
f(2(x+y))&=\log_e\left(\frac{2(x+y)}{2}\right)\\
&=\log_e(x+y)\\
f(2x)+f(2y)&=\log_e(x)+\log_e(y)\\
&\neq\log_e(x+y)\\
\text{If } f(x)&=x-2,\\
f(2(x+y))&=2(x+y)-2\\
&=2x+2y-2\\
f(2x)+f(2y)&=2x-2+2y-2\\
&=2x+2y-4\\
&\neq 2x+2y-2\\
\text{If } f(x)&=e^{\frac{x}{2}}\\
f(2(x+y))&=e^{x+y}\\
f(2x)+f(2y)&=e^x+e^y\\
&\neq e^{x+y}\\
\text{If } f(x)&=\frac{x}{2}\\
f(2(x+y))&=x+y\\
f(2x)+f(2y)&=x+y
\end{aligned}$$

The answer is D.

Question 13

$$y = \frac{a}{x^2}$$
$$= ax^{-2}$$
$$\frac{dy}{dx} = -2ax^{-3}$$
$$= \frac{-2a}{x^3}$$

At $x = -1$, $\frac{dy}{dx} = \frac{-2a}{-1}$
$$= 2a$$

At $x = -1$, $y = \frac{a}{(-1)^2} = a$

Point of tangency is $(-1, a)$.
Tangent also passes through the point $(0, c)$.

So, gradient of tangent is $\frac{a-c}{-1-0} = c - a$.

So $2a = c - a$
$$c = 3a$$

The answer is E.

Question 14

$$\frac{dx}{dt} = \frac{2}{\sqrt{t}}$$

$$= 2t^{-\frac{1}{2}}$$

$$x = 2\int t^{-\frac{1}{2}}dt$$

$$= 4t^{\frac{1}{2}} + c$$

$$x = 4\sqrt{t} + c$$

When $t = 1,\ \ x = 4$

$4 = 4 \times 1 + c$ so $c = 0$

So $x = 4\sqrt{t}$

Also, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Since $y = \frac{\tan(x)}{4}$

$$\frac{dy}{dx} = \frac{\sec^2(x)}{4}$$

and since $\frac{dx}{dt} = \frac{2}{\sqrt{t}}$ and $x = 4\sqrt{t}$

then $\frac{dx}{dt} = 2 \times \frac{4}{x}$

$$= \frac{8}{x}$$

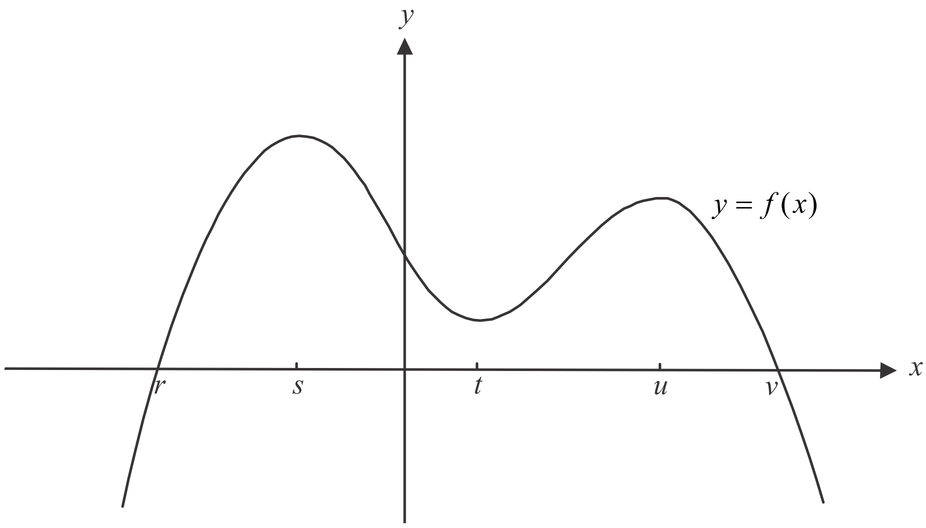
So $\frac{dy}{dt} = \frac{\sec^2(x)}{4} \times \frac{8}{x}$

$$= \frac{2\sec^2(x)}{x}$$

The answer is A.

Question 15

Do a quick sketch.

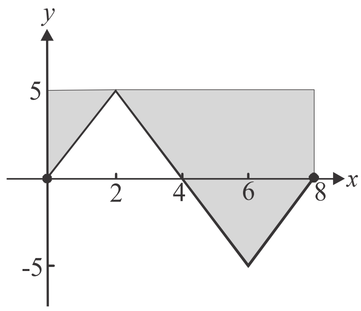


Since the coefficient of the x^4 is negative, the quartic graph comes up from both the left and right sides.
$f(r) = f(v) = 0$ tells us that the graph crosses the x-axis at $x = r$ and $x = v$. We don't know whether r and v are positive or negative but we do know that $r < v$.
Similarly, $f'(s) = f'(t) = f'(u) = 0$ tells us that there are stationary points, in this case maximum and minimum turning points at $x = s, x = t$ and $x = u$.
Again we don't know whether s, t, or u are positive or negative but we do know that $r < s < t < u < v$.
So, from the graph we see that the gradient is positive for $x \in (-\infty, s) \cup (t, u)$.
Note that at $x = s, x = t$ and $x = u$ the gradient is zero (and therefore not positive), so we don't include these endpoints.
The answer is B.

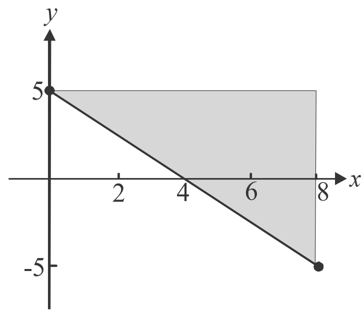
Question 16

To find the average value of a function from a graph, shade the areas enclosed by the graph, the line $y = 5$, the y-axis and the line $x = 8$.
If the area above the line $y = 5$ is equal to the area below the line $y = 5$, then the average value of the function is 5.
Only option C offers this feature.
The answer is C.

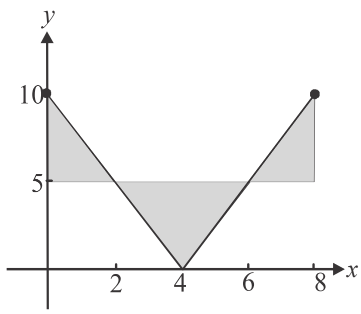
A.



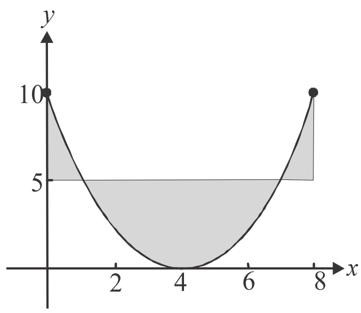
B.



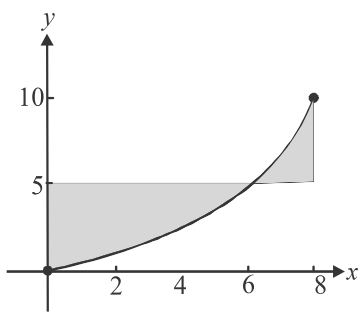
C.



D.



E.



Question 17

For $x < 0$, the shaded area $= \int_{-2}^{0} (g(x) - f(x))dx$

For $x > 0$, the shaded area $= -\int_{0}^{3} g(x)dx$ (negative because the region is below the x-axis).

Total area $= \int_{-2}^{0} (g(x) - f(x))dx - \int_{0}^{3} g(x)dx$.

The answer is D.

Question 18
The system will have infinitely many solutions or no solutions when $\Delta = 0$ where Δ is the determinant of the 2×2 matrix.

So $1 \times 4 - (k+3) \times k = 0$

$$4 - k^2 - 3k = 0$$
$$k^2 + 3k - 4 = 0$$
$$(k+4)(k-1) = 0$$
$$k = -4 \text{ or } k = 1$$

Now, the system of equations is

$$x + ky = 8$$
$$(k+3)x + 4y = 2k$$

When $k = -4$, the system becomes

$$x - 4y = 8 \quad -(1)$$
$$-x + 4y = -8 \quad -(2)$$

$(2) \times {}^{-}1$ gives $x - 4y = 8$

This is the same as equation (1) so the lines lie on top of one another and therefore there are infinitely many solutions.

When $k = 1$, the system becomes

$$x + y = 8 \quad -(1)$$
$$4x + 4y = 2 \quad -(2)$$

$(2) \div 4$ gives $x + y = \frac{2}{4} \quad -(3)$

So the graphs of equation (1) and (3) are parallel lines and therefore there are no points of intersection and hence the system has no solutions.

So infinitely many solutions exist for $k = -4$.

The answer is A.

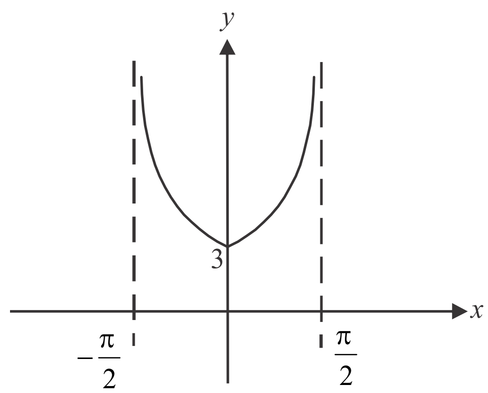
Question 19
Sketch $y = f(x)$ on your CAS. $f(0) = 3$, so the function is defined at $x = 3$. Option A is incorrect.

The function *f* is decreasing for $x < 0$. Option B is incorrect.

The function *f* is continuous at $x = 0$ but not smoothly continuous so the gradient function f' is not defined there.

Note that at $x = 0$ there is a sharp point.

The answer is C.



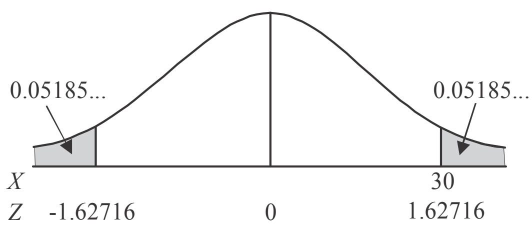
Question 20

$$\Pr(X \geq 30) = \frac{140}{2700}$$
$$= 0.05185...$$

Using CAS and the inverse Normal function,

$\Pr(Z \leq -1.62716) = 0.05185...$

So $\Pr(Z \geq 1.62716) = 0.05185...$



Since $z = \frac{x - \mu}{\sigma}$

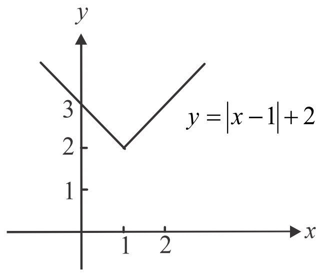
$$1.62716 = \frac{30 - \mu}{3.4}$$
$$\mu = 24.467...$$

The closest answer is 24.47.

The answer is C.

Question 21

Method 1 – graphically



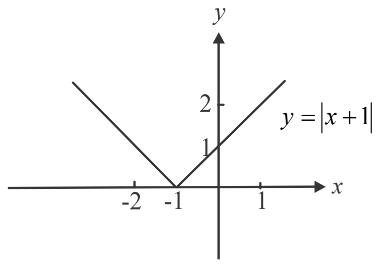
The transformation T involves

- a reflection in the y-axis
- a translation of 2 units up

The graph of the image function $y=|x-1|+2$ is shown.

Reverse these two transformations to obtain the original function by

- translating 2 units down
- reflecting in the y-axis



The graph of the original function is shown.
The rule for this original function is $y=|x+1|$.
The answer is B.

Method 2 – algebraically
The transformation T involves

- a reflection in the y-axis
- a translation of 2 units up

The rule of the image function of f is $y=|x-1|+2$.
To "undo" the transformation and get back to the original function, we

- translate 2 units down. Algebraically we do this by replacing y with $y+2$
- reflect in the y-axis. Algebraically we do this by replacing x with $-x$.

So the image function

of $y=|x-1|+2$

becomes $y+2=|-x-1|+2$

$y=|-1(x+1)|$

$y=|x+1|$

The answer is B.

Method 3 – algebraically
Let (x', y') be an image point.
Image equation is given by $y'=|x'-1|+2$.

Now, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

so, $x'=-x$

and $y'=y+2$

Since $y'=|x'-1|+2$

$y+2=|-x-1|+2$

$y=|-1(x+1)|$

$y=|x+1|$

The answer is B.

Question 22

Stationary points occur when $g'(x)=0$

$$g(x)=-ax^3+bx^2+c$$

$$g'(x)=-3ax^2+2bx$$

When $g'(x)=0$,

$-3ax^2+2bx=0$

This equation is quadratic so we can use the quadratic formula.

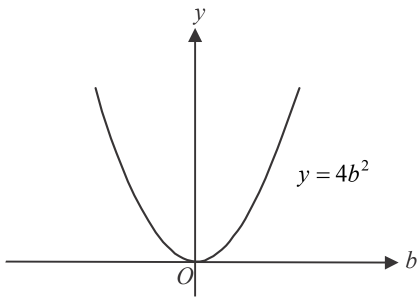
$$x=\frac{-2b\pm\sqrt{4b^2-4\times{}^-3a\times 0}}{-6a}$$

$$=\frac{-2b\pm\sqrt{4b^2}}{-6a}$$

The quadratic equation has 2 solutions
(i.e. more than 1 solution)
when $4b^2>0$

$$b\in R\backslash\{0\}$$

The answer is E.



SECTION 2

Question 1 (14 marks)

a. **i.** The point *A* is the *x*-intercept of the graph of $y = \log_e(2x)$.

<u>*x*-intercepts</u> occur when $y = 0$

$0 = \log_e(2x)$

$e^0 = 2x$

$1 = 2x$

$x = \frac{1}{2}$

A is the point $(0.5, 0)$.

(1 mark)

ii. $y = \log_e(2x)$

$\frac{dy}{dx} = \frac{2}{2x}$

$= \frac{1}{x}$

When $x = 1$, $\frac{dy}{dx} = 1$

(1 mark)

iii. Point *C* is the *x*-intercept of the tangent to the curve $y = \log_e(2x)$ at the point $P(1, \log_e(2))$.

Equation of tangent is given by

$y - \log_e(2) = 1(x-1)$ (using the gradient found in part ii.)

$y = x - 1 + \log_e(2)$

<u>*x*-intercepts</u> occur when $y = 0$

$0 = x - 1 + \log_e(2)$

$x = 1 - \log_e(2)$

The *x*-coordinate of point *C* is $1 - \log_e(2)$ as required.

(1 mark)

iv. Point *D* is the point of intersection of the line $x = 1.5$ and the tangent with equation $y = x - 1 + \log_e(2)$ (from part **iii.**).

When $x = 1.5$, $y = 1.5 - 1 + \log_e(2)$

$= 0.5 + \log_e(2)$

So the *y*-coordinate of point *D* is $0.5 + \log_e(2)$ as required.

(1 mark)

b. The gradient of *CD* is 1 (from part **a**.). The gradient of *EP* is therefore – 1 (since $1 \times -1 = -1$). *P* is the point $(1, \log_e(2))$.

The equation of *EP* is given by

$y - \log_e(2) = -1(x-1)$

$y = -x + 1 + \log_e(2)$

(1 mark)

At point *E*, $x = 0$, so $y = 1 + \log_e(2)$.

Point *E* is the point $(0, 1 + \log_e(2))$.

The rectangular piece of public land has

area = length × width

$= 1.5 \times (1 + \log_e(2))$ km^2

(1 mark)

c. **i.** A is the point $(0.5, 0)$
B is the point $(1.5, \log_e(3))$

The gradient of $AB = \dfrac{\log_e(3) - 0}{1.5 - 0.5}$

$= \log_e(3)$

(1 mark)

The equation of AB is given by (using point A)

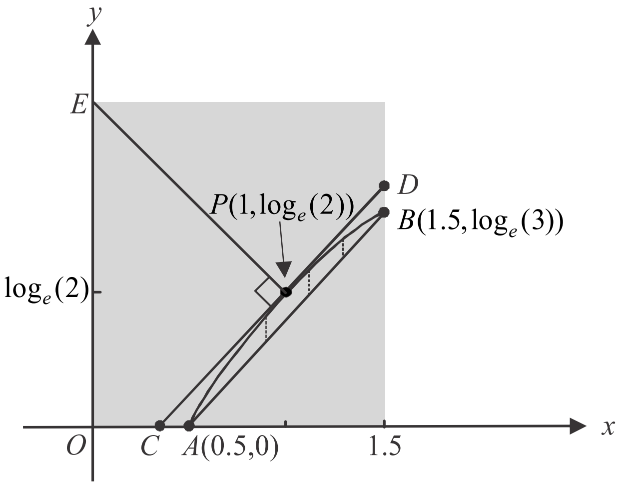
$y - 0 = \log_e(3)(x - 0.5)$

$y = x\log_e(3) - 0.5\log_e(3)$

as required

(1 mark)

ii.



A few dotted indicative distances are shown in the north-south direction, between the dry creek bed and this new straight track between A and B.
A function $d(x)$ which gives the north-south distance between the curved path with equation $y = \log_e(2x)$ and the straight path with equation $y = x\log_e(3) - 0.5\log_e(3)$ is given by

$d(x) = \log_e(2x) - (x\log_e(3) - 0.5\log_e(3))$

$= \log_e(2x) - x\log_e(3) + 0.5\log_e(3)$

(1 mark)

Find the x value when this maximum distance occurs.

Method 1 – using CAS
Solve $d'(x) = 0$ for x.

$x = \dfrac{1}{\log_e(3)}$ **(1 mark)**

Method 2 – by hand

$$d'(x) = \frac{2}{2x} - \log_e(3)$$
$$= \frac{1}{x} - \log_e(3)$$

Solve $0 = \frac{1}{x} - \log_e(3)$ for max.

$$\frac{1}{x} = \log_e(3)$$
$$1 = x\log_e(3)$$
$$x = \frac{1}{\log_e(3)}$$

(1 mark)

So the maximum occurs when $x = \frac{1}{\log_e(3)}$.

Substitute this into $d(x)$ to find the maximum distance.

So $d\left(\frac{1}{\log_e(3)}\right) = 0.148405...$

$= 0.15$ km (correct to 2 dcimal places)

(1 mark)

d. C is the point $(1-\log_e(2), 0)$ (from part **a. iii**)

D is the point $(1.5, 0.5+\log_e(2))$ (from part **a. iv.**)

$$P' = \text{midpoint of } CD$$
$$= \left(\frac{1-\log_e(2)+1.5}{2}, \frac{0+0.5+\log_e(2)}{2}\right)$$
$$= (1.25 - 0.5\log_e(2),\ 0.25 + 0.5\log_e(2))$$

(1 mark)

P is the point $(1, \log_e(2))$.

The x-coordinate of P, i.e. 1, has been multiplied by $1.25 - 0.5\log_e(2)$ to become the x-coordinate of P'.
This represents a dilation from the y-axis by a factor of 0.903426…
So $q = 0.90$ (correct to 2 decimal places).

(1 mark)

The y-coordinate of P, i.e. $\log_e(2)$, has been multiplied by $\frac{0.25+0.5\log_e(2)}{\log_e(2)}$, to become the y-coordinate of P'.
This represents a dilation from the x-axis by a factor of 0.86067....
So $n = 0.86$ (correct to 2 decimal places).

(1 mark)

Question 2 (16 marks)

a. **i.** $0.9 \times 0.9 \times 0.9 \times 0.9 = 0.6561$

(1 mark)

ii. We have a binomial distribution with $n = 30$ and $p = 0.9$.

(1 mark)

$\Pr(X \geq 25) = 0.9268$ (correct to 4 decimal places)

(1 mark)

iii. This is a conditional probability question.

$\Pr(X > 28 | X \geq 25)$

$= \dfrac{\Pr(X > 28 \cap X \geq 25)}{\Pr(X \geq 25)}$ (conditional probability formula) **(1 mark)**

$= \dfrac{\Pr(X > 28)}{\Pr(X \geq 25)}$

$= \dfrac{0.183695}{0.92681}$ (the denominator comes from part a. ii.) **(1 mark)** for 0.183695

$= 0.198$ (correct to 3 decimal places)

(1 mark)

b. **i.** Method 1

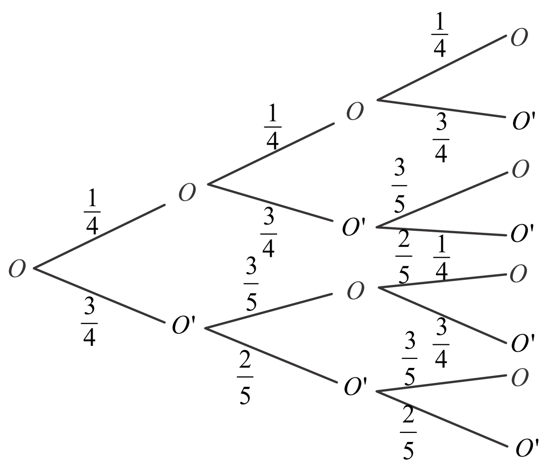
Pr(out at least once)

$= 1 - \Pr(O', O', O')$ **(1 mark)**

$= 1 - \dfrac{3}{4} \times \dfrac{2}{5} \times \dfrac{2}{5}$

$= \dfrac{22}{25}$ **(1 mark)**

Method 2 – use a diagram



Pr(out at least once)

$= 1 - \Pr(O', O', O')$ **(1 mark)**

$= 1 - \dfrac{3}{4} \times \dfrac{2}{5} \times \dfrac{2}{5}$

$= \dfrac{22}{25}$

(1 mark)

ii. Create a transition matrix T and an initial state matrix S_1.

$$\begin{array}{c} \text{this week} \\ \begin{matrix} O & O' \end{matrix} \end{array}$$

$$T = \begin{bmatrix} \frac{1}{4} & \frac{3}{5} \\ \frac{3}{4} & \frac{2}{5} \end{bmatrix} \begin{matrix} O \\ O' \end{matrix} \text{ next week} \qquad S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} O \\ O' \end{matrix}$$

(1 mark)

$$\begin{aligned} S_{10} &= T^9 S_1 \\ &= \begin{bmatrix} 0.444400... \\ 0.555599... \end{bmatrix} \end{aligned}$$

The probability he puts it out in the 10^{th} week is 0.444 (correct to 3 decimal places).

(1 mark)

c. Y has a binomial distribution with $n = 30$ and $p = p$.

Given $\Pr(Y \geq 29) = 11 \times \Pr(Y = 30)$

$$\begin{aligned} LS &= \Pr(Y \geq 29) \\ &= \Pr(Y = 29) + \Pr(Y = 30) \\ &= {}^{30}C_{29} p^{29}(1-p)^1 + {}^{30}C_{30} p^{30}(1-p)^0 \\ &= 30p^{29}(1-p) + p^{30} \\ &= 30p^{29} - 30p^{30} + p^{30} \\ &= 30p^{29} - 29p^{30} \end{aligned}$$

(1 mark)

$$\begin{aligned} RS &= 11 \times \Pr(Y = 30) \\ &= 11 \times {}^{30}C_{30} p^{30}(1-p)^0 \\ &= 11p^{30} \end{aligned}$$

So, $30p^{29} - 29p^{30} = 11p^{30}$

$$\begin{aligned} p^{29}(30 - 29p) &= 11p^{30} \\ 30 - 29p &= 11p, \quad p \neq 0 \\ 30 &= 40p \\ p &= \frac{3}{4} \end{aligned}$$

as required.

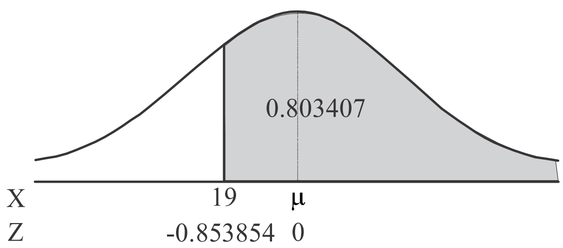
(1 mark)

d. $X \sim N(\mu, \sigma)$

$Y \sim Bi\left(30, \frac{3}{4}\right)$ from part c.

Now $\Pr(Y \ge 21) = 0.803407$ (using CAS and binomCdf)

Since $\Pr(X \ge 19) = \Pr(Y \ge 21)$, then $\Pr(X \ge 19) = 0.803407$.



Using CAS and the inverse normal function the corresponding value of Z is -0.853854.

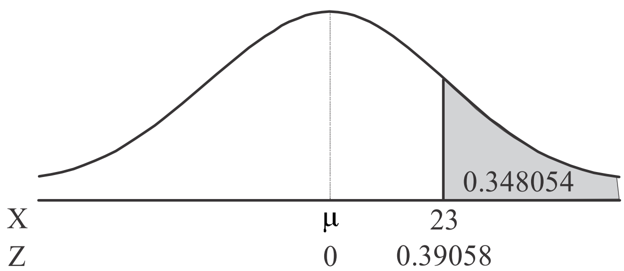
Since $z = \frac{x-\mu}{\sigma}$,

$$-0.853854 = \frac{19-\mu}{\sigma} \qquad -(1)$$

(1 mark)

Also, $\Pr(Y \ge 24) = 0.348054$ (using CAS and binomCdf).

Since $\Pr(X \ge 23) = \Pr(Y \ge 24)$, then $\Pr(X \ge 23) = 0.348054$.



Using CAS and inverse normal function, the corresponding value of Z is 0.39058.

Since $z = \frac{x-\mu}{\sigma}$

$$0.39058 = \frac{23-\mu}{\sigma} \qquad -(2)$$

(1 mark)

Solve equations (1) and (2) simultaneously using CAS or by hand:

$$-0.853854\sigma = 19 - \mu \quad (1)$$

$$0.39058\sigma = 23 - \mu \quad (2)$$

$(2)-(1)$ $1.24443\sigma = 4$

$$\sigma = 3.21431$$

In (2)

$$\mu = 21.7446$$

So $\mu = 21.74$ and $\sigma = 3.21$ (both correct to 2 decimal places)

(1 mark) for μ

(1 mark) for σ

Question 3 (15 marks)

a. $h(t) = 14 - 14\cos\left(\frac{\pi t}{2}\right)$

i. amplitude is 14 (Note that the amplitude is always positive).

(1 mark)

ii.

$$\begin{aligned}\text{period} &= \frac{2\pi}{n}\\ &= 2\pi \div \frac{\pi}{2}\\ &= 2\pi \times \frac{2}{\pi}\\ &= 4\end{aligned}$$

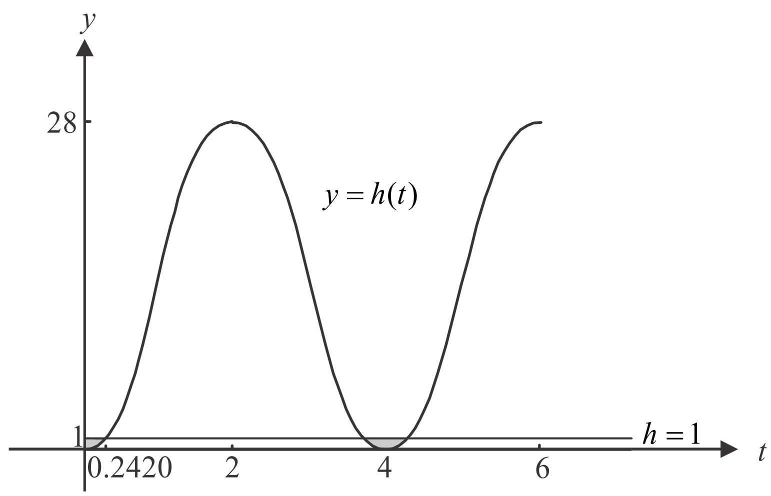
(1 mark)

b. **i.** Solve $h(t) = 1$ for t.

$$\begin{aligned}t &= 0.2420\ldots\text{hours}\\ &= 0.2420\ldots \times 60 \text{ minutes}\\ &= 14.5245\ldots\\ &= 14.52 \text{ minutes (correct to 2 decimal places)}\end{aligned}$$

(1 mark)

ii. Do a quick sketch of the function $h(t)$.



Using this graph, because of the symmetry of the curve, the total time available to Victoria during the six hours is

$3 \times 0.2420\ldots$ hours

(1 mark)

$= 0.7262\ldots$

$= 0.7262\ldots \times 60$ minutes

$= 43.5735\ldots$

So total time available is 43.57 minutes (correct to 2 decimal places)

(1 mark)

c. **i.** Since $V(30)=0,$

$$p(30)^2 - q(30)^3 = 0$$
$$900p = 27000q$$
$$p = 30q$$

(1 mark)

ii. To Show: $V(20)=2V(10)$

$$\begin{aligned} LS &= V(20) \\ &= 400p - 8000q \\ &= 400\times 30q - 8000q \quad \text{(using part i.)} \\ &= 4000q \end{aligned}$$

(1 mark)

$$\begin{aligned} RS &= 2V(10) \\ &= 2(100p - 1000q) \\ &= 2(100\times 30q - 1000q) \quad \text{(using part i.)} \\ &= 4000q \\ &= LS \end{aligned}$$

Have shown.

(1 mark)

iii. Victoria's speed is a maximum when

$$V'(x)=0$$

(1 mark)

$$\begin{aligned} V'(x) &= 2px - 3qx^2 = 0 \\ x(2p - 3qx) &= 0 \\ x = 0 \text{ or } 2p - 3qx &= 0 \\ x &= \frac{2p}{3q} \\ &= \frac{2\times 30q}{3q} \quad \text{(using part i.)} \\ &= 20 \end{aligned}$$

(1 mark)

Maximum speed occurs when $x=20$.
(Note that the minimum speed occurs when $x=0$ and $x=30$,since $V(30)=0$.)

d. From part **c. iii.** we know that Victoria's maximum speed occurs when $x=20$. We also know from part **c. i.** that $p=30q$ so when $p=0.015, q=0.0005$.

(1 mark)

$$\begin{aligned} V(x) &= px^2 - qx^3 \\ V(20) &= 0.015\times 20^2 - 0.0005\times 20^3 \\ &= 2 \end{aligned}$$

Maximum speed when $p=0.015$ is 2 m/s.

(1 mark)

e. The maximum speed that Victoria can travel at is $\frac{8}{3}$ m/s.

We know that the maximum speed occurs when $x = 20$ and we know that $p = 30q$.

$$V(x) = px^2 - qx^3$$

$$\frac{8}{3} = p \times 20^2 - \left(\frac{p}{30}\right) \times 20^3$$

Solve for *p*.

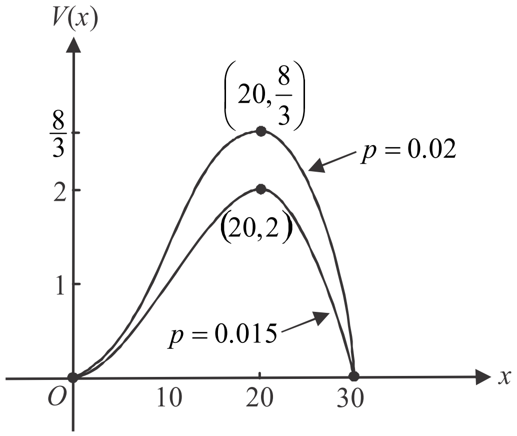
$$p = 0.02$$

(1 mark)

Sketch the functions:

$V(x) = 0.015x^2 - 0.0005x^3$, from part **d.** when $p = 0.015$ and

$V(x) = 0.02x^2 - \frac{0.02}{30}x^3$, from part **e.** when $p = 0.02$.



So the maximum value *p* can take is 0.02. Since *p* is a positive integer, the values *p* can take are $0 < p \le 0.02$.

(1 mark) - $p > 0$
(1 mark) - $p \le 0.02$

Question 4 (13 marks)

a. P is the y-intercept of the graph of f.

<u>y-intercepts</u> occur when $x=0$

$$\begin{aligned} y &= k\sqrt{1-x} \\ &= k\sqrt{1} \\ &= k \end{aligned}$$

P is the point $(0,k)$.

(1 mark)

$f(x)=k\sqrt{1-x}$ (remember k is a constant)

$$f'(x)=\frac{-k}{2\sqrt{1-x}}$$

At $x=0$,

$$f'(0)=\frac{-k}{2}$$

The equation of the tangent at point $P(0,k)$ with gradient $-\frac{k}{2}$ is

$$\begin{aligned} y-k &= -\frac{k}{2}(x-0) \\ y &= -\frac{k}{2}x+k \end{aligned}$$

(1 mark)

b. Point B is the x-intercept of the tangent found in part **a.**

<u>x-intercepts</u> occur when $y=0$

$$\begin{aligned} 0 &= -\frac{k}{2}x+k \\ -k &= -\frac{k}{2}x \\ x &= 2 \end{aligned}$$

So B is the point $(2,0)$ as required.

(1 mark)

c. The point A is the x-intercept of the graph of f.

<u>x-intercepts</u> occur when $y=0$

$$\begin{aligned} 0 &= k\sqrt{1-x} \\ x &= 1 \end{aligned}$$

area = area of ΔBPO – area of region enclosed by the x and y axes and the graph of f between $x=0$ and $x=1$

$$=\frac{1}{2}\times 2\times k-\int_0^1 f(x)dx$$ **(1 mark)**

$$=k-\frac{2k}{3} \quad \text{(using CAS)}$$

$$=\frac{k}{3} \text{ square units}$$ **(1 mark)**

d. average value $= \dfrac{1}{1-d}\displaystyle\int_d^1 k\sqrt{1-x}\,dx$

Solve $k = \dfrac{1}{1-d}\displaystyle\int_d^1 k\sqrt{1-x}\,dx$ for d using CAS. **(1 mark)**

$d = -\dfrac{5}{4}$

(1 mark)

e. The gradient of the tangent to f at point P is $-\dfrac{k}{2}$ (from part **a.**).

The gradient of the normal is therefore $\dfrac{2}{k}$.

The equation of the normal at $P(0,k)$ is

$$y-k=\frac{2}{k}(x-0)$$

$$y=\frac{2}{k}x+k$$ **(1 mark)**

At $x=2,\ y=\dfrac{4}{k}+k$

So C is the point $\left(2,\dfrac{4}{k}+k\right)$ and P is $(0,k)$ and B is $(2,0)$.

(1 mark)

Method 1

area of $\Delta BCP = \dfrac{1}{2}\times BC\times PX$ where X is the point $(2,k)$ **(1 mark)**

$$=\frac{1}{2}\times\left(\frac{4}{k}+k\right)\times 2$$

$=k+\dfrac{4}{k}$ as required **(1 mark)**

Method 2

area of $\Delta BCP = \dfrac{1}{2}\times BP\times CP$

$=\dfrac{1}{2}\times\sqrt{k^2+4}\times\sqrt{\left(\dfrac{4}{k}+k-k\right)^2+2^2}$ (distance formula) **(1 mark)**

So $A(k)=\dfrac{1}{2}\sqrt{(k^2+4)\left(\dfrac{16}{k^2}+4\right)}$

$$=\frac{1}{2}\sqrt{(k^2+4)\frac{4}{k^2}(4+k^2)}$$

$$=\frac{1}{2}\times(k^2+4)\times\frac{2}{k}$$

$$=\frac{k^2+4}{k}$$

$=k+\dfrac{4}{k}$ as required

(1 mark)

f. $A(k) = k + \frac{4}{k}$ from part **e.**

$A'(k) = 1 - \frac{4}{k^2}$ using CAS

(1 mark)

g. Solve $A'(k) = 1 - \frac{4}{k^2} = 0$ for k using CAS.

$k = \pm 2$ but $k > 0$ so $k = 2$

$A(2) = 4$

The minimum area of ΔBCP is 4 square units. **(1 mark)**

