The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
SOLUTIONS: Trial Exam 2014

Written Examination 1

Question 1

$mx + 2y = 6$

$x + (m-1)y = -3$

Let $A = \begin{bmatrix} m & 2 \\ 1 & m-1 \end{bmatrix}$

$\det(A) = m(m-1) - 2 = 0$ **1M**

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2$ or $m = -1$ **1A**

Check the ratios.

$\frac{m}{1} = \frac{6}{-3} = -2$ for infinitely many solutions **1M**

Hence $m = 2$ or $m = -1$ for no real solutions

(OR

Let $m = 2$: The equations are $2x + 2y = 6$ and $x + y = -3$ ….parallel lines with different y intercepts and so no real solutions for this m value.

Let $m = -1$: The equations are $-x + 2y = 6$ and $x - 2y = -3$ ….parallel lines with different y intercepts again and so no real solutions for this m value. **1M**

Hence $m = 2$ or $m = -1$ for no real solutions)

OR

$y = \frac{6 - mx}{2} \quad (1)$

$y = \frac{-3 - x}{m-1} \quad (2)$

If there are no solutions, the lines must be parallel (but not coincident) and hence have the same gradient.

Therefore,

$\frac{-m}{2} = \frac{-1}{m-1}$ **1M**

$m(m-1) = 2$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$\therefore m = 2, -1$ **1A**

Check the ratios.

$\frac{m}{1} = \frac{6}{-3} = -2$ for infinitely many solutions **1M**

Hence $m = 2$ or $m = -1$ for no real solutions

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Hence $m = 2$ or $m = -1$ for no real solutions)

Question 2

$2\sin^2(2x) - 1 = 0$

$\sin^2(2x) = \frac{1}{2}$

$\sin(2x) = \pm\frac{1}{\sqrt{2}}$ **1M**

Reference angle is $\frac{\pi}{4}$.

Need answers for $x \in [0, \pi]$

$0 \le x \le \pi$ **1M**

$\therefore 0 \le 2x \le 2\pi$

$2x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Giving the answer $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ **1A**

Question 3

$2\log_2(2x+1) + \log_2(3) = 3$

$\log_2(2x+1)^2 + \log_2(3) = 3$

$\log_2\left(3(2x+1)^2\right) = 3$ **1M**

$3(2x+1)^2 = 8$ **1M**

$(2x+1)^2 = \frac{8}{3}$

$2x + 1 = \frac{2\sqrt{2}}{\sqrt{3}},\ 2x + 1 > 0$

$x = \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{2}$ **1A**

$= \frac{\sqrt{6}}{3} - \frac{1}{2}$

Question 4

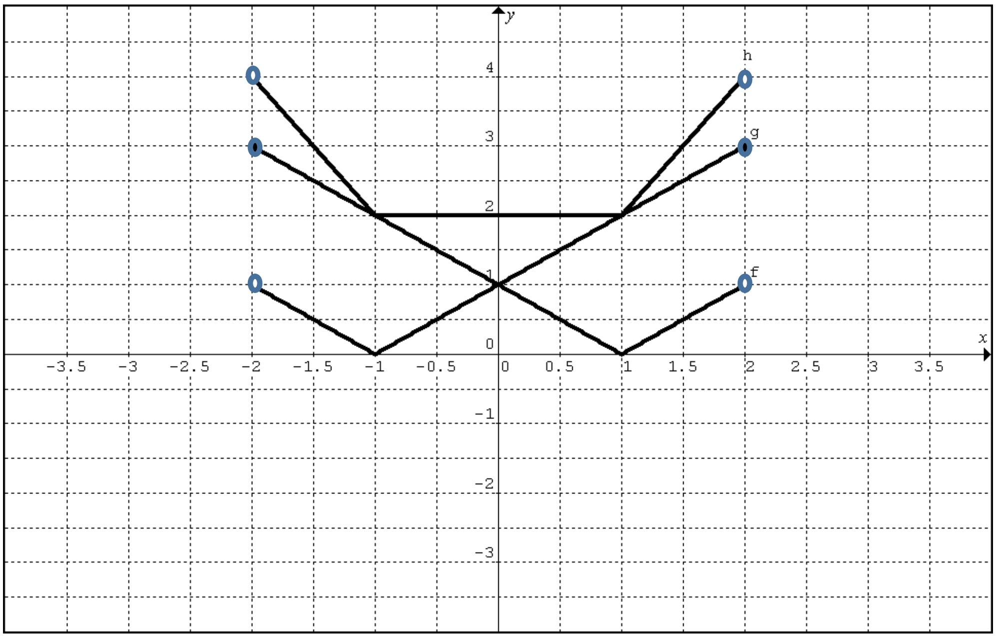
a. f drawn correctly with closed circle at $(-2,3)$ and open circle at (2,1). **1A**

g drawn correctly with closed circle at (2, 3) and open circle at $(-2,1)$. **1A**

b. $\text{dom } h(x) = \text{dom} f(x) \cap \text{dom } g(x)$

h drawn correctly **1A**

with open circles at (2, 4) and $(-2,4)$. **1A**



Question 5

a. $f(x) = kx^2 \tan(2x)$

Using the Product Rule $f'(x) = \left(\tan(2x) \times 2kx\right) + \left(kx^2 \times 2\sec^2(2x)\right)$ **1M**

$f'(x) = 2kx\tan(2x) + 2kx^2\sec^2(2x)$ **1A**

b. $f'\left(\frac{\pi}{8}\right) = 2k\left(\frac{\pi}{8}\right)\tan\left(2\left(\frac{\pi}{8}\right)\right) + 2k\left(\frac{\pi}{8}\right)^2 \sec^2\left(2\left(\frac{\pi}{8}\right)\right)$

Giving

$$f'\left(\frac{\pi}{8}\right) = k\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) + k\frac{\pi^2}{32} \times \frac{1}{\cos^2\left(\frac{\pi}{4}\right)}$$

$$f'\left(\frac{\pi}{8}\right) = k\frac{\pi}{4} + k\frac{\pi^2}{32} \times 2$$

$$f'\left(\frac{\pi}{8}\right) = k\frac{\pi}{4} + k\frac{\pi^2}{16}$$

$f'\left(\frac{\pi}{8}\right) = \frac{k\pi(4+\pi)}{16}$ as required **1M**

c. From **b**. $f'\left(\frac{\pi}{8}\right)=\frac{k\pi(4+\pi)}{16}$

Solve $\frac{k\pi(4+\pi)}{16}=\frac{3}{4}$ **1M**

$k\pi(4+\pi)=12$

$k\pi=\frac{12}{4+\pi}$

Giving $k=\frac{12}{\pi(4+\pi)}$ **1A**

Question 6

a $f:(-\infty,A]\to R, f(x)=x^2+\frac{2}{3}x+3$.

Find the x value of the turning point.

$x=\frac{-b}{2a}=\frac{-2}{3\times 2}=-\frac{1}{3}$

$A=-\frac{1}{3}$ **1A**

OR

Complete the square.

$f(x)=x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}+3$

$=\left(x+\frac{1}{3}\right)^2+\frac{26}{9}$

$A=-\frac{1}{3}$ **1A**

OR

Find the derivative.

$f'(x)=2x+\frac{2}{3}=0$

$x=-\frac{1}{3}=A$ **1A**

b. Let $y=x^2+\frac{2}{3}x+3$

Inverse swap x and y. **1M**

$x=y^2+\frac{2}{3}y+3$

Complete the square.

$x=\left(y+\frac{1}{3}\right)^2-\frac{1}{9}+3$

$x=\left(y+\frac{1}{3}\right)^2+\frac{26}{9}$

$y=\pm\sqrt{\left(x-\frac{26}{9}\right)}-\frac{1}{3}$

$y = -\sqrt{\left(x - \frac{26}{9}\right)} - \frac{1}{3}$

Negative square root because of the domain of *f*.
Domain of f^{-1} is the same as the range of *f*.

$f\left(-\frac{1}{3}\right) = \frac{1}{9} - \frac{2}{9} + 3 = \frac{26}{9}$

$\left[\frac{26}{9}, \infty\right)$

OR

$x - \frac{26}{9} \geq 0, x \geq \frac{26}{9}$

$f^{-1}: \left[\frac{26}{9}, \infty\right) \to R, f^{-1}(x) = -\sqrt{\left(x - \frac{26}{9}\right)} - \frac{1}{3}$

1A Domain, **1A** Rule

OR

Use the Quadratic Formula.

$0 = y^2 + \frac{2}{3}y + 3 - x$

$y = -\frac{2}{6} \pm \frac{\sqrt{\left(\frac{4}{9} - 4(3-x)\right)}}{2}$

$y = -\frac{2}{6} \pm \sqrt{\frac{1}{9} - 3 + x}$

$y = -\frac{1}{3} - \sqrt{x - \frac{26}{9}}$

Negative square root because of the domain of *f*.
Domain of f^{-1} is the same as the range of *f*.

$f\left(-\frac{1}{3}\right) = \frac{1}{9} - \frac{2}{9} + 3 = \frac{26}{9}$

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$x - \frac{26}{9} \geq 0, x \geq \frac{26}{9}$

$f^{-1}: \left[\frac{26}{9}, \infty\right) \to R, f^{-1}(x) = -\sqrt{\left(x - \frac{26}{9}\right)} - \frac{1}{3}$

1A Domain, **1A** Rule

Question 7

a. $f'(x) = 3e^{3x+2}$

$m_T = f'(0) = 3e^2$

$m_N = -\frac{1}{3e^2}$

$f(0) = e^2$, $\left(0, e^2\right)$

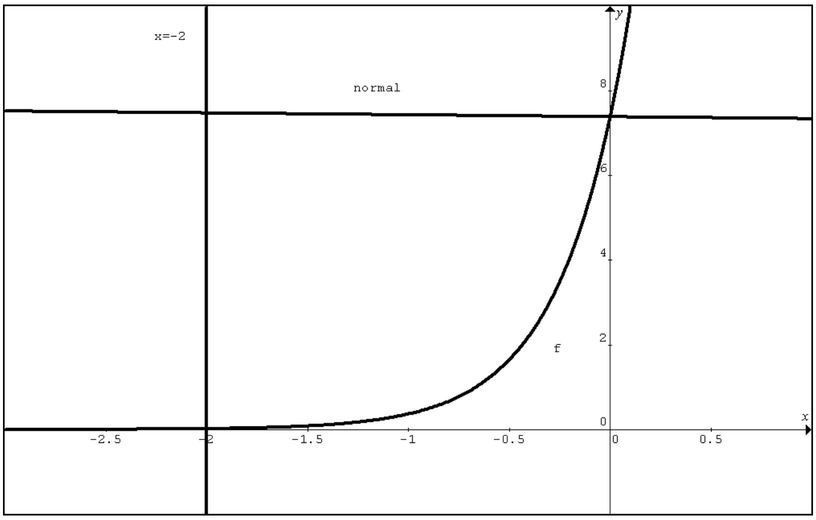
$y = -\frac{1}{3e^2}x + e^2$ **1M**

b. $\int_{-2}^{0}\left(-\frac{1}{3e^2}x + e^2 - e^{3x+2}\right)dx$ **1M**

$= \left[-\frac{1}{6e^2}x^2 + e^2x - \frac{e^{3x+2}}{3}\right]_{-2}^{0}$ **1M**

$= -\frac{e^2}{3} - \left(-\frac{4}{6e^2} - 2e^2 - \frac{e^{-4}}{3}\right)$

$= \frac{5e^2}{3} + \frac{2}{3e^2} + \frac{1}{3e^4}$ **1A**



Question 8

$\Pr(A) = \frac{1}{4}$ and $\Pr\left(A \cap B\right) = \frac{1}{5}$.

a. Given $\Pr(A' \cap B) = \Pr(A)$

Karnaugh Map **1M**

	A	A′	
B	0.20	0.25	0.45
B′	0.05	0.50	0.55
	0.25	0.75	1

$\Pr(B) = 0.45$ **1A**

b. Given events A and B are independent.

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\frac{1}{5} = \frac{1}{4} \times \Pr(B)$

$\therefore \Pr(B) = \frac{4}{5}$

Karnaugh Map **1M**

	A	A′	
B	0.20	0.60	0.80
B′	0.05	0.15	0.20
	0.25	0.75	1

Need $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A \cup B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5}$

$\Pr(A \cup B) = \frac{17}{20}$ **1A**

Question 9

$$f(x)=\begin{cases}\dfrac{k}{2x} & 1\le x\le e^3\\ 0 & \text{elsewhere}\end{cases}.$$

a. For a PDF $\int_1^{e^3}\left(\dfrac{k}{2x}\right)dx=1$

$$\int_1^{e^3}\left(\frac{k}{2x}\right)dx=\frac{k}{2}\left[\log_e(x)\right]_1^{e^3}$$

$$=\frac{k}{2}\left(\log_e\left(e^3\right)-\log_e(1)\right)$$

$$=\frac{k}{2}\log_e\left(e^3\right)$$ **1M**

$$=\frac{3k}{2}$$

Let $\dfrac{3k}{2}=1$

Giving $k=\dfrac{2}{3}$ **1M**

b. From **a**. $k=\dfrac{2}{3}$ giving $f(x)=\begin{cases}\dfrac{1}{3x} & 1\le x\le e^3\\ 0 & \text{elsewhere}\end{cases}$

$$\Pr\left(X<10\mid X\ge 1\right)=\frac{\Pr\left(1\le X<10\right)}{\Pr(X\ge 1)}$$

$$\Pr\left(X<10\mid X\ge 1\right)=\frac{\int_1^{10}\left(\frac{1}{3x}\right)dx}{1}$$ **1M**

$$=\frac{\frac{1}{3}\left[\log_e(x)\right]_1^{10}}{1}$$

$$=\frac{\log_e(10)-\log_e(1)}{3}$$

Giving $\Pr\left(X<10\mid X\ge 1\right)=\dfrac{1}{3}\log_e(10)$ **1A**

c. $E(X) = \int_1^{e^3} \left(x \times \frac{1}{3x} \right) dx$

$E(X) = \int_1^{e^3} \left(\frac{1}{3} \right) dx$

$= \left[\frac{x}{3} \right]_1^{e^3}$ **1M**

$= \frac{1}{3}\left(e^3 - 1\right)$

Giving $E(X) = \frac{e^3}{3} - \frac{1}{3}$ **1A**

Question 10

a. $\frac{d}{dx}\left(\frac{5}{2x^2 - 1} \right) = \frac{-5 \times 4x}{\left(2x^2 - 1\right)^2} = -\frac{20x}{\left(2x^2 - 1\right)^2}$ **1A**

b. $\int_1^a \left(\frac{20x}{\left(2x^2 - 1\right)^2} + 1 \right) dx = \frac{37}{7}$

$\frac{d}{dx}\left(\frac{5}{2x^2 - 1} \right) = -\frac{20x}{\left(2x^2 - 1\right)^2}$, therefore, $\int \left(\frac{20x}{\left(2x^2 - 1\right)^2} \right) dx = \frac{-5}{2x^2 - 1} + c$

$\left[\frac{-5}{2x^2 - 1} + x \right]_1^a = \frac{37}{7}$ **1M**

$\left(\frac{-5}{2a^2 - 1} + a \right) - (-4) = \frac{37}{7}$

$\frac{-5}{2a^2 - 1} + a - \frac{9}{7} = 0$

Multiply both sides of the equation by $7(2a^2 - 1)$

$14a^3 - 18a^2 - 7a - 26 = 0$ **1M**

Let $f(x) = 14a^3 - 18a^2 - 7a - 26$

$f(1) \neq 0, f(-1) \neq 0, f(2) = 0$

Hence $a = 2$ **1A**