Neap:

Trial Examination 2014

VCE Mathematical Methods (CAS) Units 3&4

Written Examination 1

Suggested Solutions

Question 1

a. $\frac{dy}{dx} = 5\left(\frac{x^2}{3} - 2x\right)^4 \times \left(\frac{2x}{3} - 2\right)$

$= 10\left(\frac{x}{3} - 1\right)\left(\frac{x^2}{3} - 2x\right)^4$ A1

b. $f'(x) = \frac{2x\cos(x) - x^2(\sin(x))}{\cos^2(x)}$ M1

$= \frac{2x\cos(x) + x^2(-\sin(x))}{\cos^2(x)}$

$f'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2\sqrt{3}}{9}$ A1

Question 2

a. $\int \frac{2}{(3x-2)^4}dx = 2\int (3x-2)^{-4}dx$

$= 2\left[-\frac{1}{3}(3x-2)^{-3} \times \frac{1}{3}\right]$ M1

$= \frac{-2}{9(3x-2)^3}$ A1

b. $\frac{1}{3}\int_3^a \frac{1}{(x-2)}dx = -1$

$\int_3^a \frac{1}{(x-2)}dx = -3$

$[\log_e(x-2)]_3^a = -3$ M1

$\log_e(a-2) - \log_e(3-2) = -3$

$\log_e(a-2) = -3$

$a - 2 = e^{-3}$

$a = e^{-3} + 2$ A1

Question 3

Finding stationary points first:

$$\frac{df(x)}{x} = \frac{d(x^2e^{-x})}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)\text{‘} \qquad \text{M1}$$

$$2x - x^2 = 0(e^{-x} \neq 0)$$

$$x = 0 \text{ or } x = 2$$

Using sign diagram, there is a minimum at 0 and maximum at 2.

$$f(0) = 0, f(2) = \frac{4}{e^2} \qquad \text{A1}$$

Find values of the function at the endpoints.

$$f(-1) = e, f(3) = \frac{9}{e^3}$$

Comparing values, we can make a conclusion that absolute minimum is $(0, 0)$ and absolute maximum is $(-1, e)$. A1

Question 4

a. $\cos\left(2x + \frac{\pi}{2}\right) = -\frac{\sqrt{3}}{2}$

$$2x + \frac{\pi}{2} = \frac{5\pi}{6} + 2\pi k \text{ or } 2x + \frac{\pi}{2} = \frac{7\pi}{6} + 2\pi k \qquad \text{M1}$$

$$2x = \frac{\pi}{3} + 2\pi k \text{ or } 2x = \frac{2\pi}{3} + 2\pi k$$

$$x = \frac{\pi}{6} + \pi k \text{ or } x = \frac{\pi}{3} + \pi k$$

Checking which solutions are in the required interval gives

$$x = -\frac{11\pi}{6}, -\frac{5\pi}{3}, -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3} \qquad \text{A1}$$

b. $[\sqrt{3} - 2, \sqrt{3} + 2]$ A1

Question 5

a. $5^{4x} - 6 \times 5^{2x} + 5 = 0$

Let $u = 5^{2x}$

$u^2 - 6u + 5 = 0$ M1

$u = 1$ or $u = 5$

$5^{2x} = 1$ or $5^{2x} = 5$

$x = 0$ or $x = 0.5$ A1

b. $\log_{\frac{1}{3}}(x) = \dfrac{\log_3(x)}{\log_3\left(\frac{1}{3}\right)}$ M1

$= -\log_3(x)$ A1

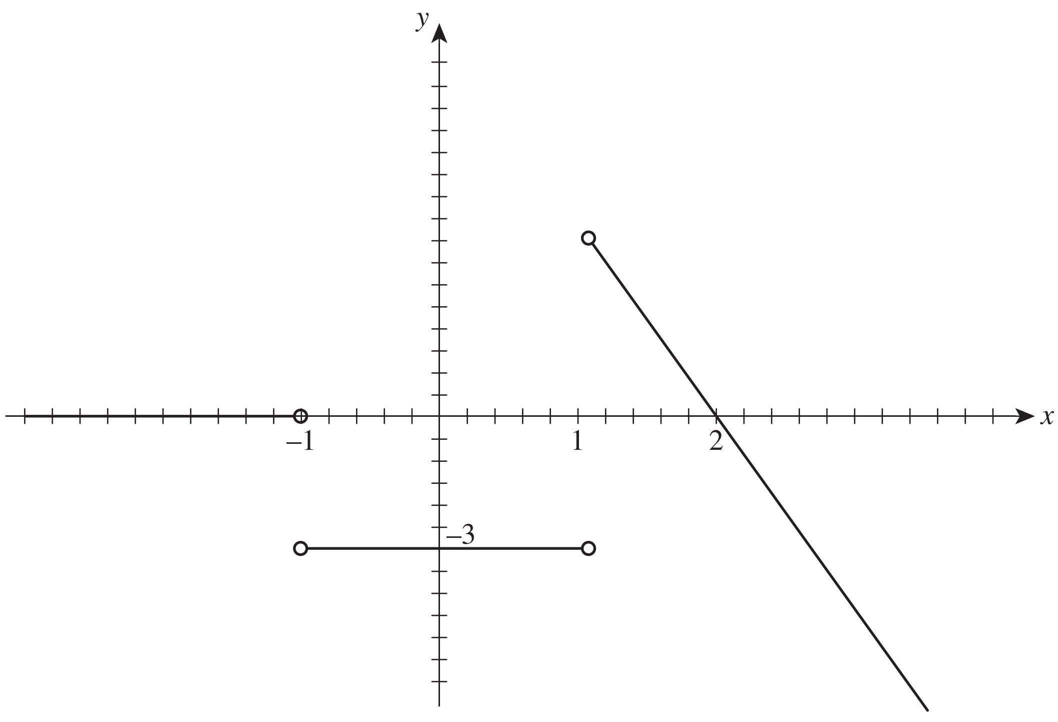
$2\log_3 x - \log_3(x) = 3$

$\log_3(x) = 3$

$x = 27$

Question 6

a.



correct shape A1
correct endpoints A1

b. Domain: $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ A1

Question 7

a. As the graph of $f(x) = 2(x-a)^2 - b - 3$ is parabola, modulus will not change the x-intercepts, it will just reflect part of the parabola below the x axis in the x-axis. So to solve the problem we can remove the modulus sign.

$2(x-a)^2 - b - 3 = 0$, substitute $-\frac{2}{3}$ and 2 instead of x and we will obtain 2 simultaneous equations

$$2(2-a)^2 - b - 3 = 0$$

$$2\left(-\frac{3}{2} - a\right)^2 - b - 3 = 0 \qquad \text{M1}$$

solve them by eliminating b (subtract first equation from second)

$$2\left(-\frac{3}{2} - a\right)^2 - 2(2-a)^2 = 0$$

$$a = \frac{1}{4} \qquad \text{A1}$$

Substitute this value back into any of the equations and solve for b

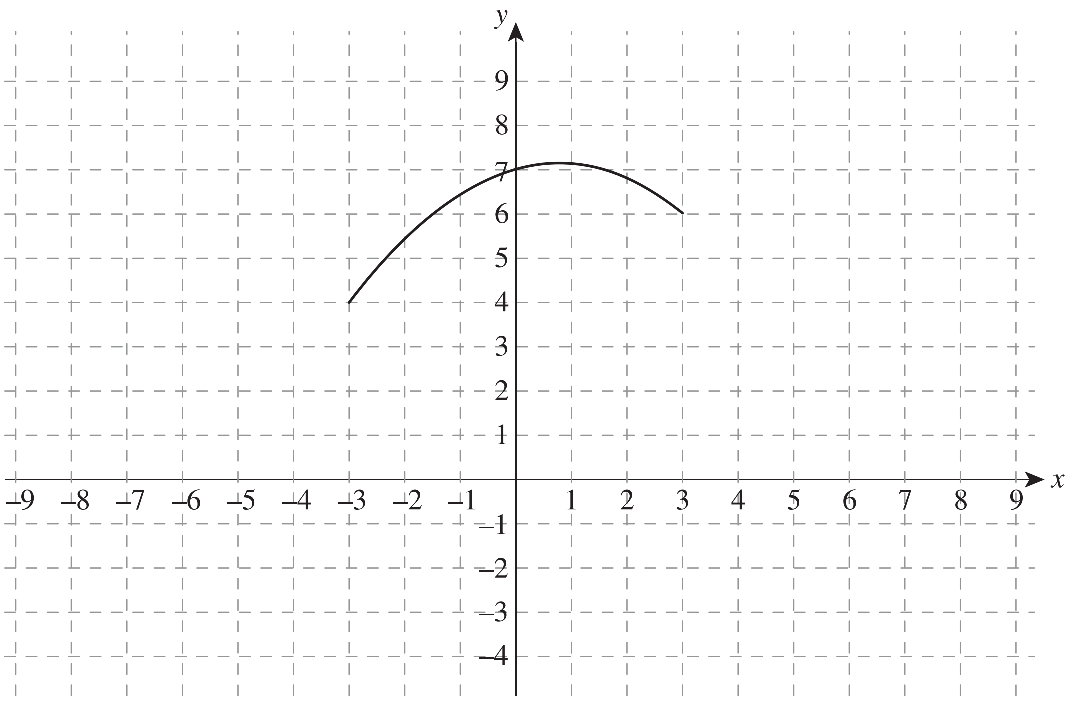
$$b = \frac{25}{8} \qquad \text{A1}$$

b. This transformation is a dilation from y-axis by factor 3 and a translation 1 unit in the positive direction of y-axis. This transformation will move x-intercepts (cusp points) 3 times further away from y-axis to –4.5 and 6 and then move the graph 1 up.

So the rule for the function will become $y = \left|2\left(\frac{x}{3} - \frac{1}{4}\right)^2 - \frac{49}{8}\right| + 1$ M1

Substituting $x = 0$, $x = -3$ and $x = 3$ will give y-intercept and end points.

$(-3, 4)$, $(0, 7)$ and $(3, 6)$. A1



Question 8

a. $f'(x) = 2x\log_e(x) + x$ A1

b. $\int_{\frac{1}{e}}^{e} 2x\log_e(x) + x \, dx = x^2\log_e(x)$

$$\int_{\frac{1}{e}}^{e} 2x\log_e(x)dx + \int_{\frac{1}{e}}^{e} xdx = \left[x^2\log_e x\right]_{\frac{1}{e}}^{e}$$

$$\int_{\frac{1}{e}}^{e} 2x\log_e(x)dx = \left[x^2\log_e x\right]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} xdx$$ M1

$$2\int_{\frac{1}{e}}^{e} x\log_e(x)dx = \left[x^2\log_e x\right]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} xdx$$

$$\int_{\frac{1}{e}}^{e} x\log_e(x)dx = \frac{1}{2}\left(\left[x^2\log_e x\right]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} xdx\right)$$

$$= \frac{1}{2}\left[x^2\log_e(x) - \frac{x^2}{2}\right]_{\frac{1}{e}}^{e}$$ M1

$$= \frac{1}{2}\left[\left(e^2\log_e(e) - \frac{e^2}{2}\right) - \left(\frac{1}{e^2}\log_e\left(\frac{1}{e}\right) - \frac{1}{2e^2}\right)\right]$$

$$= \frac{1}{2}\left[\left(e^2 - \frac{e^2}{2}\right) - \left(-\frac{1}{e^2} - \frac{1}{2e^2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{e^2}{2} + \frac{3}{2e^2}\right)$$

$$= \frac{1}{4}\left(e^2 + \frac{3}{e^2}\right)$$ A1

Question 9

a. As all probabilities should add up to 1

$$3p^2 + 2p = 1$$

$$3p^2 + 2p - 1 = 0$$ M1

$$(3p - 1)(p + 1) = 0$$

Solve for p: $p = \frac{1}{3}$ or $p = -1$. Reject –1 as probability cannot be negative. A1

b. Subtract probability of Jacob not winning any games from 1.

$$\Pr(X > 0) = 1 - \Pr(X = 0) = 1 - p^2$$

$$\Pr(X > 0) = 1 - p^2 = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$$ A1

c. Recognise that this is conditional probability – it is given that Jacob won at least 2 games.

So $\Pr(X \ge 3 | X \ge 2) = \dfrac{\Pr(X \ge 3)}{\Pr(X \ge 2)}$ M1

$$\Pr(X \ge 3 | X \ge 2) = \frac{\frac{p}{2} + \frac{2p^2}{3}}{2p + \frac{2p^2}{3}} = \frac{\frac{13}{54}}{\frac{40}{54}}$$

$$\Pr(X \ge 3 | X \ge 2) = \frac{13}{40}$$ A1

Question 10

a. $f(x) = k(2x^3 - x^2 - 7x + 6)$

$$\int_0^1 k(2x^3 - x^2 - 7x + 6)\,dx = 1$$ M1

$$k\left[\frac{x^4}{2} - \frac{x^3}{3} - \frac{7x^2}{2} + 6x\right]_0^1 = 1$$

$$k\left(\frac{1}{2} - \frac{1}{3} - \frac{7}{2} + 6\right) = 1$$

$$k = \frac{3}{8}$$ A1

b. i. $\Pr(X > 43)$ is required

$$Z = \frac{43 - 50}{5} = -1.4$$

$\Pr(X > 43) = \Pr(Z > -1.4)$ M1

$\Pr(Z > -1.4) = \Pr(Z < 1.4)$ by symmetry

Therefore $\Pr(X > 43) = 0.75$ A1

ii. $\Pr(X > 57 | X > 50)$ is required (conditional probability)

$$\Pr(X > 57 | X > 50) = \frac{\Pr(X > 57)}{\Pr(X > 50)}$$ M1

$$= \frac{\Pr(Z > 1.4)}{\Pr(Z > 0)}$$

$$= \frac{1 - 0.75}{0.5}$$

$$= 0.5$$ A1