Technology active end-of-year examination
Functions and graphs, algebra
Section A: 15 marks
Section B: 25 marks
Suggested writing time: 60 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

$\log_{10}(1000) =$

A 1

B 2

C 3

D 4

E 10^3

$\log_{10}(1000) = \log_{10}(10^3) = 3\log_{10}(10) = 3$

QUESTION 2

An equivalent statement for $a = \log_2(3 - 6x)$ is

A $6x = \dfrac{6 - 2^a}{3}$

B $x = 2^a - 3$

C $x = 1 - \dfrac{2^a}{6}$

D $2 = a^{3-6x}$

E $x = \dfrac{3 - 2^a}{6}$

$a = \log_2(3 - 6x) \Leftrightarrow 2^a = 3 - 6x$

so $x = \dfrac{1}{6}(3 - 2^a) = \dfrac{3 - 2^a}{6}$

QUESTION 3

The rule for the inverse of the function $f(x) = 2 + \log_e(3 - x)$ is $f^{-1}(x) =$

A $2 + \log_e(3 - y)$

B $3 + \log_e(2 - x)$

C $3 - e^{x-2}$

D $2 + e^{3-x}$

E $\dfrac{1}{2 + \log_e(3 - x)}$

$y = 2 + \log_e(3 - x)$

Swap x and y and rearrange to find the inverse.

$x = 2 + \log_e(3 - y)$

$\Rightarrow \log_e(3 - y) = x - 2$

$\Leftrightarrow e^{x-2} = 3 - y$

$y = 3 - e^{x-2}$

Thus, $f^{-1}(x) = 3 - e^{x-2}$

QUESTION 4

For $f(x) = \begin{cases} e^x, & x \geq 0 \\ \sin(x), & x < 0 \end{cases}$

$f\left(-\dfrac{\pi}{2}\right) =$

A $e^{-\frac{\pi}{2}}$ and -1

B $e^{-\frac{\pi}{2}}$

C -1

D e^{-1} and $-\dfrac{\pi}{2}$

E 1

$f(x) = \begin{cases} e^x, & x \geq 0 \\ \sin(x), & x < 0 \end{cases}$

$x = -\dfrac{\pi}{2}$ is in domain $x < 0$

$f\left(-\dfrac{\pi}{2}\right) = \sin\left(-\dfrac{\pi}{2}\right) = -1$

ISBN 9780170354141

QUESTION 5

Consider $f(x) = \cos(x)$ and $g(x) = x^2 + 2x$ for their maximal domains. The domain of $f(g(x))$ is

A $\boldsymbol{R}$

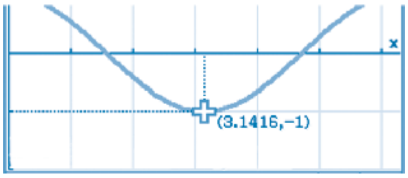
B $[-1, \infty)$

C $(2, \infty)$

D $[-1, 1]$

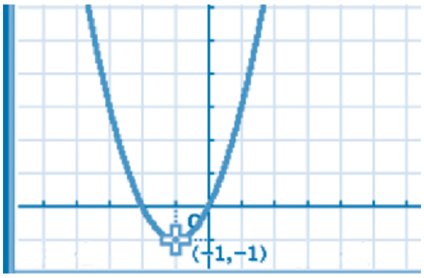
E $[-1, 0]$

$f(x) = \cos(x)$



and

$g(x) = x^2 + 2x$



Test $f(g(x))$.

ran (inner) $\subseteq$ dom (outer)

ran $(g) \subseteq$ dom (f)

$[-1, \infty) \subseteq \boldsymbol{R}$

Dom $f(g(x))$ = dom $g(x) = \boldsymbol{R}$

QUESTION 6

Consider $f(x) = \cos(x)$ and $g(x) = x^2 + 2x$ for their maximal domains. The rule and domain respectively for $g(f(x))$ is

A $y = \cos(x^2 + 2x)$, $[-1, 1]$

B $y = \cos(x)$, $[-1, 1]$

C $y = \cos^2(x) + 2\cos(x)$, $[-1, \infty)$

D $y = \cos(x^2 + 2x)$, $\boldsymbol{R}$

E $y = \cos^2(x) + 2\cos(x)$, $\boldsymbol{R}$

$f(x) = \cos(x)$ and $g(x) = x^2 + 2x$.

Test $g(f(x))$.

ran (inner) $\subseteq$ dom (outer)

ran $(f) \subseteq$ dom (g)

$[-1, 1] \subseteq \boldsymbol{R}$

dom $g(f(x))$ = dom $f(x) = \boldsymbol{R}$

Rule $g(f(x)) = \cos^2(x) + 2\cos(x)$

QUESTION 7

$P(x) = -0.5x^3 + x^2 + 0.5x - 1$ is a cubic polynomial. Factorised into the product of linear factors, it can be expressed as

A $P(x) = (x + 1)(x - 1)(x - 2)$

B $P(x) = x(x + 1)(x - 2)$

C $P(x) = 0.5x(x + 1)(2 - x)$

D $P(x) = 0.5x(x + 1)(x - 2)$

E $P(x) = 0.5(1 - x)(x + 1)(x - 2)$

$P(x) = -0.5x^3 + x^2 + 0.5x - 1$

$= -0.5(x + 1)(x - 1)(x - 2)$

$= 0.5(1 - x)(x + 1)(x - 2)$

QUESTION 8

The solution of the equation $e^{x+1} - 2 = 1$ for x is

A $e^{x+1} = 3$

B $x = \log_e(2) - 1$

C $x = \log_e(3) - 1$

D $x + 1 = \log_e(3)$

E $x = \log_e(y + 2) - 1$

$e^{x+1} - 2 = 1 \Rightarrow e^{x+1} = 3$

$\Leftrightarrow x + 1 = \log_e(3)$

$x = \log_e(3) - 1$

QUESTION 9

The point(s) of intersection between the graph of $y = \log_e(3 - 2x)$ and its inverse is/are closest to

A $x = -2, y = -2$

B $x = 1, y = 1$

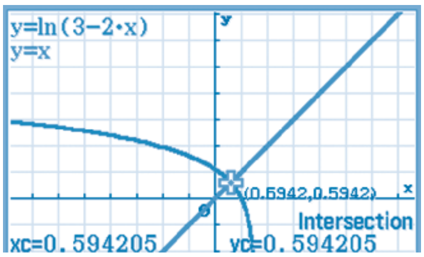
C $x = 0.6, y = 0.6$

D $x = 0.594, y = 0.594$

E $x = 0.058, y = 0.594$

Equate $y = \log_e(3 - 2x)$ with $y = x$, over which the inverse is reflected.

Solve $\log_e(3 - 2x) = x$



This gives $x = 0.594$, $y = 0.594$.

ISBN 9780170354141

QUESTION 10

The graph of $y = -2\cos(2x - 1) + 7$ has an amplitude and period respectively of

A -2 and π

B 10 and 2π

C 2 and $\frac{\pi}{2}$

D 10 and π

E 2 and π

> $y = -2\cos(2x - 1) + 7$
>
> Amplitude $= 2$
>
> Period $= \frac{2\pi}{2} = \pi$

QUESTION 11

The coordinates of A and B are $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(5, 1)$ respectively. The equation of the line perpendicular to AB, which goes through the point B is

A $11y = x + 6$

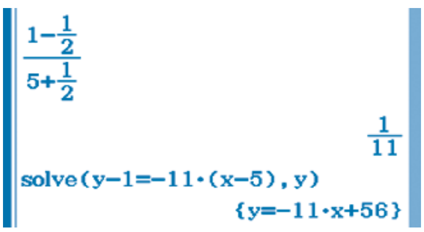
B $9y = 14 - x$

C $y = 9x - 44$

D $y + 11x = 56$

E $y = -9x + 46$

> $A\left(-\frac{1}{2}, \frac{1}{2}\right)$, $B(5, 1)$.
>
> Gradient $AB = \dfrac{1 - \frac{1}{2}}{5 + \frac{1}{2}} = \dfrac{1}{11}$
>
> $m_T = \frac{1}{11}$, $m_N = -11$
>
> For the equation of the line, use $y - y_1 = m(x - x_1)$ with gradient $= -11$ and point $B(5, 1)$.
>
> $y - 1 = -11(x - 5)$
>
> $y = -11x + 56$
>
> $y + 11x = 56$



QUESTION 12

For the polynomial $P(x) = x^3 - x^2 + 2x$, the remainder when $P(x)$ is divided by $(2x + 1)$ is

A $-\frac{1}{2}$

B $-\frac{11}{8}$

C $\frac{11}{8}$

D $\frac{15}{8}$

E -3

> $P(x) = x^3 - x^2 + 2x$
>
> $P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) = -\frac{11}{8}$ = remainder

QUESTION 13

The functionality equation $f(x) + f(y) = f(x + y)$ is satisfied by

A $f(x) = e^{2x}$

B $f(x) = \frac{2}{x}$

C $f(x) = \log_e(2x)$

D $f(x) = 2x$

E $f(x) = 2x^2$

> $f(x) + f(y) = f(x + y)$
>
> By inspection, it is not A (exp) or C (log).
>
> It looks linear.
>
> Test D.
>
> If $f(x) = 2x$, show that $f(x) + f(y) = f(x + y)$.
>
> LHS $= f(x) + f(y)$
>
> $= 2x + 2y$
>
> $= 2(x + y)$
>
> $= f(x + y) =$ RHS

ISBN 9780170354141

QUESTION 14

Simplified, the expression

$\frac{1}{2}\log_a(y) + 3\log_a(x) - \frac{1}{3}\log_a(a)$

can be expressed as

A $\log_a\left(x^3\sqrt{y}\right) - \frac{1}{3}$

B $\frac{1}{6}\log_a\left(ax^3\sqrt{y}\right)$

C $\frac{19}{6}\log_a(a)$

D $\log_e\left(\frac{x^3\sqrt{y}}{\sqrt[3]{a}}\right)$

E $\log_a\left(x^3\sqrt{y}\right)$

$$\frac{1}{2}\log_a(y) + 3\log_a(x) - \frac{1}{3}\log_a(a) = \log_a\left(y^{\frac{1}{2}}\right) + \log_a\left(x^3\right) - \frac{1}{3}$$
$$= \log_a\left(x^3\sqrt{y}\right) - \frac{1}{3}$$

QUESTION 15

The system of linear graphs of $3x - my = m$ and $6mx - 8y = 2$, where m is a constant, has no solutions for

A $m \in R\backslash\{-2, 2\}$

B $m \in R\backslash\{-2\}$

C $m = -2, 2$

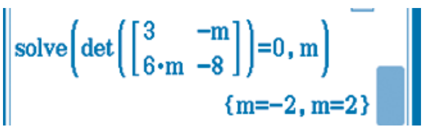
D $m = -2$

E $m = 2$

$3x - my = m$ and $6mx - 8y = 2$

$\det\begin{bmatrix} 3 & -m \\ 6m & -8 \end{bmatrix} = -24 + 6m^2 = 0$ for a singular matrix.

This gives $m = -2, 2$.



Test both $m = -2$ and $m = 2$ in the equations of the lines.

$3x - my = m$, $6mx - 8y = 2$

$m = 2 \Rightarrow 3x - 2y = 2 \quad 12x - 8y = 2$ parallel lines

$m = -2 \Rightarrow 3x + 2y = -2 \quad -12x - 8y = 2$ parallel lines

For parallel lines with no solutions, $m = -2, 2$.

Alternative solution

Equate gradients for lines

$3x - my = m$ and $6mx - 8y = 2$

$\frac{3}{m} = \frac{6m}{8}$

Gives $m = 2, -2$

ONE ANSWER PER LINE | **USE PENCIL ONLY**

1	A	B	■	D	E
2	A	B	C	D	■
3	A	B	■	D	E
4	A	B	■	D	E
5	■	B	C	D	E
6	A	B	C	D	■
7	A	B	C	D	■
8	A	B	■	D	E
9	A	B	C	■	E
10	A	B	C	D	■
11	A	B	C	■	E
12	A	■	C	D	E
13	A	B	C	■	E
14	■	B	C	D	E
15	A	B	■	D	E

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 16

Two lines are defined by $y = 3x - k$ and $y = 3kx + k$ for their maximal domains, where k is a real constant.

a What are the value(s) of k for the two lines to have no point of intersection?

$y = 3x - k$ and $y = 3kx + k$

Rearranging, we have $3x - y = k$ and $3kx - y = -k$

$\det\begin{bmatrix} 3 & -1 \\ 3k & -1 \end{bmatrix} = -3 + 3k = 0$ for a singular matrix.

Thus, $k = 1$.

Test $k = 1$ in the equations of the lines.

$y = 3x - k \quad y = 3kx + k$

$y = 3x - 1 \quad y = 3x + 1$ parallel lines

So there is no point of intersection for $k = 1$.

Alternative solution

Two lines are defined by $y = 3x - k$ (line 1) and $y = 3kx + k$ (line 2).

The gradient of line 1 is 3.

The gradient of line 2 is $3k$.

Solving $3 = 3k$ gives $k = 1$.

3 marks

ISBN 9780170354141

b If the lines $y = 3x - k$ and $y = 3kx + k$ are now perpendicular, show that the value of k is equal to $-\frac{1}{9}$.

Perpendicular lines for $m_1 \times m_2 = -1$

$3 \times 3k = -1$

$k = -\frac{1}{9}$

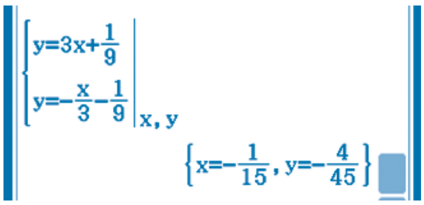
1 mark

c Hence, using $k = -\frac{1}{9}$, find the coordinates of the point of intersection of the lines.

$y = 3x - k$ and $y = 3kx + k$ with $k = -\frac{1}{9}$

become $y = 3x + \frac{1}{9}$ and $y = -\frac{1}{3}x - \frac{1}{9}$

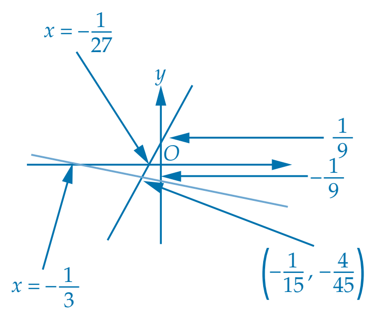
Solve for the intersection point.



The intersection point is $\left(-\frac{1}{15}, -\frac{4}{45}\right)$.

2 marks

d Sketch the graphs of the lines $y = 3x + \frac{1}{9}$ and $y = -\frac{1}{3}x - \frac{1}{9}$, labelling the point of intersection found in part **c**.



2 marks

e Find the area of the shape bounded by the x-intercept of the line with the –ve gradient, the x-intercept of the line with the +ve gradient and the x-axis.

x-intercept of line with –ve gradient, $x = -\frac{1}{3}$

x-intercept of line with +ve gradient, $x = -\frac{1}{27}$

Area of triangle $= \frac{1}{2}$ base × height

$= \frac{1}{2} \times \frac{8}{27} \times \frac{4}{45}$

$= \frac{16}{1215}$ square units

2 marks

(Total: 10 marks)

QUESTION 17

Consider $f: [-2, 2] \to R$, $f(x) = ax^4 - 2bx^2 + c$

a It is known that both $x - 2$ and $2x + 1$ are factors of $f(x)$, and when $f(x)$ is divided by $x + 1$, the remainder is 6. Find a, b and c.

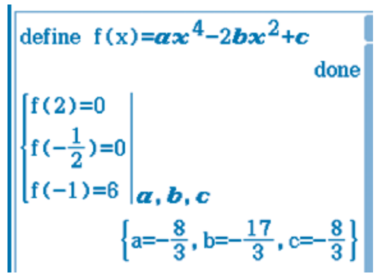
$f(x) = ax^4 - 2bx^2 + c$

$f(2) = 0$

$f\left(-\frac{1}{2}\right) = 0$

$f(-1) = 6$

Solve simultaneously.



Thus $a = -\frac{8}{3}$, $b = -\frac{17}{3}$, $c = -\frac{8}{3}$

4 marks

b Hence, fully factorise $f(x)$.

$$f(x) = ax^4 - 2bx^2 + c$$

$$f(x) = -\frac{8}{3}x^4 + \frac{34}{3}x^2 - \frac{8}{3}$$

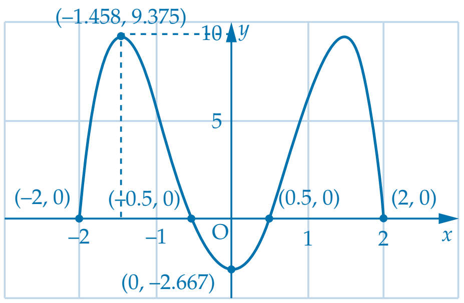
$$= -\frac{2}{3}(4x^4 - 17x^2 + 4)$$

$$= -\frac{2}{3}(x^2 - 4)(4x^2 - 1)$$

$$= -\frac{2}{3}(x + 2)(x - 2)(2x + 1)(2x - 1)$$

1 mark

c Sketch the graph of $f(x)$ for $x \in [-2, 2]$, labelling exact x-intercepts as well as the turning points, correct to 2 decimal places.



3 marks

d Find a new function, $g(x)$, if it is the image of $f(x)$ after it is translated 2 units in the +ve direction of the x-axis, $\frac{8}{3}$ units in the +ve direction of the y-axis, and then dilated by 2 units from the y-axis.

$$f(x - 2) + \frac{8}{3} = -\frac{2}{3}[4(x - 2)^4 - 17(x - 2)^2 + 4] + \frac{8}{3}$$

$$g(x) = -\frac{2}{3}\left[4\left(\frac{1}{2}x - 2\right)^4 - 17\left(\frac{1}{2}x - 2\right)^2 + 4\right] + \frac{8}{3}$$

3 marks

e Express this new function, $g(x)$, in fully factorised form, hence showing that one quadratic factor has rational roots and the other quadratic factor has irrational roots.

$\text{factor}\left(g\left(\frac{1}{2}x-2\right)+\frac{8}{3}\right)$

$$\frac{-(x^2-8\cdot x-1)\cdot(x-4)^2}{6}$$

$$g(x) = -\frac{1}{6}(x - 4)^2(x^2 - 8x - 1)$$

$\text{rFactor}(x^2-8\cdot x-1)$

$$(x+\sqrt{17}-4)\cdot(x-\sqrt{17}-4)$$

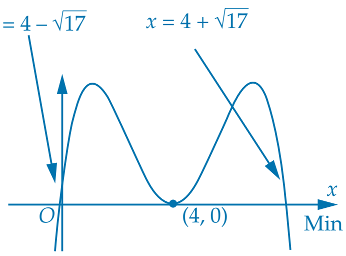
$$g(x) = -\frac{1}{6}(x - 4)^2(x - 4 + \sqrt{17})(x - 4 - \sqrt{17})$$

Rational and irrational roots are shown.

Alternatively, find the discriminant of $(x^2 - 8x - 1)$.

2 marks

f Sketch a graph of $g(x)$, labelling all axial intercepts.



2 marks
(Total: 15 marks)

 ISBN 9780170354141