TEST 5

Technology active end-of-year examination
Functions and graphs, algebra and calculus
Section A: 20 marks
Section B: 40 marks
Suggested writing time: 90 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

If $y = \frac{e^{2x}}{x}$, $\frac{dy}{dx} =$

A $\frac{2e^{2x}}{x}$

B $2e^{2x}$

C $e^{2x}\left(\frac{2x-1}{x}\right)$

D $e^{2x}\left(\frac{2x-1}{x^2}\right)$

E $e^{2x}(2x+1)$

$$y = \frac{e^{2x}}{x}$$

$$\frac{dy}{dx} = \frac{2xe^{2x} - e^{2x}}{x^2} = e^{2x}\left(\frac{2x-1}{x^2}\right)$$

QUESTION 2

Find $f'(x)$ if $f(x) = 3e^{-x}(x+1)$

A $f'(x) = -3e^{-x}$

B $f'(x) = -3xe^{-x}$

C $f'(x) = 3xe^{-x}$

D $f'(x) = 3e^{-x}$

E $f'(x) = 1 + 3xe^{-x}$

$$f(x) = 3e^{-x}(x+1)$$

$$f'(x) = (x+1) \times (-3e^{-x}) + 3e^{-x} \times 1$$

$$= -3xe^{-x} - 3e^{-x} + 3e^{-x}$$

$$= -3xe^{-x}$$

QUESTION 3

Find $f'(1)$ if $f(x) = 2\log_e(1+x^2)$

A $\frac{4x}{x^2+1}$

B $\frac{8}{5}$

C 2

D 1

E 0

$$f(x) = 2\log_e(1+x^2)$$

$$f'(x) = 2 \times 2x \times \frac{1}{x^2+1} = \frac{4x}{x^2+1}$$

$$f'(1) = 2$$

QUESTION 4

If $f'(x) = 3x\cos(x)$, the anti-derivative function $f(x)$ is equal to

A $3(\sin(x) + \cos(x))$

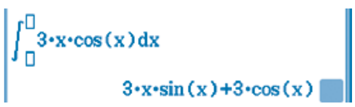
B $3x\sin(x) + 3\cos(x)$

C $\frac{3x^2}{2}\sin(x)$

D $\frac{3x^2}{2}\cos(x)$

E $-3x\sin(x) + 3\cos(x)$

If $f'(x) = 3x\cos(x)$



$f(x) = 3x\sin(x) + 3\cos(x)$

QUESTION 5

For $f(x) = \begin{cases} x+1, & x \le 0 \\ e^x, & x > 0 \end{cases}$

The equation of the tangent at $x = 1$ is

A $y = x + 1$

B $y = xe^x + 1$

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C $y = xe^2$

D $y = xe$

E $y = xe^x$

$$f(x) = \begin{cases} x+1, & x \le 0 \\ e^x, & x > 0 \end{cases}$$

$x = 1$ is in domain $x > 0$.

The gradient of the tangent at $x = 1$ is e^1.

To find the equation of the tangent at $x = 1$,

$y - y_1 = m(x - x_1)$

$y - e = e(x - 1)$

$y = ex - e + e$

This gives

$y = xe$

QUESTION 6

If $f'(x) = 4x^3 - 6x^2 + 3x$ and $f(0) = 1$, $f(x) =$

A $4x^3 - 6x^2 + 3x + 1$

B $x^4 - 2x^3 + \frac{3}{2}x^2 + 1$

C $x^4 - 2x^3 + \frac{3}{2}x^2$

D $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}$

E $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x$

If $f'(x) = 4x^3 - 6x^2 + 3x$, then

$$f(x) = \frac{4x^4}{4} - \frac{6x^3}{3} + \frac{3x^2}{2} + c$$

Also, $f(0) = 1$ so $c = 1$.

$\therefore f(x) = x^4 - 2x^3 + \frac{3}{2}x^2 + 1$

QUESTION 7

The area under the curve $f(x) = -(x - 1)(2x - 1)(x + 1)$ from $x = -1$ to $x = \frac{1}{2}$ is

A $\frac{7}{96}$

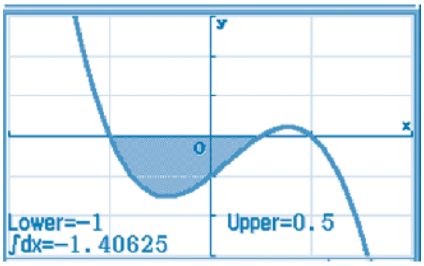
B $\frac{4}{3}$

C $-\frac{4}{3}$

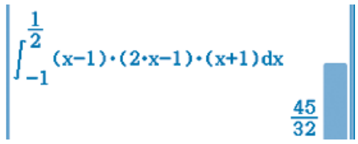
D $\frac{45}{32}$

E $-\frac{45}{32}$

$f(x) = -(x - 1)(2x - 1)(x + 1)$ from $x = -1$ to $x = \frac{1}{2}$ is the area below the x-axis.



$$\text{Area} = -\int_{-1}^{\frac{1}{2}} -(x-1)(2x-1)(x+1)\,dx = \frac{45}{32}$$



QUESTION 8

$f(x) = 3\cos(x)$ and $g(x) = 2\log_e(x)$ for maximal domains. The function $h = f(g(x))$ has the derivative

A $h'(x) = 3\cos(2\log_e(x))$

B $h'(x) = 2\cos^2(3\log_e(x))$

C $h'(x) = 3\cos(2\log_e(x))$

D $h'(x) = -\frac{6}{x}\sin(2\log_e(x))$

E $h'(x) = -\frac{6}{x}\cos(2\log_e(x))$

$f(x) = 3\cos(x)$ and $g(x) = 2\log_e(x)$.

$h = f(g(x)) = 3\cos(2\log_e(x))$

$h'(x) = -3\sin(2\log_e(x)) \times \frac{2}{x} = -\frac{6}{x}\sin(2\log_e(x))$

QUESTION 9

$$\int \frac{2}{3x+1}\,dx =$$

A $\frac{2}{3}\log_e|(3x+1)| + c$

B $2\log_e|(3x+1)| + c$

C $2\log_e|(3x+1)|$

D $3\log_e|(3x+1)|$

E $\frac{3}{2}\log_e|(3x+1)| + c$

$$\int \frac{2}{3x+1}\,dx = \frac{2}{3}\log_e|(3x+1)| + c$$

QUESTION 10

The area enclosed between the curves $y = x^2 - 1$ and $y = -x^2 + 1$ can be found by using the expression

A $\int_{-1}^{1}(x^2 - 1)\,dx$

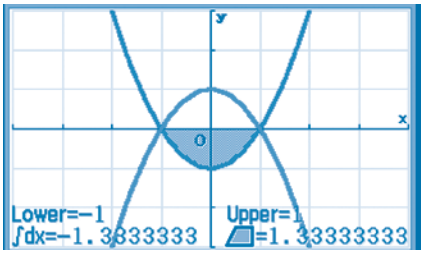
B $\int_{-1}^{1}(-x^2 + 1)\,dx$

C $2\int_{-1}^{1}(-x^2 + 1)\,dx$

D $\int_{-1}^{1}(2x^2 - 2)\,dx$

E $-4\int_{0}^{1}(x^2 - 1)\,dx$

$y = x^2 - 1$ and $y = -x^2 + 1$



Area is double the shading $= 2 \times \frac{4}{3} = \frac{8}{3}$

Or $4 \times$ area between $x = 0$ and $x = 1$ of upper curve

Or $-4 \times$ area between $x = 0$ and $x = 1$ of lower curve

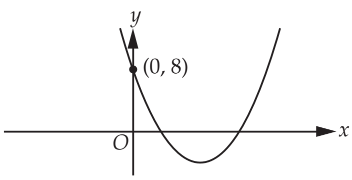
Area $= -4\int_{0}^{1}(x^2 - 1)\,dx$

QUESTION 11

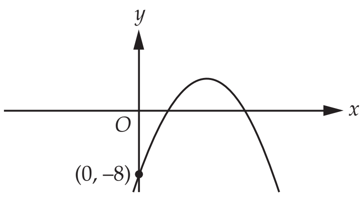
Let $g(x) = x^3 - 6x^2 + 8x$.

The graph that best represents $g'(x)$ is

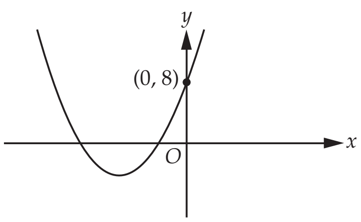
A



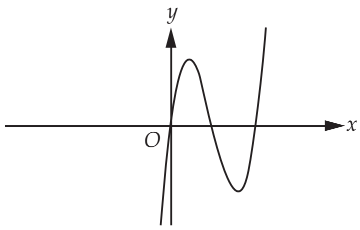
B



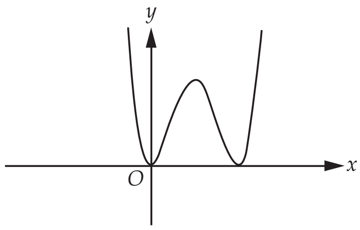
C



D



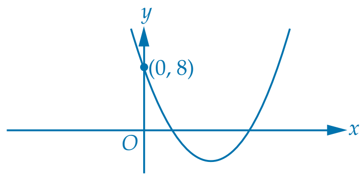
E



Let $g(x) = x^3 - 6x^2 + 8x$.

So $g'(x) = 3x^2 - 12x + 8$

The graph that best represents $g'(x)$ is



QUESTION 12

If $f(x) = 2x^3 - 3x + 3$, the value(s) of x for which the gradient of the curve is equal to 3 is

A $x = -1, x = 1$

B $x = -\frac{\sqrt{2}}{2}, x = \frac{\sqrt{2}}{2}$

C $x = \frac{\sqrt{2}}{2}$ only

D $x = 0$ only

E $x = 1$ only

$f(x) = 2x^3 - 3x + 3$

$f'(x) = 6x^2 - 3$

Letting $f'(x) = 3$ gives

$6x^2 - 3 = 3$

$6x^2 = 6$

$x = -1, x = 1$

QUESTION 13

The velocity of a particle is described by the function $v(t) = 3t^3 - t - 2$, where $v(t)$ is in m/s and t is in seconds, $t \geq 0$. The position of the particle after 3 seconds is

A 20 m

B 50.25 m

C 53.75 m

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D 1.75 m

E 52 m

$v(t) = 3t^3 - t - 2$.

The position of the particle after 3 seconds

$= \int_0^3 (3t^3 - t - 2)\,dt$

$= 50.25$ m

QUESTION 14

The average value of the function $y = x^2 + 3$ from $x = 1$ to $x = 4$ is

A 15

B $33\frac{1}{3}$

C 10

D 5

E $53\frac{1}{3}$

Average value of the function $y = x^2 + 3$ from $x = 1$ to $x = 4$

$= 10$

QUESTION 15

The average rate of change of the function $y = x^2 + 3$ from $x = 1$ to $x = 4$ is

A 15

B $33\frac{1}{3}$

C 10

D 5

E $53\frac{1}{3}$

Average rate of change of the function $y = x^2 + 3$ from $x = 1$ to $x = 4$

$= 5$

QUESTION 16

The gradient of the curve $f(x) = x^2 - 4x$ from $x = 1$ to $x = 1 + h$ can be expressed as

A $\lim_{x \to 0} \frac{f(1+h) - f(1)}{h}$

B $\frac{f(1) - f(1+h)}{h}$

C $\lim_{x \to h} \frac{f(1+h) - f(1)}{h}$

D $\lim_{h \to 0} \frac{(h+1)^2 - 4(h+1) + 3}{h}$

E $h - 2$

From first principles, the gradient of the curve $f(x) = x^2 - 4x$ from $x = 1$ to $x = 1 + h$

$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h)^2 - 4(1+h) + 3}{h}$

QUESTION 17

The velocity of a particle is described by the function $v(t) = 3t^3 - t - 2$, where $v(t)$ is in m/s and t is in seconds, $t \geq 0$. The acceleration of the particle after 3 seconds is

A $9t^2 - 1$ m/s^2

B $18t$ m/s^2

C 54 m/s^2

D 80 m/s^2

E 76 m/s^2

$v(t) = 3t^3 - t - 2$

Acceleration $= 9t^2 - 1$

At $t = 3$, $a = 80$ m/s^2

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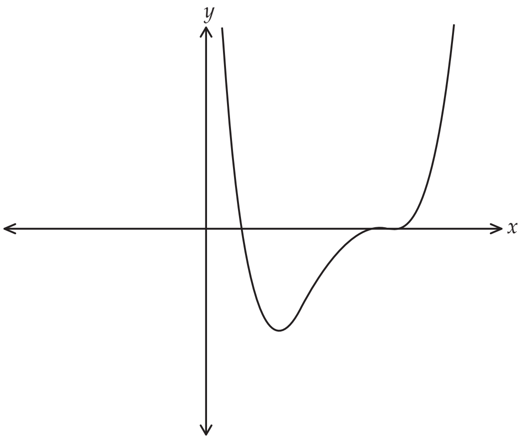
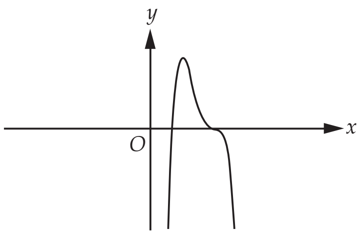
QUESTION 18

A graph is shown below.

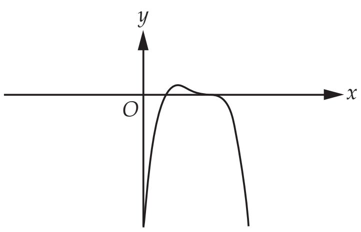


The graph of its derivative looks like

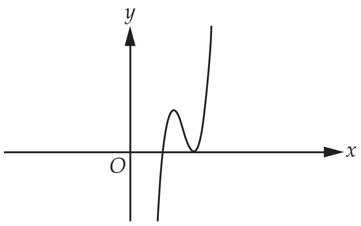
A



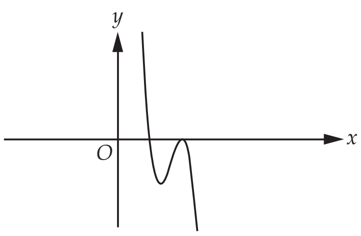
B



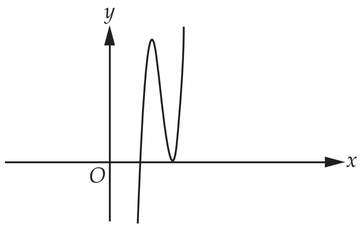
C



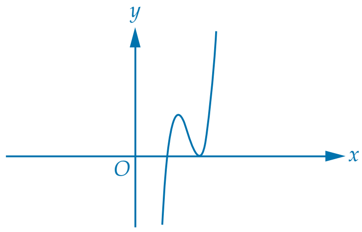
D



E



For the graph of the derivative, we expect a +ve cubic. This is graph C as the graph of E has a local maximum that is too high.



QUESTION 19

Consider $f: \left(-\frac{\pi}{2}, \frac{\pi}{4}\right) \to \boldsymbol{R}, f(x) = 3\cos(x)$.

The maximum value of f is

A 3

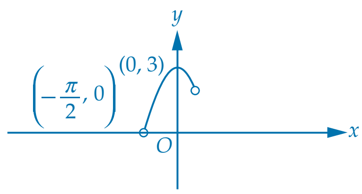
B 0

C $\frac{3\sqrt{2}}{2}$

D $\frac{\pi}{4}$

E $-\frac{\pi}{2}$

$f: \left(-\frac{\pi}{2}, \frac{\pi}{4}\right) \to \boldsymbol{R}, f(x) = 3\cos(x)$.



Maximum value = 3

QUESTION 20

Consider $f: \left[-\frac{\pi}{2}, \frac{\pi}{4}\right) \to \boldsymbol{R}, f(x) = 3\cos(x)$.

The minimum value of f is

A 3

B 0

C $\frac{3\sqrt{2}}{2}$

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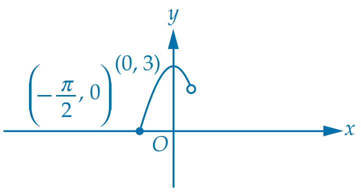
D $\frac{\pi}{4}$

E $-\frac{\pi}{2}$

$f:\left[-\frac{\pi}{2}, \frac{\pi}{4}\right) \to \boldsymbol{R}, f(x) = 3\cos(x).$



Minimum value = 0

ONE ANSWER PER LINE **USE PENCIL ONLY**

1	A	B	C	■	E	11	■	B	C	D	E
2	A	■	C	D	E	12	■	B	C	D	E
3	A	B	■	D	E	13	A	■	C	D	E
4	A	■	C	D	E	14	A	B	■	D	E
5	A	B	C	■	E	15	A	B	C	■	E
6	A	■	C	D	E	16	A	B	C	■	E
7	A	B	C	■	E	17	A	B	C	■	E
8	A	B	C	■	E	18	A	B	■	D	E
9	■	B	C	D	E	19	■	B	C	D	E
10	A	B	C	D	■	20	A	■	C	D	E

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 21

The function $f(x) = x^3 + ax^2 + bx + c$ has a stationary point at (1, 300).

a Find the values of a and b in terms of c.

$f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$f(1) = 300$ and $f'(1) = 0$

Solving simultaneously,

```
define f(x)=x^3+a·x^2+b·x+c
                              done
define g(x)=d/dx(f(x))
                              done
solve({f(1)=300, g(1)=0}, a, b)
        {a=c−302, b=−2·c+601}
```

$a = c - 302$, $b = -2c + 601$

3 marks

A company models its monthly profits using the function $f(x) = x^3 + ax^2 + bx + c$, where $f(x)$ is the monthly profit in \$ and x is the day of the month, where, for example, $x = 1$ represents the 1st of the month and $x = 20$ represents the 20th day of the month. The mathematicians in the company find that profits are satisfactory when $a = b$.

b Find the value of c for which $a = b$.

$a = b$ using $a = c - 302$ and $b = -2c + 601$ gives

$c - 302 = -2c + 601$

$\therefore c = 301$

1 mark

c For the domain $x \geq 1$, and using your values of a, b and c, find $f'(x)$, and hence show that the local maximum or minimum point(s) occur at the end of the domain.

When $c = 301$, $b = a = -1$

$f(x) = x^3 + ax^2 + bx + c$

$f(x) = x^3 - x^2 - x + 301$

$f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$

$f'(x) = 0$ for $x = -\frac{1}{3}, x = 1$

For the domain $x \geq 1$, the only stationary point is at $x = 1$, the endpoint of the domain.

3 marks

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d Find the minimum and maximum profits for a 30-day month, and the days of the month on which they occur.

There are no stationary points within the domain.

So the endpoints are the maximum and minimum.

Since $f(x)$ is an increasing function over the domain, we have

Minimum at (1, 300)

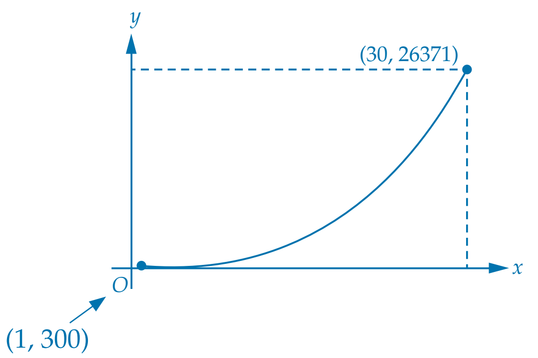
Maximum at (30, 26 371)

So on the 1st day of the 30-day month, the profit is at the minimum of \$300.

On the 30th day, the profit is at the maximum of \$26 371.

3 marks

e Sketch the graph of $f: [1, 30] \rightarrow \mathbf{R}, y = f(x)$, labelling the coordinates of the endpoints.



3 marks

f Find the value of c for which $a = 2b$.

```
{f(1)=300
{g(1)=0      |a, b
          {a=c-302, b=-2·c+601}
solve(c-302=2(-2·c+601), c)
                        {c=300.8}
```

$c = 300.8$

1 mark

g Using the value of c found in part **f**, describe what this does to the profit of the company.

The function now is

$f_1(x) = x^3 - x^2 - x + 300.8$

Since $f_1(x)$ is an increasing function over the interval, we have

Minimum at (1, 299.8)

Maximum at (30, 26 370.8)

So on the 1st day of the 30-day month, the profit is at the minimum of \$299.8.

On the 30th day, the profit is at the maximum of \$26 370.8.

The profit is reduced at both endpoints by 0.2 = 20 cents.

The profit will be lower for every day of the month by 20 cents.

2 marks
(Total: 16 marks)

QUESTION 22

The pitch of a roof in the northern states of America is, by standard, a certain value so that the heavy snow in the area doesn't weigh too heavily on the roof. In general, as the pitch of the roof increases, the snow slides off more easily, and the load on the roof decreases. But a steeper roof is more expensive to build. To encourage snow to slide off, the roof should have a minimum pitch of 3 : 12, meaning that it drops more than 3 feet for every 12 feet of roof, but a pitch of 4 : 12 is better.

a What is the gradient of a 3 : 12 roof?

Gradient $= \dfrac{3}{12} = \dfrac{1}{4}$

1 mark

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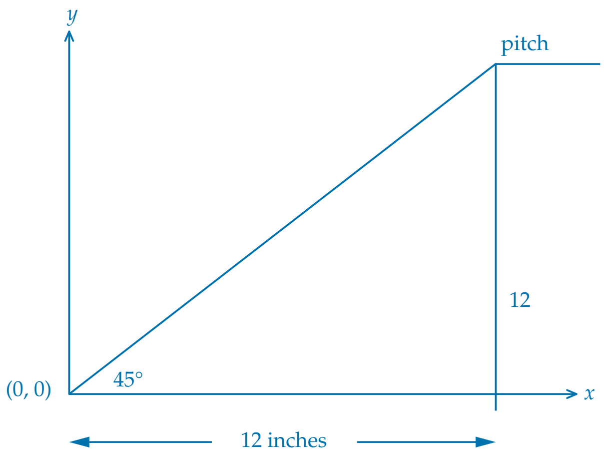
b What is the gradient of a 4 : 12 roof?

Gradient = $\frac{4}{12} = \frac{1}{3}$

Consider the diagram below, where a 12 : 12 pitch forms an angle of 45°.



A pitch of 3 : 12 has an angle of 14°.

A pitch of 4 : 12 has an angle of 18.5°.

1 mark

c If tan (45°) = 1 and equals the ratio $\frac{12}{12}$, what approximate ratio in the form of $\frac{x}{12}$ does tan (14°) equal?

tan (14°) = $\frac{3}{12}$

1 mark

d If tan (45°) = 1 = the ratio $\frac{12}{12}$, what approximate ratio in the form of $\frac{x}{12}$ does tan (18.5°) equal?

tan (18.5°) = $\frac{4}{12}$

1 mark

e A straight line of roofing has a pitch of 4 : 12 and goes through the point (0, 0). What is the equation of the line?

Pitch of 4 : 12

Gradient = $\frac{1}{3}$

The equation of the line is

$y = \frac{1}{3}x$

1 mark

f A straight line of roofing has a pitch of 3 : 12 and goes through the point (2, 10). What is the equation of the line?

Pitch of 3 : 12

Gradient = $\frac{1}{4}$

To find the equation of line at (2, 10),

$y - y_1 = m(x - x_1)$

$y - 10 = \frac{1}{4}(x - 2)$

$4y - 40 = x - 2$

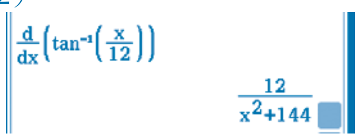
$4y = x + 38$

2 marks

Let $\tan(\theta) = \frac{x}{12}$ so that $\theta = \tan^{-1}\left(\frac{x}{12}\right)$, where θ° is the angle of the roof to the horizontal and x is the vertical height of the roof, in inches.

g Find $\frac{d}{dx}(\theta)$ and hence find the value of x for which the angle of the roof is at its maximum.

$\theta = \tan^{-1}\left(\frac{x}{12}\right)$

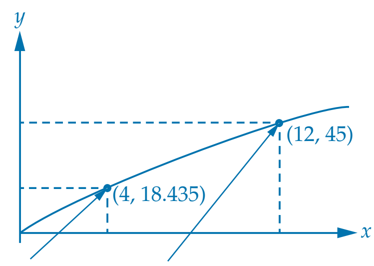


$\frac{d}{dx}(\theta) = \frac{12}{x^2 + 144}$

Solve $\frac{d}{dx}(\theta) = \frac{12}{x^2 + 144} = 0$ for the stationary point.

There is no solution for x.

The angle will be at maximum at the end of the domain, that is, for the maximum magnitude of the vertical height x.



Graph of $\theta = \tan^{-1}\left(\frac{x}{12}\right)$

3 marks

(Total: 10 marks)

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QUESTION 23

Let $f(x) = x + 2$ and $g(x) = 2x^2 + ax + 4$ for maximal domains, where a is a real constant.
If $f(x) = g(x)$, find the value(s) of a for which there is

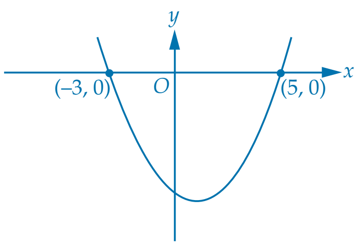
a exactly one solution to the equation $f(x) = g(x)$

$f(x) = g(x)$
$x + 2 = 2x^2 + ax + 4$
$\Rightarrow 2x^2 + ax + 4 - x - 2 = 0$
$\Rightarrow 2x^2 + x(a - 1) + 2 = 0$
$\Delta = (a - 1)^2 - 16 = 0$ for one solution
$(a - 1)^2 = 16$
$a - 1 = \pm 4$
$a = 5, -3$

2 marks

b more than one solution to the equation $f(x) = g(x)$.

We need $\Delta > 0$ for two solutions



$a > 5, a < -3$

2 marks

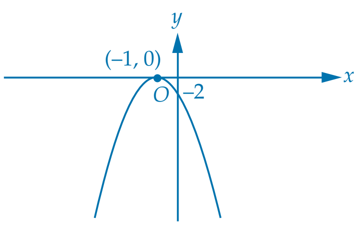
c no solution to the equation $f(x) = g(x)$.

We need $\Delta < 0$ for no solutions
$-3 < a < 5$

1 mark

d For the case of exactly one solution to the equation $f(x) = g(x)$, where $a > 0$, sketch the graph of $h(x) = f(x) - g(x)$.

For exactly one solution, $a = 5, -3$
Since $a > 0$, $a = 5$
$h(x) = f(x) - g(x) = x + 2 - (2x^2 + ax + 4)$, where $a = 5$
$\Rightarrow x + 2 - 2x^2 - 5x - 4$
$\Rightarrow -2x^2 - 4x - 2$
$h(x) = -2x^2 - 4x - 2 = -2(x^2 + 2x + 1)$
So $h(x) = -2(x + 1)^2$



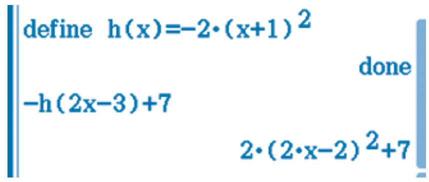
3 marks

e Find $h'(x)$, and hence find the coordinates of any stationary point in the graph of $h(x)$.

$h(x) = -2(x + 1)^2$
$h'(x) = -4(x + 1) = 0$ for stationary point
This gives $x = -1$
Coordinates are $(-1, 0)$.

3 marks

f Find the simplified transformed equation for $h_T(x) = -h(2x - 3) + 7$.



$h_T(x) = -h(2x - 3) + 7$
$h_T(x) = 2(2x - 2)^2 + 7$

1 mark

g Describe in words the step-by-step transformation to get to the image $h_T(x)$.

$h(x) = -2(x + 1)^2$	
Dilate from the y-axis by 0.5 units	$y = -2(2x + 1)^2$
Reflect over the x-axis	$y = 2(2x + 1)^2$
Translate in +ve direction of the x-axis by 1.5 units	$y = 2(2(x - 1.5) + 1)^2$ $\Rightarrow y = 2((2x - 3) + 1)^2$ $\Rightarrow y = 2(2x - 2)^2$
Translate in +ve direction of the y-axis by 7 units	$y = 2(2x - 2)^2 + 7$ so $h_T(x) = 2(2x - 2)^2 + 7$

2 marks
(Total: 14 marks)

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