TEST 2

Functions and graphs
Technology-free end-of-year examination
Total marks: 25
Suggested writing time: 40 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

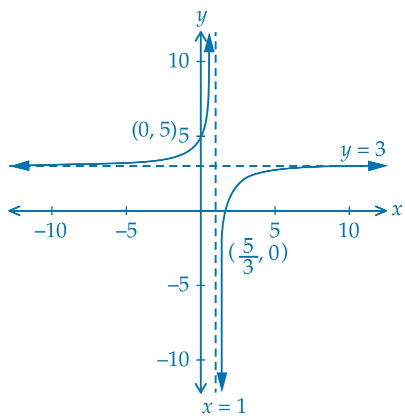
a State the maximal domain for the function

$y = -\dfrac{2}{x-1} + 3$

$x \in \mathbf{R} \backslash \{1\}$

1 mark

b Hence sketch the graph of $y = -\dfrac{2}{x-1} + 3$, labelling equations of asymptotes and axial intercepts with their coordinates.



2 marks
(Total: 3 marks)

QUESTION 2

a The coordinates of A and B are (2, 3) and (4, –1) respectively. Find the equation of the line joining A and B.

Use the equation of the line $y - y_1 = m(x - x_1)$.

Gradient of the line $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1-3}{4-2} = -2$

Using (2, 3), $y - y_1 = m(x - x_1)$ gives

$y - 3 = -2\,(x - 2)$

$y = -2x + 7$

1 mark

b Hence find the equation of the line perpendicular to the line found in part **a**, which passes through the midpoint of A and B.

$m_T = -2$ so $m_N = \dfrac{1}{2}$

Midpoint of A and $B = \left(\dfrac{2+4}{2}, \dfrac{3-1}{2}\right) = (3, 1)$

Using (3, 1), $y - y_1 = m(x - x_1)$ gives

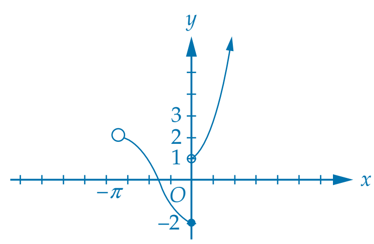
$y - 1 = \dfrac{1}{2}(x - 3)$

$2y = x - 1$

2 marks
(Total: 3 marks)

QUESTION 3

a Sketch the graph of $y = \begin{cases} -2\cos(x), & -\pi < x \le 0 \\ e^x, & x > 0 \end{cases}$



2 marks

b State the value of $f(0)$.

$f(0) = -2$

1 mark
(Total: 3 marks)

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QUESTION 4

Find the image of the point (1, –6) under each of the following transformations:

a reflection in the y-axis

By matrix methods, the solutions are:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

So $(1, -6) \rightarrow (-1, -6)$, i.e., positive x becomes negative x.

1 mark

b dilation by factor 2 from the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$$

So $(1, -6) \rightarrow (1, -12)$; i.e. each y value is multiplied by 2.

1 mark

c dilation by factor 3 from y-axis

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

So $(1, -6) \rightarrow (3, -6)$; i.e., each x value is multiplied by 3.

1 mark

(Total: 3 marks)

QUESTION 5

a Find the rule and domain for the inverse of the function

$f: (-\infty, 2] \rightarrow \boldsymbol{R}, f(x) = 2(x-2)^2 + 1$

To find f^{-1}, swap x and y in $y = 2(x-2)^2 + 1$.

$x = 2(y-2)^2 + 1$

$(y-2)^2 = \dfrac{x-1}{2}$

$y = 2 \pm \sqrt{\dfrac{x-1}{2}}$ select lower branch

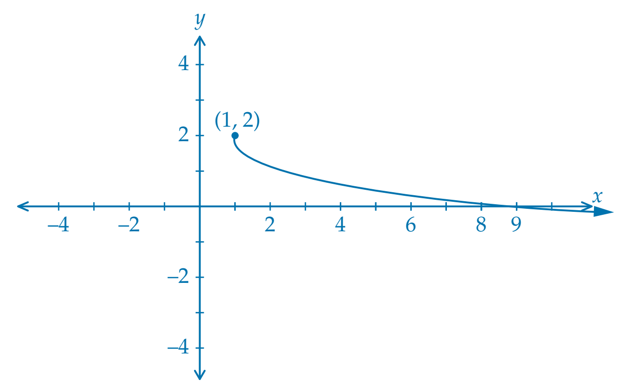
Rule: $f^{-1}(x) = 2 - \sqrt{\dfrac{x-1}{2}}$

Domain: $f^{-1}(x)$ = range $f(x) = [1, \infty)$

$f^{-1}: [1, \infty) \rightarrow \boldsymbol{R}, f^{-1}(x) = 2 - \sqrt{\dfrac{x-1}{2}}$

2 marks

b Sketch the graph of $f^{-1}(x)$



2 marks

(Total: 4 marks)

QUESTION 6

Solve for x in the equation $e^{2x} - 2e^x - 3 = 0$.

$e^{2x} - 2e^x - 3 = 0$

Let $A = e^x$

This gives $A^2 - 2A - 3 = 0$

$(A-3)(A+1) = 0$

$A = 3, A = -1$

So $e^x = 3$, $e^x = -1$ (not a possible solution)

$e^x = 3 \Leftrightarrow x = \log_e(3)$

Answer: $x = \log_e(3)$

2 marks

QUESTION 7

Solve for x in the equation

$2\log_e(x+2) - 2\log_e(2) = 3\log_e(4)$

$2\log_e(x+2) - 2\log_e(2) = 3\log_e(4)$

$\Leftrightarrow \log_e(x+2)^2 = 2\log_e(2) + 3\log_e(2^2)$

$\Leftrightarrow \log_e(x+2)^2 = 2\log_e(2) + 6\log_e(2)$

$\Leftrightarrow \log_e(x+2)^2 = 8\log_e(2) = \log_e(2^8)$

Hence, we have

$(x+2)^2 = 2^8$

$x^2 + 4x + 4 = 2^8$

$x^2 + 4x - 252 = 0$

$(x+18)(x-14) = 0 \rightarrow x = -18, x = 14$

$x = -18$ gives $\log_e(-16)$, which doesn't exist.

Answer: $x = 14$

2 marks

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QUESTION 8

For $f: (1, \infty) \rightarrow \boldsymbol{R}, f(x) = -\log_e(x - 1)$ and $g: [-1, 10] \rightarrow \boldsymbol{R}$, $g(x) = e^x$

a find the rule and domain for $(f + g)(x)$

The **sum** is defined as $(f + g)(x) = f(x) + g(x)$

Rule: $(f + g)(x) = -\log_e(x - 1) + e^x$

Domain $(f + g)(x)$ = intersection of domain of $f(x)$ and $g(x)$.

Domain $(f + g)(x) = x \in (1, 10]$

2 marks

b find the rule and domain for $f(g(x))$ to exist.

$f(g(x))$ exists if ran $(g) \subseteq$ dom (f).

ran $(g) \subseteq$ dom $(f) \rightarrow [e^{-1}, e^{10}] \not\subset (1, \infty)$

Restrict the range of g so that ran $(g) \subseteq$ dom (f).

$(e^0, e^{10}] \subseteq (1, \infty)$

Rule: $f(g(x)) = -\log_e(e^x - 1)$

Domain $f(g(x))$ = restricted dom $(g(x)) = (0, \infty)$

3 marks
(Total: 5 marks)

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