B

Free Exam
for 2006-15 VCE study design

Engage
Education
Foundation

Units 3 and 4 Maths Methods (CAS): Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1

The correct answer is C.

At $x = 2, y = 2, \frac{d}{dx}\sqrt{2x} = \frac{1}{\sqrt{2x}} = \frac{1}{2}$

Hence the gradient of the normal is $-1/\frac{1}{2} = -2$, and its equation is $y - 2 = -2(x - 2)$ or $y = 6 - 2x$.

Question 2

The correct answer is E.

$$\sigma_x^2 = E[X^2] - (E[X])^2 = \left[1^2\frac{1}{21} + 2^2\frac{2}{21} + \cdots + 6^2\frac{6}{21}\right] - \left[1\frac{1}{21} + 2\frac{2}{21} + \cdots + 6\frac{6}{21}\right]^2 = \frac{20}{9}$$

$$\sigma_x = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3}$$

Question 3

The correct answer is B.

$$\text{average value } = \frac{1}{5-1}\int_1^5 \frac{1}{x}dx = \frac{1}{4}[log_e x]_1^5 = \frac{log_e 5}{4}$$

Question 4

The correct answer is A.

$$2\cos(\text{x}) = \sqrt{3} \Rightarrow \cos(x) = \frac{\sqrt{3}}{2} \Rightarrow x = 2k\pi \pm \frac{\pi}{6}, k \in \mathbb{Z}$$

Question 5

The correct answer is E.

f is one to one, $f(1) = 1$ and $f(4) = 8$ hence $D = [1,4) = \{x: x \geq 1\} \cap \{x: x < 4\}$

Question 6

The correct answer is D.

Question 7

The correct answer is B.

$$\text{probability} = 4 \times \frac{C_5^{13}}{C_5^{52}} = \frac{\mathbf{4\times13!\times47!}}{\mathbf{8!\times52!}}$$

Question 8

The correct answer is C.

This can be observed by graphing f.

Question 9

The correct answer is E.

$$\int 2^x dx = \int e^{x log_e 2} dx = \frac{2^x}{log_e 2} + c$$

Question 10

The correct answer is A.

This is most easily obtained by drawing a Venn diagram.

Question 11

The correct answer is B.

This is given by the formula for linear approximation, noting that $\sqrt[3]{343} = 7$.

Question 12
The correct answer is E.
$f'(x) = \frac{2x\log_e x - x}{(\log_e x)^2} = 0 \text{ when } x = \sqrt{e}, f(\sqrt{e}) = 2e$

Question 13
The correct answer is D.
The period of the tangent function given is $\left|\frac{\pi}{a}\right|$, and it attains all real values once within each period, hence the number of solutions is given by $\frac{\text{interval width}}{\text{period}} = |a|$.

Question 14
The correct answer is B.
Independence of events A and B requires that $\Pr(A|B) = \Pr(A)$ and vice versa. This only holds for response B.

Question 15
The correct answer is C.
Taking the derivative by first principles from both sides of $x = 2$ yields values of different signs.

Question 16
The correct answer is B.
This follows from the general probability mass function of binomially distributed random variables.

Question 17
The correct answer is E.
The intersections of the curves are found by setting $3x = x(x^2 - 1) \Rightarrow x = 0, \pm 2$
Observation of the graph shows that $y = 3x$ lies above $y = x(x^2 - 1)$ over $(0,2)$, and the second enclosed region on $(-2,0)$ can be seen to have the same area by symmetry.

Question 18
The correct answer is A.
a is found at the turning point of the parabola, restricting it such that it is one to one. Reasons for taking only the positive root in our rule can be seen by reflecting our restricted f in the line $y = x$.

Question 19
The correct answer is D.
Differentiating gives a local maximum at $x = \frac{2\pi}{3}$ which is higher than the endpoint at $x = 1$. The minimum is found at the left endpoint..

Question 20
The correct answer is E.

Question 21
The correct answer is C.
The $\Pr(68 \leq X \leq 84) = \Pr(-3 \leq Z \leq 1) = \Pr(Z \leq 1) - \Pr(Z \leq -3) = 0.840$

Question 22
The correct answer is C.
g must be restricted such that its range is a subset of the domain of f.

Section B – Short-answer questions

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a
$x, y > 0$ [1]

Question 1b
$S = x^2 + 4xy = 108$
$\Rightarrow y = \frac{27}{x} - \frac{x}{4}$ [1]
$V = x^2 y$
$= x^2 \left(\frac{27}{x} - \frac{x}{4}\right) = 27x - \frac{1}{4}x^3$ [1]

Question 1c
$\frac{dV}{dx} = 27 - \frac{3}{4}x^2$ [1]
$\frac{dV}{dx} = 0$ at stationary points
$\Rightarrow \frac{3}{4}x^2 = 27 \Rightarrow x^2 = 36 \Rightarrow x = 6 \; as \; x > 0$ [1]
$V(6) = 108cm^3$, and a sign diagram must be used to show that this is a maximum [1]
dimensions are $6cm \times 6cm \times 3cm$ [1]

Question 1d
Graph should be sinusoidal and:
be centred around $h = 1cm$ with period 2 seconds (3 periods are observed in given domain) [2]
have stationary points $(0,0.8)$, $(1,1.2)$, $(2,0.8)$, $(3,1.2)$, $(4,0.8)$, $(5,1.2)$, $(6,0.8)$ [2]

Question 1e
First solve $\frac{cos(\pi(x-3))}{5} + 1 = 0.9$ to obtain $t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}, \frac{13}{3}, \frac{17}{3}$ [2]
Then note from the graph that $h < 0.9$ on $\left(0, \frac{1}{3}\right), \left(\frac{5}{3}, \frac{7}{3}\right)\left(\frac{11}{3}, \frac{13}{3}\right) \; and \; \left(\frac{17}{3}, 6\right)$ [1]
Hence $\text{total time unsafe} = \text{sum of interval widths} = \frac{1}{3} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3} = 2$ seconds [1]

Question 2a
$dom(f) = (0, \infty)$, $ran(f) = \mathbb{R}$, $dom(g) = \mathbb{R}$, $ran(g) = \mathbb{R}$ [1]

Question 2b
Existence requires $ran(g) \subseteq dom(f)$ ie. $(a + 1, \infty) \subseteq (0, \infty)$
$\therefore a > -1, a_{min} = -1$ [1]
Rule is $f[g(x)] = 2log_e(x + 1)$ [1]

Question 2c
The curve should:
be of (rising) logarithmic shape [1]
have an asymptote at $x = -1$ and an intercept at $(0,0)$ [1]

Question 2d
Let $x = 2log_e(y + 1) \Rightarrow (f \circ g)^{-1}(x) = y = e^{\frac{x}{2}} - 1$ [1]
Domain is $\mathbb{R}$, range is $(-1, \infty)$ [1]

Question 2e

$\int_0^{2\pi} e^{\frac{x}{2}} - 1dx = \left[2e^{\frac{x}{2}} - x\right]_0^{2\pi} = (2e^{\pi} - 2\pi) - (2e^0 - 0) = 2e^{\pi} - 2\pi - 2$ [1]

Equivalent area is that bounded by $(f \circ g)^{-1}(x)$, the y axis, and the line $y = 2\pi$ [1]

Question 2f

$\int_0^{e^{\pi}-1} (f \circ g)(x)\, dx = [(e^{\pi} - 1) \times (f \circ g)(e^{\pi} - 1)] - \int_{(f\circ g)(0)}^{(f\circ g)(e^{\pi}-1)} (f \circ g)^{-1}(x)dx$ [2]

$= (e^{\pi} - 1) \times 2\pi - \int_0^{2\pi} (f \circ g)^{-1}(x)dx = (2\pi e^{\pi} - 2\pi) - (2e^{\pi} - 2\pi - 2) = e^{\pi}(2\pi - 2) + 2$ [1]

$= 101.116\ (3DP)$ [1]

Question 3a

Circle $x^2 + y^2 = 9$ [1]

Top half $y = \sqrt{9 - x^2}$, bottom half $y = -\sqrt{9 - x^2}$ [1]

Question 3b

Consider first the tangent touching the circle above the x-axis

$\frac{d}{dx}\left(\sqrt{9 - x^2}\right) = -\frac{x}{\sqrt{9-x^2}}$ [1]

$\text{gradient} = \frac{0-\sqrt{9-x^2}}{5-x} = \frac{d}{dx}\left(\sqrt{9 - x^2}\right)$ [1]

$$\Rightarrow \frac{\sqrt{9 - x^2}}{x - 5} = -\frac{x}{\sqrt{9 - x^2}}$$

$\Rightarrow 9 - x^2 = -x(x - 5) \Rightarrow 9 - x^2 = 5x - x^2 \Rightarrow x = \frac{9}{5}$, at which point[1]

$\text{gradient} = -\frac{\frac{9}{5}}{\sqrt{9-\left(\frac{9}{5}\right)^2}} = -\frac{\frac{9}{5}}{3\sqrt{1-\frac{9}{25}}} = -\frac{\frac{9}{5}}{3\sqrt{\frac{16}{25}}} = -\frac{\frac{9}{5}}{3\times\frac{4}{5}} = -\frac{9}{12} = -\frac{3}{4}$ and [1]

$y = \sqrt{9 - \left(\frac{9}{5}\right)^2} = \frac{12}{5}$ [1]

Hence the tangent has the equation: $y - 0 = -\frac{3}{4}(x - 5) \Rightarrow y = -\frac{3}{4}x + \frac{15}{4}$ [1]

By symmetry (or otherwise), we deduce that the equation of the tangent touching the circle below the x-axis is $y = \frac{3}{4}x - \frac{15}{4}$ [2]

Question 3c

Recognising that the area bounded by the two tangents and the x-axis is equivalent to that of a rectangle, and that the area under each half of the circle is equal, we obtain

$$\text{area} = \left(5 - \frac{9}{5}\right)\left(\frac{12}{5} - 0\right) - 2\int_{\frac{9}{5}}^{3} \sqrt{9 - x^2}dx$$

OR using only areas between curves and symmetry properties,

$\text{area} = 2\left\{\int_{\frac{9}{5}}^{3} \left[\left(-\frac{3}{4}x + \frac{15}{4}\right) - \left(\sqrt{9 - x^2}\right)\right] dx + \int_3^5 \left[-\frac{3}{4}x + \frac{15}{4}\right] dx\right\}$ [2]

This area is approximately $3.654\ unit^2$ [1]

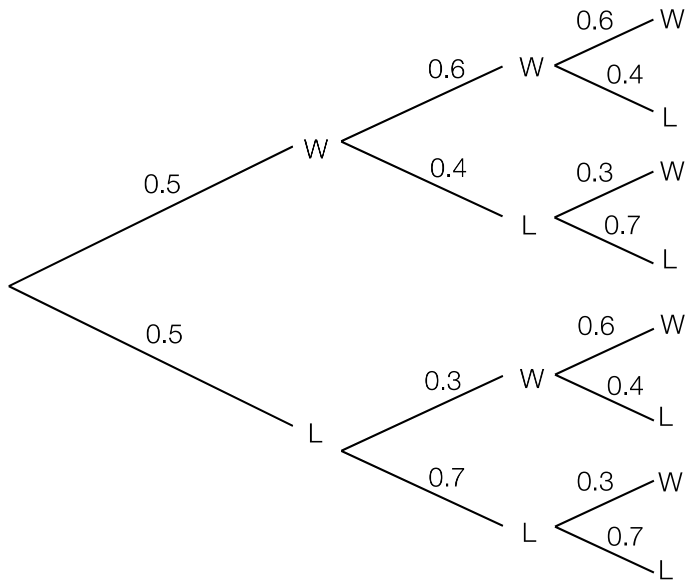
Question 4a

$T = \begin{bmatrix} P(W_{i+1}|W_i) & P(W_{i+1}|L_i) \\ P(L_{i+1}|W_i) & P(L_{i+1}|L_i) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ [1]

$S_5 = T^4 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.429 \\ 0.571 \end{bmatrix}\ (3DP)$

$Pr(W_5) = 0.429\ (3DP)$ [1]

Question 4b



$\Pr(WWW) = (0.5)(0.6)(0.6)$

$\Pr(WWL) = (0.5)(0.6)(0.4)$

$\Pr(WLW) = (0.5)(0.4)(0.3)$

$\Pr(WLL) = (0.5)(0.4)(0.7)$

$\Pr(LWW) = (0.5)(0.3)(0.6)$

$\Pr(LWL) = (0.5)(0.3)(0.4)$

$\Pr(LLW) = (0.5)(0.7)(0.3)$

$\Pr(LLL) = (0.5)(0.7)(0.7)$

Correct tree/sample space [2]
Correct probabilities [2]

Question 4c

$Pr(W_1|L_3) = \frac{Pr(W_1 \cap L_3)}{Pr(L_3)}$ [1]

$= \frac{Pr(WWL)+Pr(WLL)}{Pr(WWL)+Pr(WLL)+Pr(LWL)+Pr(LLL)}$ [1]

$= 0.460\ (3DP)$ [1]

Question 4d

In the long run we expect our proportion of wins to be $\frac{0.3}{0.4+0.3} = \frac{3}{7}$ (from transition matrix) [1]

Question 4e

For this question we shall use $invnorm(x)$ to denote the inverse standard normal of x

$invnorm(0.9) = \frac{80-\mu}{\sigma}$ and $invnorm(0.05) = \frac{30-\mu}{\sigma}$ [1]

$invnorm(0.9) - invnorm(0.05) = \frac{80-\mu}{\sigma} - \frac{30-\mu}{\sigma}$

$\sigma = \frac{80-30}{invnorm(0.9)-invnorm(0.05)} = 17.083\ (3DP)$ [1]

$\mu = 80 - \sigma \times invnorm(0.9) = 58.105\ (3DP)$ [1]

Question 4f

$E[P_d] = \frac{7}{8}E[P] + 10 = 60.842\ (3DP)$ [1]

$sd(P_d) = \sqrt{Var(P_d)} = \sqrt{\left(\frac{7}{8}\right)^2 Var(P)} = 14.947\ (3DP)$ [1]

$Pr(P_d < 30) = 0.020\ (3DP), Pr(P_d > 80) = 0.100\ (3DP)$ [1]

$Pr(P_d < 30) + Pr(P_d > 80) < Pr(P < 30) + Pr(P > 80) \Rightarrow$ Callum should take the drug [1]