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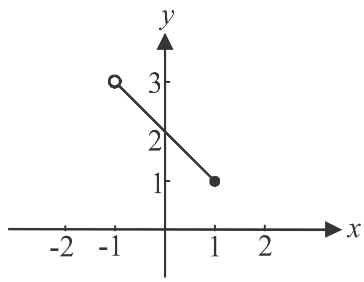
MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2015

SECTION 1 – Multiple-choice answers

1. E	9. D	17. A
2. C	10. C	18. D
3. E	11. E	19. E
4. A	12. D	20. C
5. B	13. D	21. B
6. C	14. D	22. C
7. D	15. D	
8. B	16. E	

SECTION 1 – Multiple-choice solutions

Question 1



Do a quick sketch.
$r_f = [1,3)$
The answer is E.

Question 2

The maximal domain occurs when $x^2 - x - 6 \geq 0$ since the square root of a negative number is undefined for $x \in R$.
Solve $x^2 - x - 6 \geq 0$ for *x*.
$x \leq -2$ or $x \geq 3$
In interval notation, this is expressed as $x \in (-\infty, -2] \cup [3, \infty)$.
The answer is C.

Question 3

Method 1 - intuitively

$$y = \tan\left(\frac{\pi x}{4}\right)$$

$$\begin{aligned}\text{period} &= \frac{\pi}{n} \text{ where } n = \frac{\pi}{4}\\ &= \pi \div \frac{\pi}{4}\\ &= \pi \times \frac{4}{\pi}\\ &= 4\end{aligned}$$

The graph of this function is dilated by a factor of 2 from the *y*-axis, that is, it is stretched horizontally by a factor of 2, so its period will be 8.
The answer is E.

Method 2 – algebraically

$$y = \tan\left(\frac{\pi x}{4}\right)$$

The graph of this function is dilated by a factor of 2 from the y-axis, so replace x with $\frac{x}{2}$ in the equation.

$$\begin{aligned} y &= \tan\left(\frac{\pi}{4} \times \frac{x}{2}\right) \\ &= \tan\left(\frac{\pi x}{8}\right) \end{aligned}$$

The period of this transformed function is

$$\begin{aligned} \frac{\pi}{n} &= \pi \div \frac{\pi}{8} \\ &= 8 \end{aligned}$$

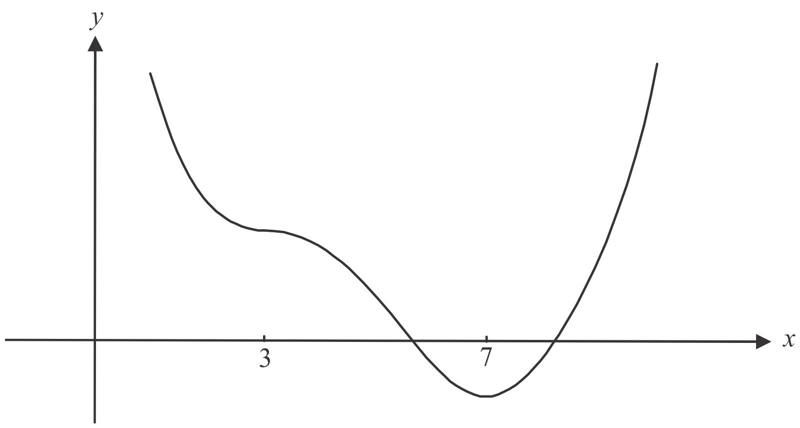
The answer is E.

Question 4

The period of the graph is π. Since the period is $\frac{2\pi}{n}, \frac{2\pi}{n} = \pi$ so $n = 2$.
The shape of the graph is that of an inverted cos graph that has been translated 1 unit down so the rule could be $g(x) = -\cos(2x) - 1$.
The answer is A.

Question 5

Do a quick sketch of a possible graph.



There is a stationary point of inflection at $x = 3$. Note that there could be an x-intercept at $x = 7$ if the graph shown was translated vertically up but there doesn't have to be.
The answer is B.

Question 6

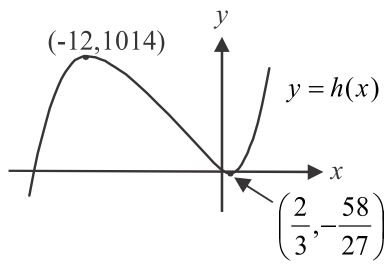
$h(x) = x^3 + 17x^2 - 24x + 6$

$h'(x) = 3x^2 + 34x - 24 = 0$

$(x+12)(3x-2) = 0$

Stationary points occur at $x = -12$ and $x = \frac{2}{3}$.

Do a quick sketch.



An inverse will exist if h is a $1:1$ function.

This will only occur if $D = \left[-12, \frac{2}{3}\right]$

The answer is C.

Question 7

$$\int_2^4 (1 - 3h(x))dx$$

$$= \int_2^4 1dx - 3\int_2^4 h(x)dx$$

$$= [x]_2^4 - 3 \times -1$$

$$= 4 - 2 + 3$$

$$= 5$$

The answer is D.

Question 8

$f'(x) = e^{\sqrt{x}}, \quad x \ge 0$

$f(x) = \int e^{\sqrt{x}}\, dx$

So $f(x) = 2e^{\sqrt{x}}\left(\sqrt{x} - 1\right) + c$ using CAS

Since $f(0) = 1$,

$1 = 2e^0(0-1) + c$

$c = 3$

So $f(x) = 2e^{\sqrt{x}}\left(\sqrt{x} - 1\right) + 3$

The answer is B.

Question 9

$E(X) = 0 \times 0.1 + 2 \times a + 3 \times b + 5 \times 0.2 = 2.6$

So $2a + 3b = 1.6 \quad -(A)$

Also, $0.1 + a + b + 0.2 = 1$

$a + b = 0.7 \quad -(B)$

$(B) \times 2 \qquad 2a + 2b = 1.4 \quad -(C)$

$(A) - (C) \qquad b = 0.2$

In $(B) \qquad a = 0.5$

The answer is D.

Question 10

Pr(no black balls)

$= \Pr(W,W) + \Pr(W,R) + \Pr(R,W)$

$= \frac{5}{10} \times \frac{4}{9} + \frac{5}{10} \times \frac{1}{9} + \frac{1}{10} \times \frac{5}{9}$

$= \frac{30}{90}$

$= \frac{1}{3}$

The answer is C.

Question 11

$y = x^{\frac{2}{3}} + 1$

$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{2}{3\sqrt[3]{x}}$

When $x = -1$, $\frac{dy}{dx} = \frac{2}{3 \times -1} = -\frac{2}{3}$.

So the gradient of the tangent is $-\frac{2}{3}$ and the gradient of the normal is therefore $\frac{3}{2}$.

Equation of normal:

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{2}(x - -1)$

$y = \frac{3}{2}x + \frac{3}{2} + 2$

$y = \frac{3}{2}x + \frac{7}{2}$

The y-intercept is $\frac{7}{2}$.

The answer is E.

Question 12

$$f(x)=e^{\frac{x}{2}},\quad x\in R$$

$$(f(2x))^2=\left(e^{\frac{2x}{2}}\right)^2$$

$$=(e^x)^2$$

$$=e^{2x}\qquad (\text{NOT } e^{x^2})$$

Now, $f\left(\frac{x}{2}\right)=e^{\frac{x/2}{2}}$

$$=e^{\frac{x}{4}}$$

$$\neq e^{2x}$$

$$f(2x^2)=e^{\frac{2x^2}{2}}$$

$$=e^{x^2}$$

$$\neq e^{2x}$$

$$f(4x^2)=e^{\frac{4x^2}{2}}$$

$$=e^{2x^2}$$

$$\neq e^{2x}$$

$$f(4x)=e^{\frac{4x}{2}}$$

$$=e^{2x}$$

So option D is correct.

Note $2(f(x))^2=2\left(e^{\frac{x}{2}}\right)^2$

$$=2e^x$$

$$\neq e^{2x}$$

The answer is D.

Question 13

Since A and B are independent events,

$$\Pr(A\cap B)=\Pr(A)\times\Pr(B)$$

$$0.2=0.5\times\Pr(B)\qquad \text{So}\quad \Pr(B)=0.4$$

$$\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}$$

$$=\frac{0.2}{0.4}$$

$$=0.5$$

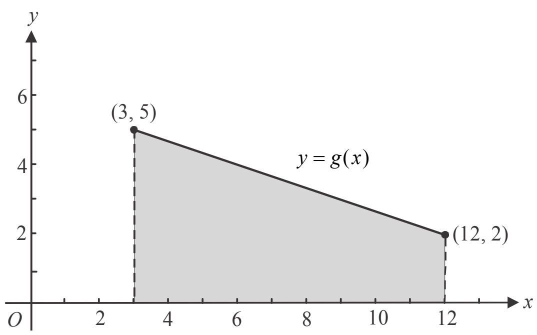
The answer is D.

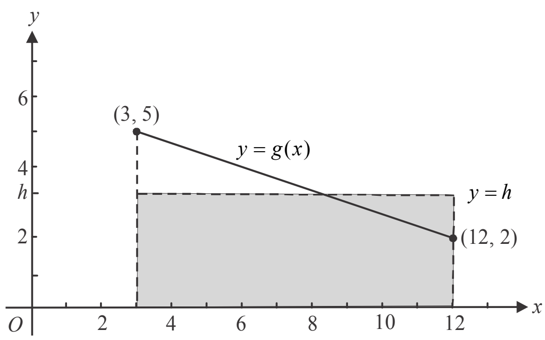
Question 14

$$\Pr(X > 4) = \int_4^6 \frac{1}{(x+2)\log_e(2)}dx$$
$$= 0.415037...$$

The closest answer is 0.4150.
The answer is D.

Question 15





The two shaded regions shown in the two diagrams above are equal in area. The average value of *g* is *h*, as indicated in the second diagram.
We want to find *h*.

shaded area in diagram 1 = shaded area in diagram 2

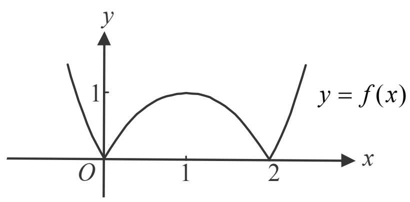
area of rectangle + area of Δ = base × height

$$9\times 2+\frac{1}{2}\times 9\times 3 = 9\times h$$
$$\frac{63}{2} = 9h$$
$$h = 3.5$$

The answer is D.

Question 16

Do a quick sketch.
For *f*, $f(1)=1$, the graph is continuous for $x \in R$, and the gradient is positive for $0 < x < 1$ and $x > 2$.
At $x=0$ and $x=2$, the graph has a sharp point; that is, it is not smoothly continuous and therefore is not differentiable at these points. Option E is not true.
The answer is E.



Question 17

$x + 2ay = 6$

$ax + 8y = a + 10$

$$\begin{bmatrix} 1 & 2a \\ a & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ a+10 \end{bmatrix}$$

$\Delta = 1 \times 8 - 2a^2 = 0$ for no solution or infinitely many solutions

$2(4 - a^2) = 0$

$2(2 - a)(2 + a) = 0$

$a = 2$ or $a = -2$

If $a = 2$, we have

$x + 4y = 6 \quad -(A)$

$2x + 8y = 12 \quad -(B)$

$(B) \div 2 \qquad x + 4y = 6 \quad -(C)$

Since (A) and (C) are the same line, there are infinitely many solutions for $a = 2$.

If $a = -2$, we have

$x - 4y = 6 \quad -(A)$

$-2x + 8y = 8 \quad -(B)$

$(B) \div {}^{-}2 \qquad x - 4y = -4 \quad -(C)$

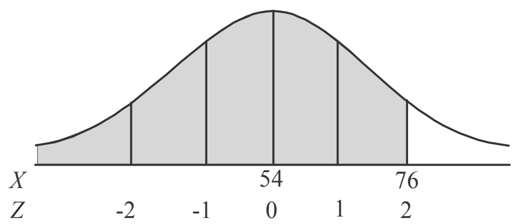
Since the lines given by (A) and (C) are parallel, there can be no solutions.

So there are infinitely many solutions only when $a = 2$.

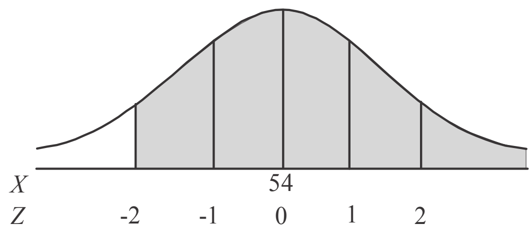
The answer is A.

Question 18

Do a quick sketch.



Pr(X < 76) is shaded



Pr(Z > -2) is shaded

Since $\Pr(X < 76) = \Pr(Z > {}^{-}2)$, then 76 must lie two standard deviations above the mean.

Now $76 - 54 = 22$ so the standard deviation is 11. So the variance of X equals $11^2 = 121$.

The answer is D.

Question 19

$y = x^2 - kx$ and $y = 3x - 4$

$x^2 - kx = 3x - 4$ (equation of intersection)

$x^2 - kx - 3x + 4 = 0$

$x^2 + x(-k - 3) + 4 = 0$

The graphs intersect at two distinct points when $\Delta > 0$.

$b^2 - 4ac > 0$

$(-k - 3)^2 - 4 \times 1 \times 4 > 0$

Solve this inequation for k using CAS $k < {}^{-}7$ or $k > 1$.

The answer is E.

Question 20

Since area of $P =$ area of Q,

solve $\int_0^5 -x^2(x-5)dx = -1 \times \int_5^u -x^2(x-5)dx$ for u.

Note that the x-intercepts of the graph of f occur at $x = 0$ and $x = 5$. Also, since area Q falls below the x-axis, we need to multiply the integral $\int_5^u -x^2(x-5)dx$ by negative one to obtain the actual area.

Using CAS, $u = 0$ or $u = \frac{20}{3}$.

Since $u > 5$, $u = \frac{20}{3}$.

The answer is C.

Question 21

The relationship $f(x+0.4) \approx f(x) + 0.4 \times f'(x)$ is an application of the approximation formula $f(x+h) \approx f(x) + hf'(x)$ where $h = 0.4$.

To approximate $\frac{1}{\sqrt[3]{1.4}}$, we use $f(x) = \frac{1}{\sqrt[3]{x}}$ and $f'(x) = \frac{-1}{3x^{\frac{4}{3}}}$

so $f(x+0.4) \approx f(x) + 0.4 \times f'(x)$

becomes $f(1+0.4) \approx f(1) + 0.4 \times \frac{-1}{3 \times 1^{\frac{4}{3}}}$

$$= 1 - \frac{0.4}{3}$$

$$= 0.86666...$$

The approximation is 0.8667

The answer is B.

Question 22

$$\begin{aligned}
A &= \text{shaded area} \\
&= \text{area of } \Delta CDE + \text{area of } \Delta ABE \\
&= \frac{1}{2} \times a \times a \times \sin(\theta) + \frac{1}{2} \times AE \times AB \\
&= \frac{a^2}{2}\sin(\theta) + \frac{1}{2} \times a\cos(\theta) \times a\sin(\theta) \\
&= \frac{a^2}{2}\sin(\theta) + \frac{a^2}{2}\cos(\theta)\sin(\theta)
\end{aligned}$$

Solve $\frac{dA}{d\theta} = 0$ for θ using CAS

$\theta = \frac{\pi}{3}$ since $0 < \theta < \frac{\pi}{2}$

The answer is C.

SECTION 2

Question 1 (6 marks)

a. At midnight on Sunday $t=0$.

$$g(0)=20-4\sin(0)$$
$$=20$$

The temperature is $20\,^\circ$C at midnight on Sunday.

(1 mark)

b.

$$\text{period}=\frac{2\pi}{n} \text{ where } n=\frac{\pi}{12}$$
$$=2\pi\div\frac{\pi}{12}$$
$$=24$$

(1 mark)

c. Since the amplitude is 4, the maximum temperature is $24\,^\circ$C. **(1 mark)**

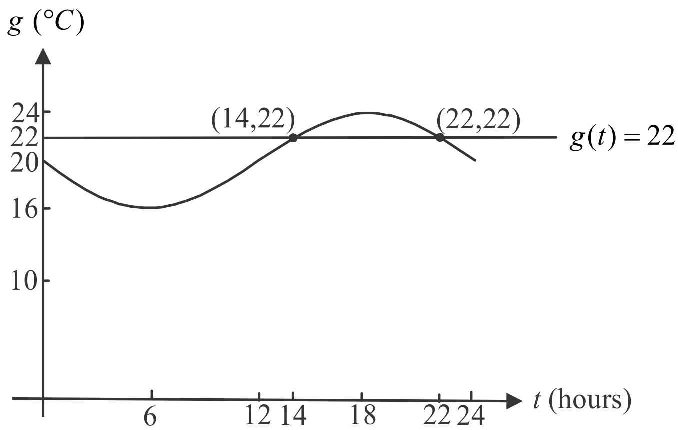
The average temperature is $20\,^\circ$C. **(1 mark)**

d. Solve $g(t)=22$ for $t\in[0,24]$

$t=14$ or 22

(1 mark)

A quick sketch of the graph of $y=g(t)$ shows that the temperature inside the greenhouse is above $22\,^\circ$C for 8 hours a day between $t=14$ and $t=22$.



$$\text{Cost}=7(5.20\times16+6.80\times8)$$
$$=\$963.20$$

(1 mark)

Question 2 (15 marks)

a. mean $= E(X) = \int_2^5 \left(x \times \frac{(x-5)^2+1}{12} \right) dx$

$= \frac{47}{16}$ hours

(1 mark)

b. Solve $\int_2^m \frac{(x-5)^2+1}{12} dx = 0.25$ for m. **(1 mark)**

$m = 2.3318...$

The maximum time was 2.332 hours (correct to 3 decimal places).

(1 mark)

c. $X \sim N(24, 0.3^2)$

$\Pr(X < x) = 0.8$

$x = 24.252486...$ (using the inverse normal function)

Filters with a maximum diameter of 24.25mm (correct to 2 decimal places) can be used in a Mini car.

(1 mark)

d. $\Pr(X < 23.5) = 0.047790...$ (using the normal cdf function)

$0.047790... \times 700 = 33.4532...$ **(1 mark)**

So 33 of the filters will fit in any car (to the nearest whole number)

(1 mark)

e. **i.** Method 1 – using transition matrices

$$\begin{array}{c} \text{this car} \\ \begin{array}{cc} \text{minor} & \text{major} \end{array} \end{array}$$

$$T = \begin{bmatrix} q & 0.6 \\ 1-q & 0.4 \end{bmatrix} \begin{matrix} \text{minor} \\ \text{major} \end{matrix} \text{ next car} \qquad S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \text{minor} \\ \text{major} \end{matrix}$$

(1 mark)

$$T^3 = T^2 S_1$$

$$= \begin{bmatrix} q^2 - 0.6q + 0.6 \\ -(q-1)(q+0.4) \end{bmatrix} \begin{matrix} \text{minor} \\ \text{major} \end{matrix}$$

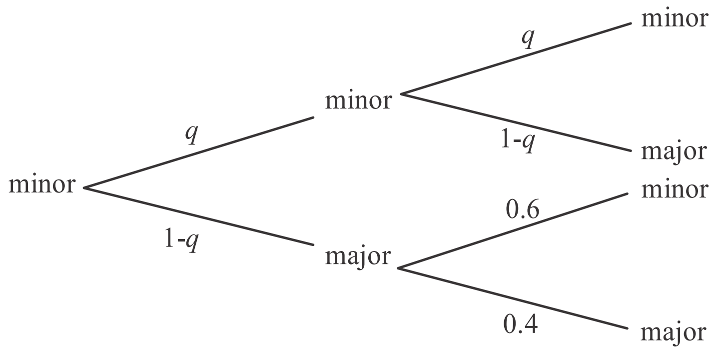
The required probability is

$-(q-1)(q+0.4)$

or $-q^2 + 0.6q + 0.4$

(1 mark)

Method 2 – using a tree diagram



(1 mark)

$\Pr(3^{rd}\text{ service is major})$
$= q(1-q)+(1-q)\times 0.4$
$= q - q^2 + 0.4 - 0.4q$
$= -q^2 + 0.6q + 0.4$

(1 mark)

ii. Solve $-q^2 + 0.6q + 0.4 = 0.2$ **(1 mark)**
$q = -0.23851\ldots$ or $q = 0.83851\ldots$
Since $0 < q < 1,\ q = 0.839$ (correct to 3 decimal places)

(1 mark)

iii. On this day $T = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$ and $S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$S_8 = T^7 S_1$ **(1 mark)**
$= \begin{bmatrix} 0.666\ldots \\ 0.333\ldots \end{bmatrix}$

The probability that the last car had a minor service was $\frac{2}{3}$.

(1 mark)

f. **i.** Solve $(0.8)^n = 0.32768$ for n
$n = 5$

(1 mark)

ii. This is a conditional probability question.
$\Pr(X > 16 | X \geq 10)$
$= \frac{\Pr(X > 16 \cap X \geq 10)}{\Pr(X \geq 10)}$
$= \frac{\Pr(X > 16)}{\Pr(X \geq 10)}$ (binom cdf(20,0.8,17,20)) (binom cdf(20,0.8,10,20)) **(1 mark)**
$= \frac{0.4114488\ldots}{0.99943\ldots}$
$= 0.4117$ (correct to 4 decimal places)

(1 mark)

Question 3 (11 marks)

a. From the diagram, we see that the maximum height occurs at the back wall of the bay where $x = 10$.
So $h(10) = 0.82822..$
Height is 0.83m (correct to 2 decimal places).

(1 mark)

b. Solve $h'(x) = 0$ for x. **(1 mark)**

$x = 0.354001...$ or $x = 1.622882...$
So the interval required is $x \in [0.35, 1.62]$ where the endpoints are expressed correct to 2 decimal places.

(1 mark)

c. Solve $h'(x) = h(x)$ for x. **(1 mark)**

$$x = 0.179568...$$

So $h(0.179568...) = 0.142515...$
The height is 0.14 metres (correct to 2 decimal places).

(1 mark)

d. Since the average rate of change of h between $(1, h(1))$ and $(p, h(p))$ is zero, then $h(1) = h(p)$.
Solve $h(1) = h(p)$ for p. **(1 mark)**

$p = 0.10541...$ or $p = 0.999999...$or $p = 2.273895...$
Since $p > 1$, $p = 2.27$ (correct to 2 decimal places).

(1 mark)

e.
$$\text{volume of topsoil} = 3 \times \int_0^{10} h(x)dx$$ **(1 mark)**

$$= 11.50244...$$

The estimate for the volume of topsoil is 11.50 cubic metres (correct to 2 decimal places).

(1 mark)

f. We are looking for the average value of the function h over the interval $x \in [0,10]$.

$$\text{average value} = \frac{1}{10-0}\int_0^{10} h(x)dx$$ **(1 mark)**

$$= 0.383414...$$

The height of the top soil would be 0.38 metres (correct to 2 decimal places).

(1 mark)

Question 4 (12 marks)

a. $f(x)=\log_e(x+a)$
f is defined for $x+a>0$

$x>-a$

So $d_f=(-a,\infty)$

(1 mark)

b. i. x-intercepts occur when $y=0$

$y=\log_e(x+a)$

$0=\log_e(x+a)$

$e^0=x+a$

$x+a=1$

$x=1-a$

If the x-intercept is negative
then $1-a<0$

$1<a$

$a>1$

(1 mark)

ii. From part i. the x-intercept occurs when $x=1-a$.
If the x-intercept is positive then
then $1-a>0$

$1>a$

$a<1$

Also, we are told in the question that a is positive so $0<a<1$.

(1 mark)

c. i. Given $f(g(x))=\frac{x}{b}$

then $\log_e(g(x)+a)=\frac{x}{b}$

$e^{\frac{x}{b}}=g(x)+a$

$g(x)=e^{\frac{x}{b}}-a$

as required.

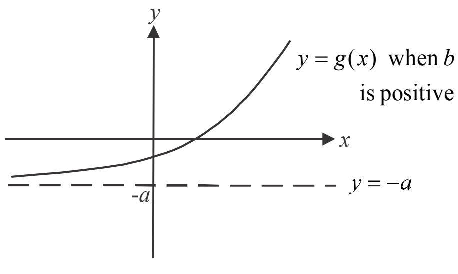
(1 mark)

ii. $f(g(x))$ is a composite function and $f(g(x))$ is defined iff $r_g\subseteq d_f$

Now $d_f=(-a,\infty)$ from part a.

To find the range of g, do a quick sketch.

$r_g=(-a,\infty)$



Since $r_g=d_f$, then $f(g(x))$ is defined.

(1 mark)

d. **i.** $f(x)=\log_e(x+a)$

Let $y=\log_e(x+a)$

Swap x and y for inverse.

$$x=\log_e(y+a)$$
$$e^x=y+a$$
$$y=e^x-a$$

So $f^{-1}(x)=e^x-a$ **(1 mark)**

$d_{f^{-1}}=r_f$

Since $r_f=R$

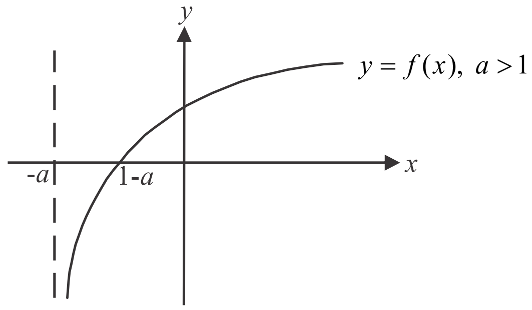
then $d_{f^{-1}}=R$

(1 mark)

ii. Since $f^{-1}(x)=e^x-a$ and $g(x)=e^{\frac{x}{b}}-a$, the transformation required to transform the graph of f^{-1} to the graph of g is a dilation from the y-axis by a factor of b units.

(1 mark)

e. Since $a>1$, the x-intercept of the graph of f will be negative from part b. i.
Do a quick sketch.



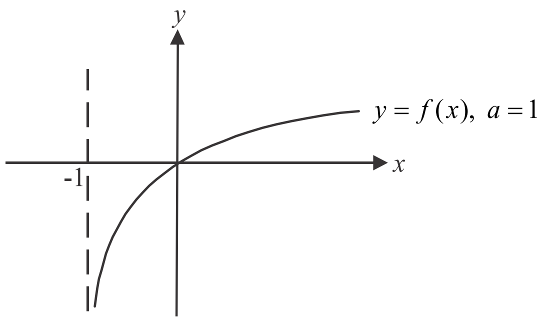
The x-intercept occurs at $x=1-a$ from part b. so

$$\text{area required} = \int_{1-a}^{0} f(x)dx$$
$$= a\log_e(a)-a+1 \text{ square units} \quad \text{(using CAS)}$$

(1 mark) – correct integrand
(1 mark) correct terminals
(1 mark) correct answer

f. From part b, if $a>1$ the graph of f has a negative x-intercept and if $0<a<1$, the graph of f has a positive x-intercept.

When $a=1$, the graph of f passes through the origin hence the area enclosed by the graph of $y=f(x)$ and the x and y axes equals zero.
Alternatively, there is no area **enclosed** by the graph of $y=f(x)$ and the x and y axes.



(1 mark)

Question 5 (14 marks)

a. The height of the land above sea level on the island, as seen from the naval ship is given by the function l. Define this function on your CAS.

At sea level, $l(x) = 0$

Solve $l(x) = 0$ for x

$x = 0$ or $x = 2000$

So the width of the island as observed from the naval ship is $2000 - 0 = 2000$ metres.

(1 mark)

b. Find $l'(x)$.

Solve $l'(x) = 0$ for x

$x = 695.519033...$ **(1 mark)**

Substitute this value into $l(x)$.

$l(695.519033...) = 217.494939...$

The highest point on the island occurs at the point $(695.5190, 217.4949)$ where coordinates are expressed correct to 4 decimal places.

(1 mark)

c. **i.**

$$d(x) = b(x) - l(x)$$
$$= \frac{x^4}{10^{10}} - \frac{x^3}{2\times10^6} + \frac{19x^2}{2\times10^4} - \frac{7x}{10} + 200$$

(1 mark)

ii.

$$d_b = [0,3000]$$
$$d_l = [0,2000]$$

So $d_d = [0,2000]$

(1 mark)

d. Define $d(x)$ (if you haven't already) on your CAS using your answer to part c. i.

Find $d'(x)$

Solve $d'(x) = 0$ for x

$x = 626.586096...$ **(1 mark)**

Substitute this into $d(x)$.

$d(626.586096...) = 26.781607...$

The minimum vertical distance between the balloon and the land is 26.78m (correct to 2 decimal places).

(1 mark)

e. Find $l'(x)$.

Substitute $x = 1000$ into $l'(x)$.

$l'(1000) = -\frac{1}{10}$ **(1 mark)**

The equation of the tangent has a gradient of $-\frac{1}{10}$ and passes through $(1000,200)$.

The equation is given by

$$y - 200 = -\frac{1}{10}(x - 1000)$$
$$y = -\frac{x}{10} + 300$$

(1 mark)

f. The missile crosses the flight path of the balloon when $b(x) = -\frac{x}{10} + 300$.

Solve this equation for x.

$x = -1000(\sqrt{2} - 2)$ or $x = 1000(\sqrt{2} + 2)$

$= 585.7864...$ $\quad$ $= 3414.2135...$

Since $d_b = [0, 3000]$, we reject this second value of x.

So $x = -1000(\sqrt{2} - 2)$. **(1 mark)**

Note that the numerical values here are used only to see whether the values of x are feasible. Since we have not been asked for decimal approximations in the question, we must use the exact values of the coordinates.

Substitute $x = -1000(\sqrt{2} - 2)$ into $b(x)$ or into $y = -\frac{x}{10} + 300$.

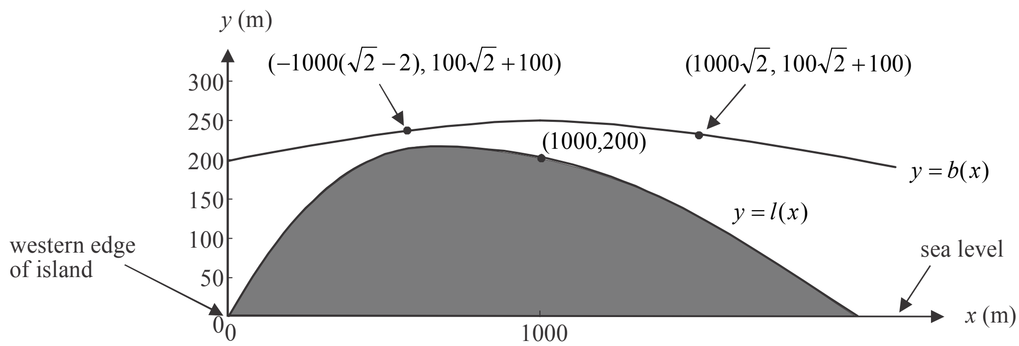
$b(-1000(\sqrt{2} - 2)) = 100\sqrt{2} + 100$

The missile crosses the flight path of the balloon at the point

$(-1000(\sqrt{2} - 2), 100\sqrt{2} + 100)$

(1 mark)

g.



To find the location of the balloon when the enemy missile crosses the flight path of the balloon, solve $b(x) = 100\sqrt{2} + 100$ for x.

$x = -1000(\sqrt{2} - 2)$ or $x = 1000\sqrt{2}$.

The balloon is located at the point $(1000\sqrt{2}, 100\sqrt{2} + 100)$. **(1 mark)**

At this instant, Victoria fires a missile to hit the point (1000, 200).

gradient of path of missile

$$= \frac{100\sqrt{2} + 100 - 200}{1000\sqrt{2} - 1000}$$

$$= \frac{1}{10}$$ **(1 mark)**

$m = \tan\theta$

So $\tan\theta = \frac{1}{10}$

$\theta = \tan^{-1}\left(\frac{1}{10}\right)$

So $p = 10$

(1 mark)