YEAR 12 *Trial Exam Paper*

2015
MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

Question 1a.

Worked solution

$$\frac{d}{dx}\left(e^x \sin(2x)\right) = e^x \sin(2x) + 2e^x \cos(2x)$$

Mark allocation: 2 marks

- 1 mark for evidence of using the product rule
- 1 mark for the correct answer

Tip

- *Always be on the look-out for product rule or quotient rule questions.*

Question 1b.

Worked solution

$$f(x) = e^{\cos(x)}$$

$$f'(x) = -\sin(x)e^{\cos(x)}$$

$$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)e^{\cos\left(\frac{\pi}{2}\right)} = -1 \times e^0 = -1$$

Mark allocation: 2 marks

- 1 mark for the correct derivative $f'(x)$
- 1 mark for the correct answer

Tip

- *Remember to re-read the question before moving on. Many students differentiate correctly but then forget to evaluate.*

Question 1c.

Worked solution

Average value of a function is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$=\frac{1}{k-1}\int_1^k \frac{1}{3+2x}\,dx$$

$$=\frac{1}{k-1}\times\frac{1}{2}\left[\log_e|3+2x|\right]_1^k$$

$$=\frac{1}{2k-2}\left[\log_e(3+2k)-\log_e(5)\right]$$

$$=\frac{1}{2k-2}\log_e\left(\frac{3+2k}{5}\right)$$

Setting $\frac{1}{2k-2}\log_e\left(\frac{3+2k}{5}\right)=\frac{1}{10}\log_e(3),$ gives $k = 6$.

Mark allocation: 3 marks

- 1 mark for setting up the correct integral with terminals from 1 to k
- 1 mark for obtaining a log equation in terms of k
- 1 mark for the answer $k = 6$

Question 2

Worked solution

To have a unique solution implies the two lines are not parallel. Therefore, they have *different* gradients.

Rearranging the lines gives

$$2y = ax - a \Rightarrow y = \frac{a}{2}x - \frac{a}{2}$$

and $y = -5x + 7$

So $\frac{a}{2} \neq -5,\ a \neq -10.$

$\therefore\ a \neq -10$

$a \in R \setminus \{-10\}$

An alternative solution involves using matrices and calculating the determinant.

Let $A = \begin{bmatrix} a & -2 \\ 5 & 1 \end{bmatrix}$

$\det A = ad - bc = a + 10$

Let $\det A \neq 0 \Rightarrow a + 10 \neq 0$

So $a \neq -10.$

$\therefore\ a \in R \setminus \{-10\}.$

Mark allocation: 2 marks

- 1 mark for equating gradients or forming the determinant
- 1 mark for the correct answer

Tip

- *Remember there are three ways this question can be asked: unique solution, no solution or infinite solutions. Be sure to be able to answer any of these.*

Question 3a.

Worked solution

Swap x and y to get $x = \frac{1}{3}\log_e\left(\frac{y-2}{3}\right)$.

Now, rearrange to make y the subject.

$$\begin{aligned} 3x &= \log_e\left(\frac{y-2}{3}\right) \\ e^{3x} &= \frac{y-2}{3} \\ 3e^{3x} &= y-2 \\ y &= 3e^{3x}+2 \end{aligned}$$

So $f^{-1}(x) = 3e^{3x}+2$.

Mark allocation: 2 marks

- 1 mark for swapping x and y
- 1 mark for the correct answer, which must be written as f^{-1}

Tip

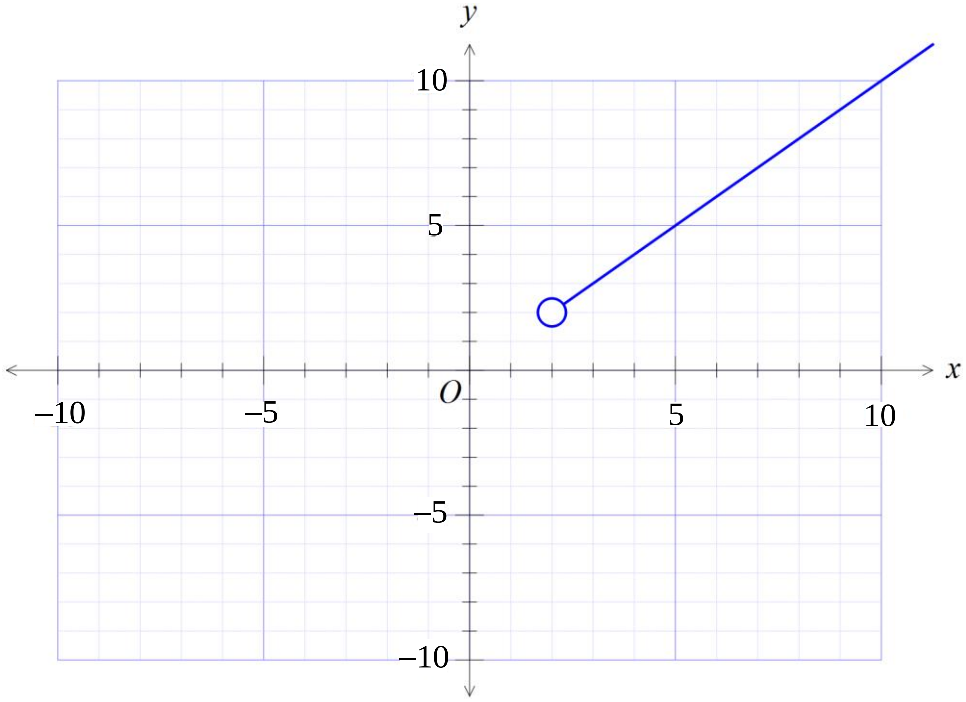
- *Remember to use correct notation – the question asked for f^{-1}, so make sure you use this in your answer.*

Question 3b.

Worked solution

The graph of $y = f^{-1}\left(f(x)\right)$ will be the graph of $y = x$ for the domain of $f(x)$, i.e. $x \in (2, \infty)$.

So the graph will be



Mark allocation: 1 mark

- 1 mark for correctly drawn graph (It must have an open circle at $x = 2$.)

Tip

- *Remember to find the domain of f. This is important as the graph of $y = f^{-1}\left(f(x)\right)$ will exist only for the domain of f.*

Question 3c.

Worked solution

The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes a transformation that produces a reflection in both the *x*- and *y*-axes.

So the new equation is $y = -\frac{1}{3}\log_e\left(\frac{-2-x}{3}\right)$.

Mark allocation: 1 mark

- 1 mark for the correct answer

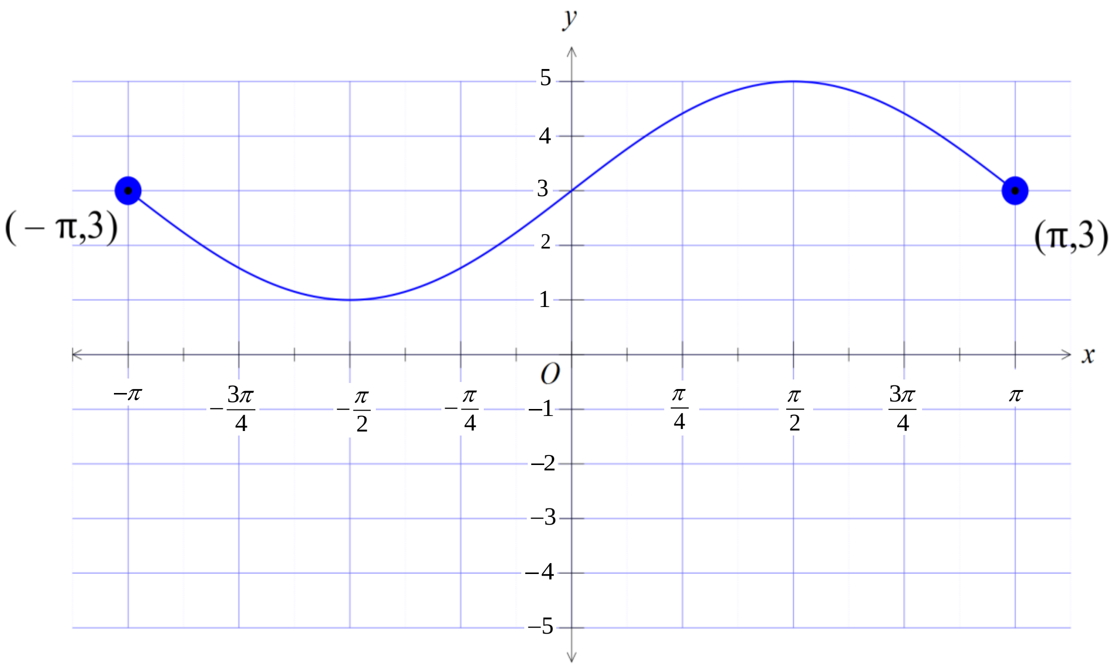
Question 4a.

Worked solution

The transformations:

- a dilation of factor 2 from the x-axis
- a translation of $+3$ units up

give a new equation of $y = 2\sin(x) + 3$.



Mark allocation: 2 marks

- 1 mark for one cycle of either a sin or cos curve that is centred around $y = 3$
- 1 mark if all aspects are correct, including correctly labelled end points

Question 4b.

Worked solution

$$\frac{dy}{dx} = 2\cos(x)$$

When $x = 0$, $\frac{dy}{dx} = 2$, so the gradient of the normal is $-\frac{1}{2}$.

The equation of the normal to the curve is

$$\therefore y - 3 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 3$$

Mark allocation: 2 marks

- 1 mark for obtaining the gradient of the normal to the curve
- 1 mark for obtaining the line equation

Tip

- *Again, re-read the question. Don't forget to find the equation, not just the gradient!*

Question 5a.

Worked solution

Max will need to drink cola, lemonade, lemonade and then cola.

$$\begin{aligned}\text{So the } \Pr(CLLC) &= 1\times 0.7\times 0.4\times 0.6\\ &= \frac{7}{10}\times\frac{2}{5}\times\frac{3}{5}=\frac{21}{125}\\ &= 0.168\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for writing CLLC for writing a product that involves 4 terms
- 1 mark for the correct answer, written as either 0.168 or $\frac{21}{125}$

Question 5b.

Worked solution

$$\begin{array}{c c} & \begin{matrix} C_i & L_i \end{matrix} \\ \begin{matrix} C_{i+1} \\ L_{i+1} \end{matrix} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \end{array}$$

$$\Pr(L \text{ long term}) = \frac{0.7}{1.3} = \frac{7}{13}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 6a.

Worked solution

Swap x and y, giving

$$x = \frac{1}{2}\log_e(y-k)$$

$$2x = \log_e(y-k)$$

$$e^{2x} = y-k$$

$$y = e^{2x} + k$$

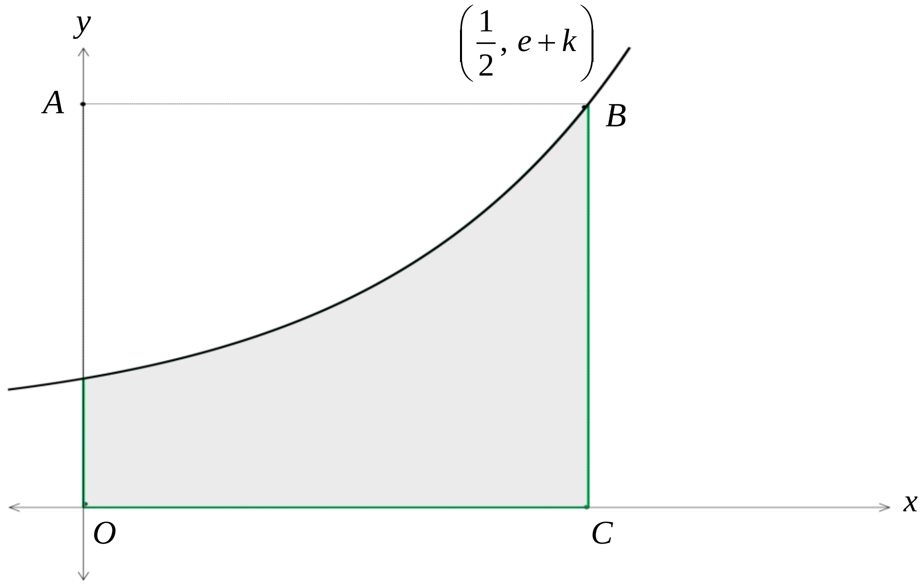
Mark allocation: 1 mark

- 1 mark for using the correct method, leading to the correct answer

Question 6b.

Worked solution

Using the inverse found in part **a**, consider the equivalent rectangle.



Area of the rectangle is $\frac{1}{2}(e+k)$.

Therefore, the area under the curve is equal to $\frac{1}{4}(e+k)$.

So $\int_0^{\frac{1}{2}} e^{2x} + k \; dx = \frac{1}{4}(e+k)$.

$$\text{LHS} = \left[\frac{1}{2}e^{2x} + kx\right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}e + \frac{1}{2}k\right) - \left(\frac{1}{2} + 0\right)$$

$$= \frac{1}{2}e + \frac{1}{2}k - \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2}e + \frac{1}{2}k - \frac{1}{2} = \frac{1}{4}(e+k)$$

$$\Rightarrow \quad \frac{1}{2}k - \frac{1}{4}k = \frac{1}{2} - \frac{1}{4}e$$

$$\Rightarrow \quad \frac{1}{4}k = \frac{1}{4}(2-e)$$

$$\Rightarrow \quad k = (2-e)$$

Mark allocation: 4 marks

- 1 mark for finding the area of the equivalent rectangle
- 1 mark for setting up the integral as $\int_0^{\frac{1}{2}} e^{2x} + k\,dx$
- 1 mark for obtaining the integral $\left[\frac{1}{2}e^{2x} + kx\right]_0^{\frac{1}{2}}$
- 1 mark for the correct answer

Tip

- *This is a 'hence' question so you must make use of your answer to part **a**.*

Question 7a.

Worked solution

$\sin(3x) = \frac{-1}{\sqrt{2}}$, 3rd and 4th quadrants

$$3x = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$3x = \frac{5\pi}{4} + 2k\pi,\ \frac{7\pi}{4} + 2k\pi$$

$$x = \frac{5\pi}{12} + \frac{2k\pi}{3},\ \frac{7\pi}{12} + \frac{2k\pi}{3},\ k \in Z \ \text{ or } \ x = \frac{(8k-1)\pi}{12},\ \frac{(8k+5)\pi}{12},\ k \in Z$$

Mark allocation: 2 marks

- 1 mark for using $\frac{\pi}{4}$ as the basic angle
- 1 mark for the correct answer

Tip

- *For trigonometry general solution questions, you must always write* $k \in Z$.

Question 7b.

Worked solution

The average rate of change of the function is given by

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{4}\right)}{\frac{\pi}{3} - \frac{\pi}{4}}$$

$$= \frac{0 - \sqrt{2}}{\frac{\pi}{12}}$$

$$= \frac{-12\sqrt{2}}{\pi}$$

Mark allocation: 2 marks

- 1 mark for correctly evaluating either $f\left(\frac{\pi}{3}\right)$ or $f\left(\frac{\pi}{4}\right)$
- 1 mark for the correct answer

Tip

- *Be careful! Students often confuse average rate of change for average value of the function.*

Question 8a.

Worked solution

Using the log laws, we get

$$\log_8 (x+1)^2 + \log_8 4 = 1$$
$$\log_8 4(x+1)^2 = 1$$
$$8^1 = 4(x+1)^2$$
$$2 = (x+1)^2$$
$$\pm\sqrt{2} = (x+1)$$
$$x = -1 \pm \sqrt{2}$$

However, $x > -1$ so $x = -1 + \sqrt{2}$.

Mark allocation: 2 marks

- 1 method mark for using log laws
- 1 mark for the correct answer

Question 8b.

Worked solution

Let $a = e^x$, as this forms the trinomial equation into a recognisable quadratic equation.

$a^2 - 8a + 7 = 0$

Factorising gives $(a-1)(a-7) = 0$.

So $a = 1$ or $a = 7$.

$\Rightarrow e^x = 1$ or $e^x = 7$

$\Rightarrow \quad x = 0$ or $x = \log_e 7$

Mark allocation: 3 marks

- 1 method mark for establishing the trinomial equation
- 1 method mark for factorising the trinomial
- 1 answer mark for both correct answers

Question 9a.

Worked solution

Converting to standard normal gives

$$\begin{aligned}\Pr(X < 30) &= \Pr\left(Z < \frac{30-60}{15}\right)\\ &= \Pr(Z < -2)\\ &= 0.0228\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 9b.

Worked solution

Using the conditional probability rule gives

$$\begin{aligned}\Pr(X < 60 \mid X > 30) &= \frac{\Pr(X < 60 \cap X > 30)}{\Pr(X > 30)}\\ &= \frac{\Pr(30 < X < 60)}{\Pr(X > 30)}\end{aligned}$$

Again, converting to standard normal gives

$$\frac{\Pr(30 < X < 60)}{\Pr(X > 30)} = \frac{\Pr(-2 < Z < 0)}{\Pr(Z > -2)}$$

Using the symmetry of the normal distribution curve, $\Pr(Z > -2) = 1 - \Pr(Z < -2)$.
So the $\Pr(-2 < Z < 0) = 0.5 - 0.0228 = 0.4772$ and $\Pr(Z > 0) = 0.5$.

Therefore, $\Pr(X < 60 \mid X > 30) = \dfrac{0.4772}{1-0.0228} = \dfrac{0.4772}{0.9772} = \dfrac{4772}{9772}$.

Mark allocation: 2 marks

- 1 mark for using conditional probability
- 1 mark for the correct answer

Tip

- *The answer must be written as a fraction and not a decimal inside a fraction.*

Question 10

Worked solution

$$f'(x) = (x-a)3(x-b)^2 + (x-b)^3$$
$$= (x-b)^2\left[3(x-a)+(x-b)\right]$$
$$= (x-b)^2[4x-3a-b]$$

For stationary points, let $f'(x) = 0$.

$\Rightarrow (x-b)^2(4x-3a-b) = 0$

$\Rightarrow x = b$ or $4x = 3a+b$

$\Rightarrow x = b$ or $x = \dfrac{3a+b}{4}$

So $b = 3$ and $4 = \dfrac{3a+3}{4}$.

$$16 = 3a+3$$
$$13 = 3a$$
$$a = \frac{13}{3}$$

Mark allocation: 3 marks

- 1 mark for $f'(x) = 0$
- 1 mark for $b = 3$
- 1 mark for $a = \dfrac{13}{3}$

END OF WORKED SOLUTIONS