YEAR 12 *Trial Exam Paper*

2015

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

SECTION 1

Question 1

Answer is E.

Worked solution

The function $y = 1 + \sqrt{a - x}$ is defined for $a - x \geq 0$, so $x \leq a$.

Question 2

Answer is D.

Worked solution

$$\begin{aligned} f(1) &= -2 \\ f(-2) &= 10 \\ f'(1) &= -3 \\ \text{and } f'(3) &= -36. \end{aligned}$$

In equation form, this can be written as:

$$\begin{aligned} a + b + c + d &= -2 \\ -8a + 4b - 2c + d &= 10 \\ 3a + 2b + c &= -3 \\ 27a + 6b + c &= -36 \end{aligned}$$

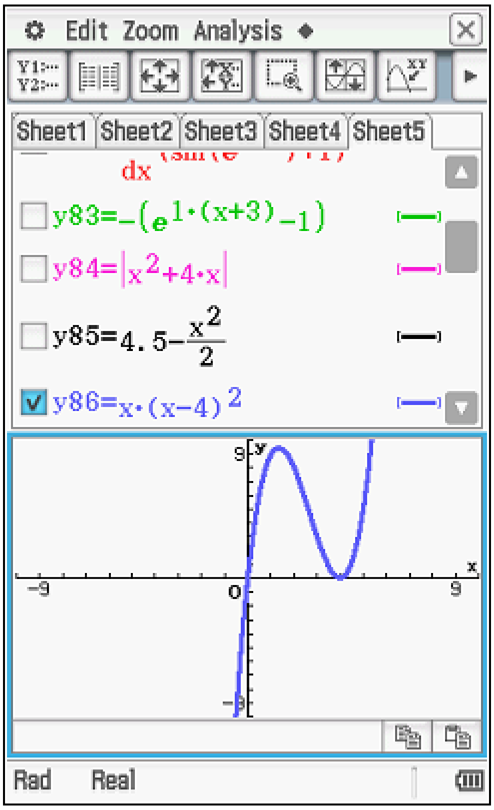
which gives $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 2 & 1 & 0 \\ 27 & 6 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ -3 \\ -36 \end{bmatrix}$.

Question 3

Answer is E.

Worked solution

A sample graph with $a = -4$ shows



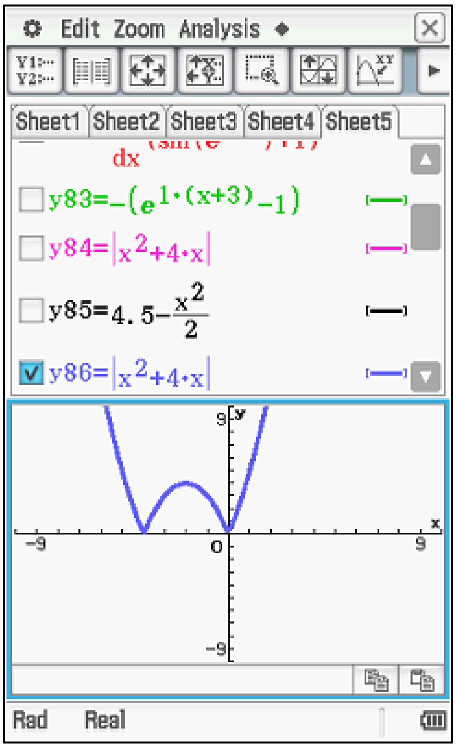
Hence, for $x < 0$, $f'(x) > 0$.

Question 4

Answer is D.

Worked solution

The graph $f(x)=|x^2+ax|$ has x-intercepts at $(0,0)$ and $(-a,0)$ and a turning point at $x=-\frac{a}{2}$, as shown below.



The gradient is positive for $x \in \left(-a,-\frac{a}{2}\right)\cup(0,\infty)$.

Question 5

Answer is B.

Worked solution

$$\begin{aligned}\int_2^4 (2+5f(x))\,dx &= \int_2^4 2\,dx + 5\int_2^4 f(x)\,dx \\ &= [2x]_2^4 + 5\times 3 \\ &= (8-4)+15 \\ &= 19\end{aligned}$$

Question 6

Answer is A.

Worked solution

$$\begin{aligned}\int_0^1 \frac{2x^2}{k}\,dx &= \frac{1}{k}\int_0^1 2x^2\,dx \\ &= \frac{1}{k}\left[\frac{2x^3}{3}\right]_0^1 \\ &= \frac{2}{3k}\end{aligned}$$

So, $\frac{2}{3k} = 1$

$$k = \frac{2}{3}$$

Question 7

Answer is C.

Worked solution

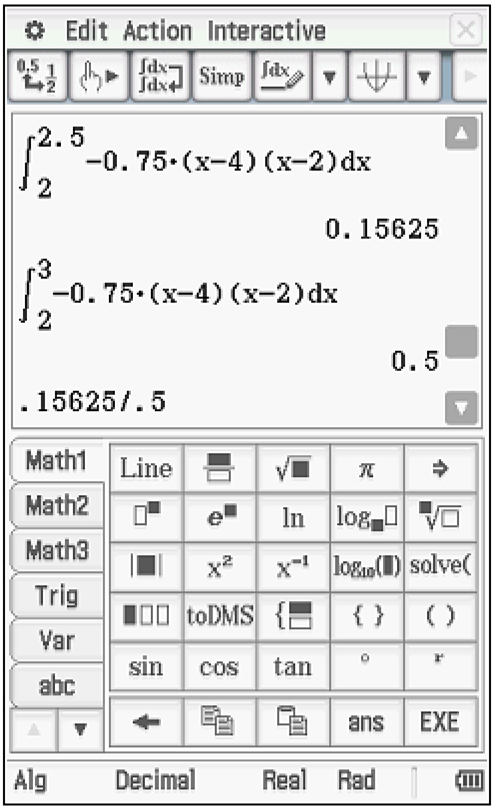
Owing to the symmetry of the distribution, the mean, median and mode occur concurrently at $x = 3$.

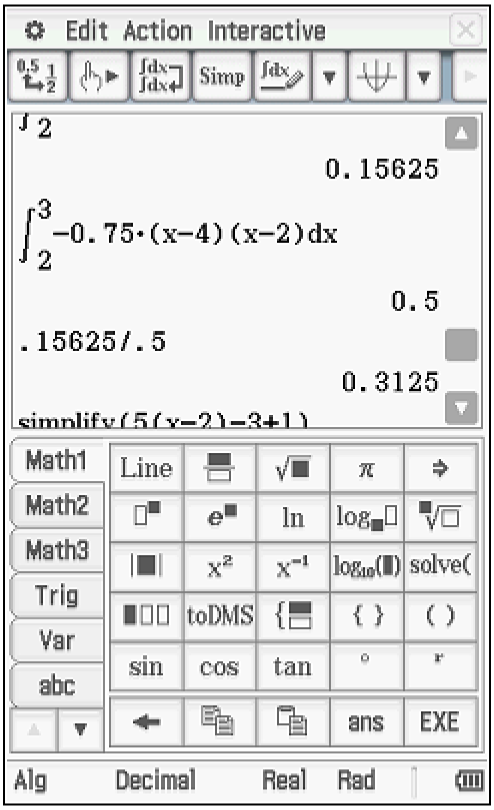
Question 8

Answer is C.

Worked solution

$$\Pr(X < 2.5 | X < 3) = \frac{\Pr(X < 2.5)}{\Pr(X < 3)}$$





Question 9

Answer is A.

Worked solution

Expanding the matrix gives

$$\left.\begin{aligned} x' &= -3x - \pi \\ y' &= 2y - 2 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{x' + \pi}{-3} \\ y &= \frac{y' + 2}{2} \end{aligned}$$

So $y = \sin(3x)$ becomes

$$\frac{y+2}{2} = \sin(-x - \pi)$$

$$y = 2\sin(-x - \pi) - 2$$

Question 10

Answer is D.

Worked solution

The graph of $y = 5x - 3$ undergoes the transformation $1 + f(x+2)$, so

$$\begin{aligned} y &= (5(x+2) - 3) + 1 \\ &= 5x + 8 \end{aligned}$$

Question 11

Answer is E.

Worked solution

Using the quotient rule gives $g(x) = \dfrac{f(x)}{e^x}$, so $g'(x) = \dfrac{e^x f'(x) - e^x f(x)}{(e^x)^2} = \dfrac{f'(x) - f(x)}{e^x}$.

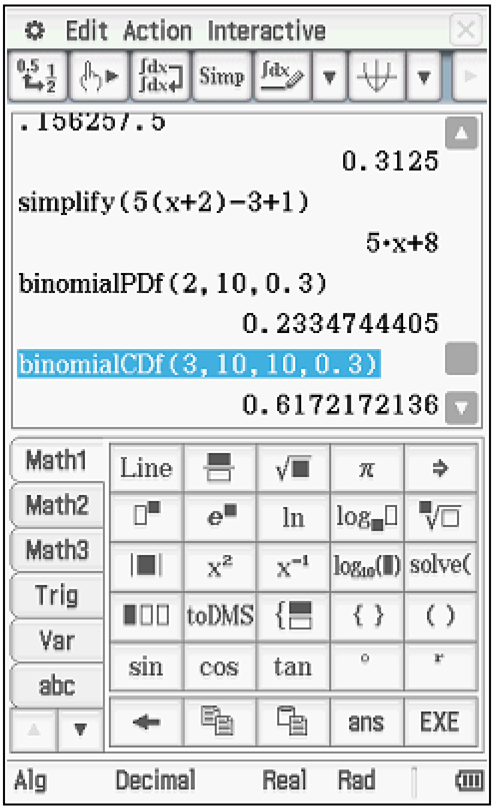
Question 12

Answer is E.

Worked solution

$X \sim Bi(n = 10, p = 0.3)$

$\Pr(X > 2) =$



Question 13

Answer is D.

Worked solution

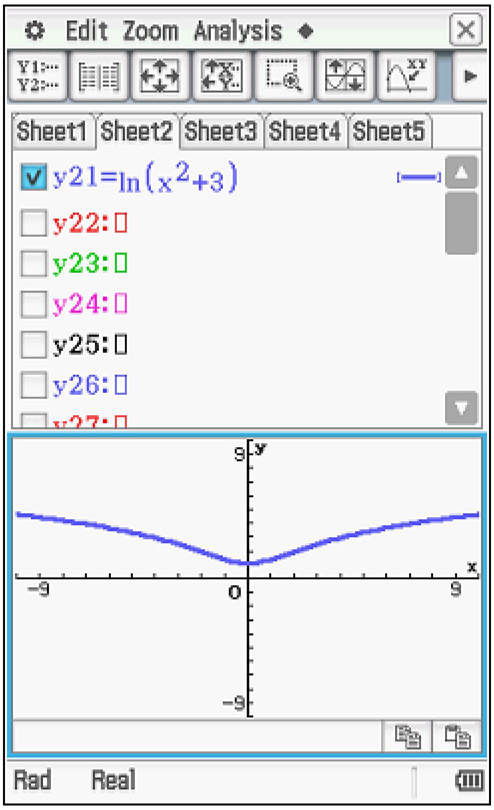
The graph has been reflected in both the *x*- and *y*-axes. The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ produces reflections in both axes.

Question 14

Answer is D.

Worked solution

The graph of $g(f(x))$ shows that the lowest point occurs at $x = 0,\ y = \log_e 3.$



The range is $[\log_e 3, \infty)$.

Question 15

Answer is C.

Worked solution

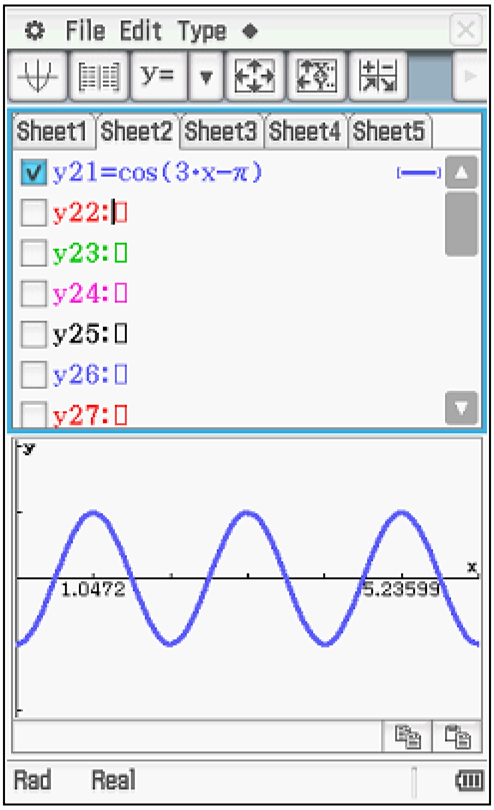
The rate of change for this function is $\begin{cases} -2 & \text{for } x < \frac{-1}{2} \\ 2 & \text{for } x > \frac{1}{2} \end{cases}$.

Question 16

Answer is D.

Worked solution

For the inverse function to exist, the function must be one-to-one. The graph shows that the function is one-to-one for $x \in [0, \frac{\pi}{3}]$.



Hence, $a = \frac{\pi}{3}$.

Question 17

Answer is C.

Worked solution

For $4.2^{\frac{3}{2}}$, $x = 4,\ h = 0.2,\ f(x) = x^{\frac{3}{2}}$ and $f'(x) = \frac{3}{2}\sqrt{x}.$

So, $f(4+h) \approx f(4) + h f'(4)$

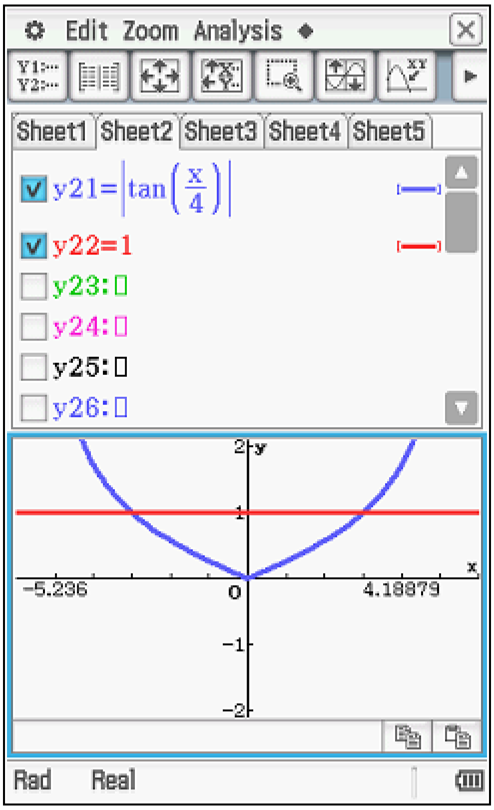
$$= 4^{\frac{3}{2}} + 0.2 \times \frac{3}{2} \times \sqrt{4}$$
$$= 8 + 0.6$$
$$= 8.6$$

Question 18

Answer is C.

Worked solution

The graph shows that there are two points where it intersects with the line $y = 1$.



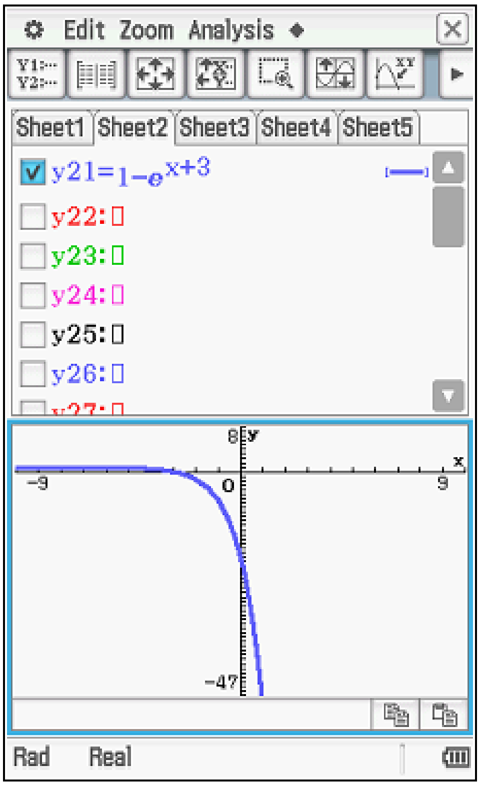
Question 19

Answer is A.

Worked solution

The graph of $y = 1 - e^{(x+3)}$ shows the region is below the *x*-axis.

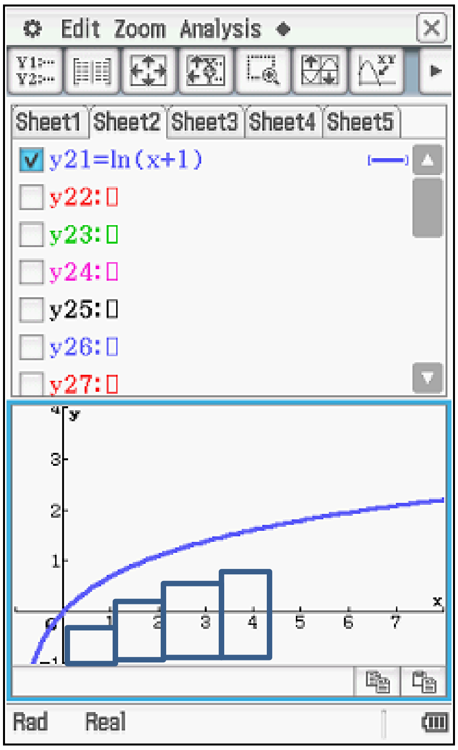
So the area is $-\int_{-3}^{0} (1 - e^{x+3})\, dx = \int_{-3}^{0} (e^{x+3} - 1)\, dx$.



Question 20

Answer is D.

Worked solution



The right rectangles have heights of:
$\log_e(1+1), \log_e(2+1), \log_e(3+1), \log_e(4+1)$
i.e. $\log_e(2), \log_e(3), \log_e(4), \log_e(5)$.
So the area is $\log_e(2)+\log_e(3)+\log_e(4)+\log_e(5)=\log_e(120)$.

Question 21

Answer is B.

Worked solution

The distance is the area under the velocity–time graph. There is an *x*-intercept at $x = 3,$ so the area is calculated in two parts as:

$$\int_0^3 (4.5 - \frac{1}{2}t^2)\,dt - \int_3^6 (4.5 - \frac{1}{2}t^2)\,dt$$

$$= 18 + 9 = 27$$

Question 22

Answer is E.

Worked solution

$$f^{-1}(x) = e^x \neq \frac{1}{\log_e(x)}$$

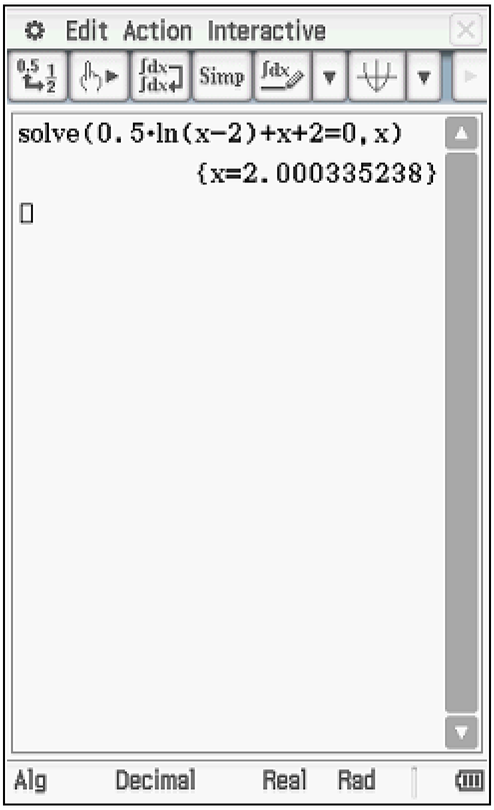
THIS PAGE IS BLANK

SECTION 2

Question 1a.

Worked solution

Using CAS:



So, $x = 2.0003$.

Mark allocation: 1 mark

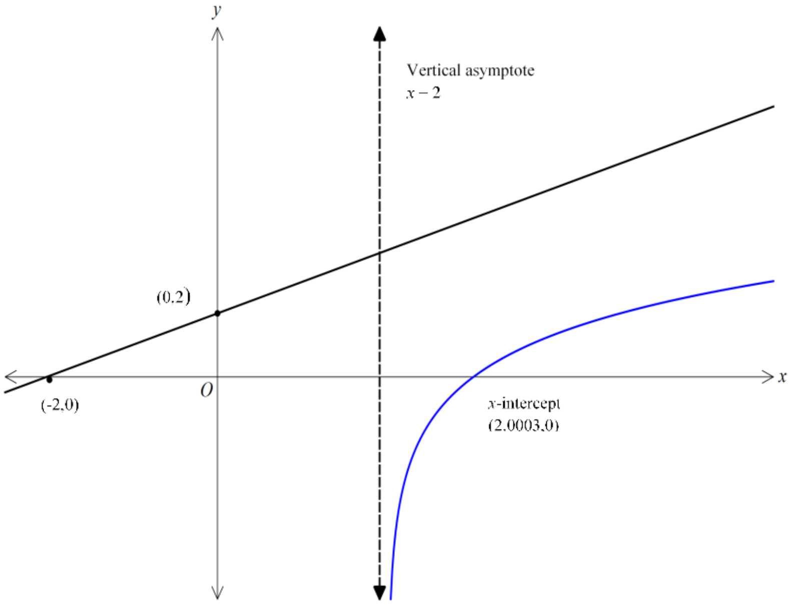
- 1 mark for 2.0003

Tip

- *Be sure to answer to the correct number of decimal places.*

Question 1b.

Worked solution



Mark allocation: 2 marks

- 1 mark for the drawing the shape and asymptote correctly
- 1 mark for labelling the *x*-intercept correctly

Tip

- *When sketching a graph, always state the equation of any asymptotes and the coordinates of any intercepts.*

Question 1c.

Worked solution

The domain of *h* is $(2,\infty)$.

Mark allocation: 1 mark

- 1 mark for $(2,\infty)$

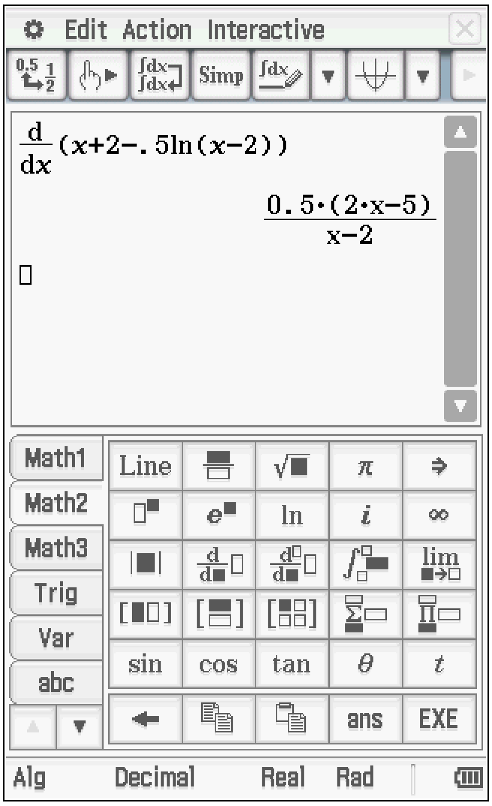
Tip

- *The domain of the added function is the domain of* $g \cap f$*. The location of the vertical asymptote is maintained so the domain is strictly greater than 2.*

Question 1d.i.

Worked solution

$\frac{d}{dx}\left(x+2-\frac{1}{2}\log_e(x-2)\right)=\frac{(2x-5)}{2(x-2)}$, as determined from CAS.



Mark allocation: 1 mark

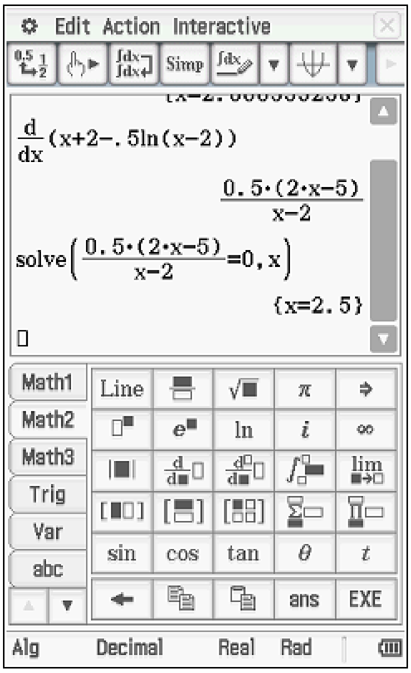
- 1 mark for the correct answer

Tip

- *Remember to use CAS – the question is worth only 1 mark and does not require working to be shown.*

Question 1d.ii.

Worked solution



$$\frac{d}{dx}(x+2-\log_e(x-2))=\frac{(2x-5)}{2(x-2)}$$

Let $h'(x)=0$:

$2x-5=0$

$x=2.5$

Mark allocation: 2 marks

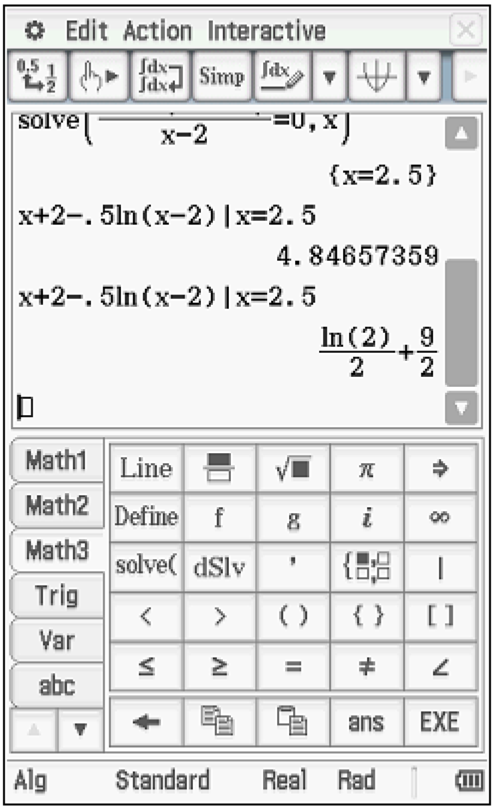
- 1 mark for getting $2x-5=0$
- 1 mark for the correct answer

Tip

- *This question is worth 2 marks so a 'thinking step' needs to be shown. It is enough to simply state* $h'(x)=0$ *and then give the answer.*

Question 1e.

Worked solution



Min distance = $\frac{1}{2}\log_e(2)+\frac{9}{2}$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2a.i.

Worked solution

$h(x) = \sin(x) + 1$

$g(x) = e^{\frac{x}{20}}$

Mark allocation: 2 marks

- 1 mark for each correct answer

Question 2a.ii.

Worked solution

For $g\left(h(x)\right)$ to exist:

Range of $h(x) \subseteq$ domain of $g(x)$

Range of $h(x) = [0, 2]$

Domain of $g(x) = R$

So, range of $h(x) \subseteq$ domain of $g(x)$.

$\therefore g(h(x))$ exists.

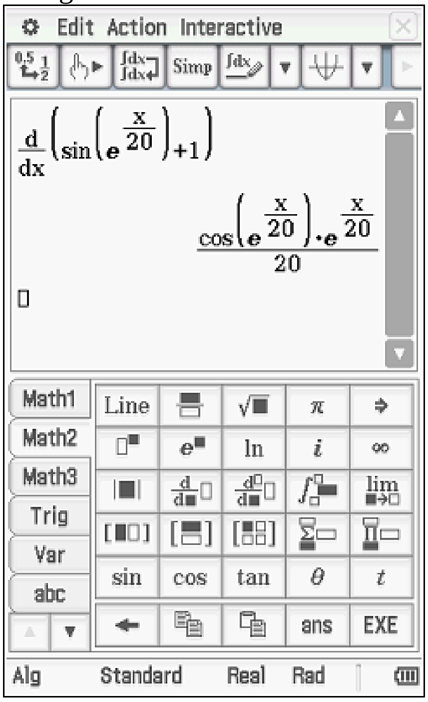
Mark allocation: 1 mark

- 1 mark for the correct answer with the correct reasons

Question 2b.

Worked solution

Using CAS:



$$\frac{d}{dx}(\sin(e^{\frac{x}{20}})+1)=\frac{e^{\frac{x}{20}}\cos(e^{\frac{x}{20}})}{20}$$

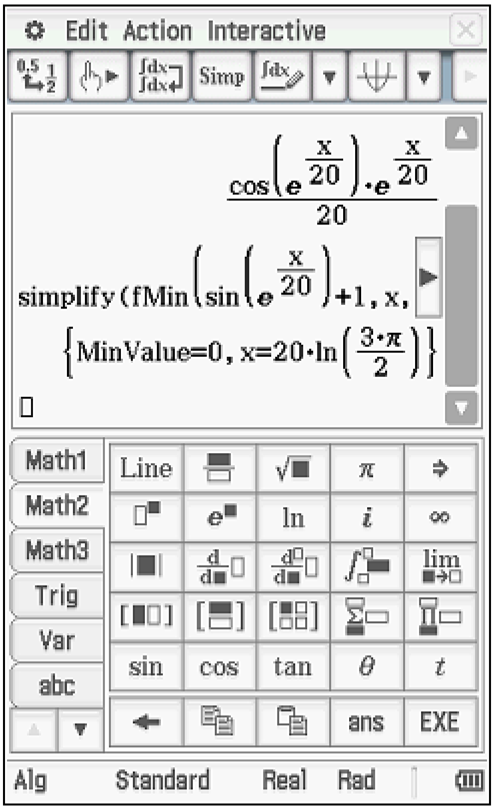
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2c.

Worked solution

Using CAS:



The first point on the base level occurs at $x = 20\log_e\left(\frac{3\pi}{2}\right)$.

Mark allocation: 1 mark

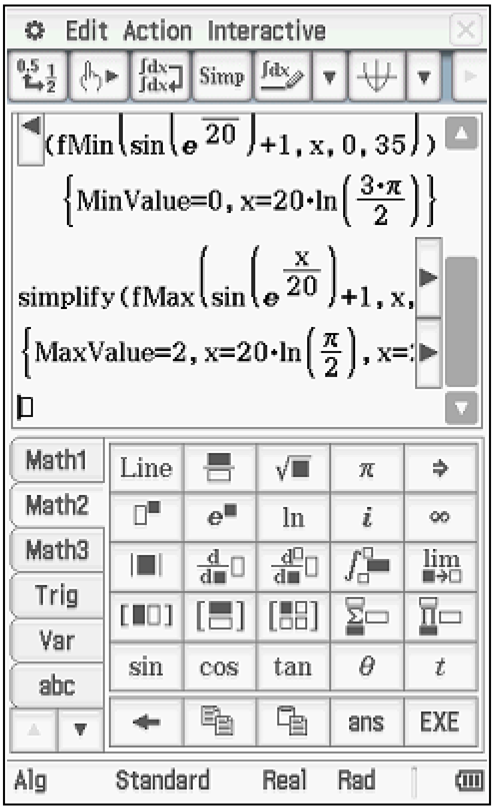
- 1 mark for the correct answer

Tip

- *To find the coordinates of the first point, specify the restriction on the domain. It is apparent from the graph given that the first minimum occurs in the domain* $[30, 35]$*, so request CAS to find the point in this domain using* $|30 \le x \le 35|$ *in the solve command.*

Question 2d.i.

Worked solution



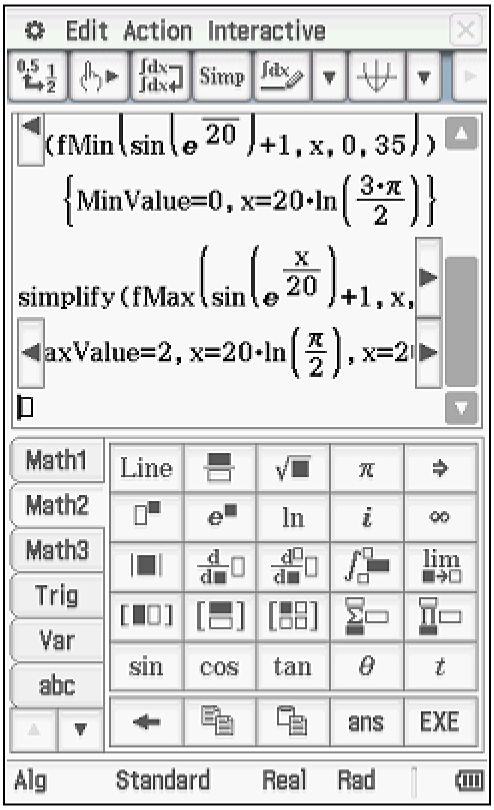
The maximum height is 2 metres.

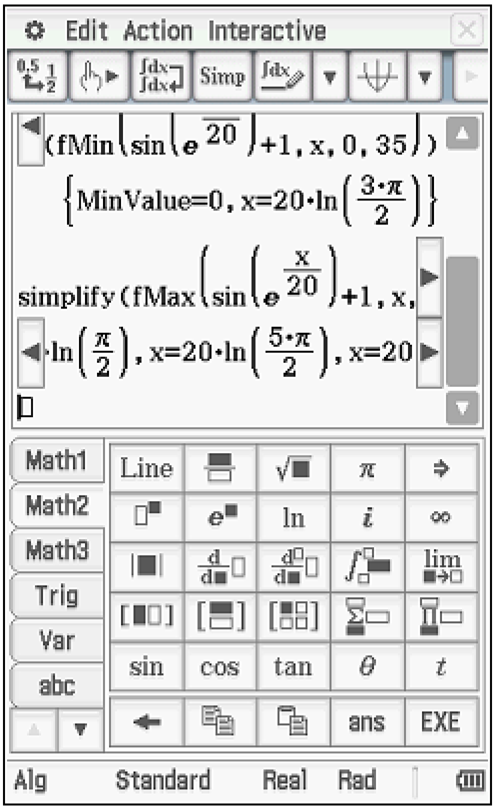
Mark allocation: 1 mark

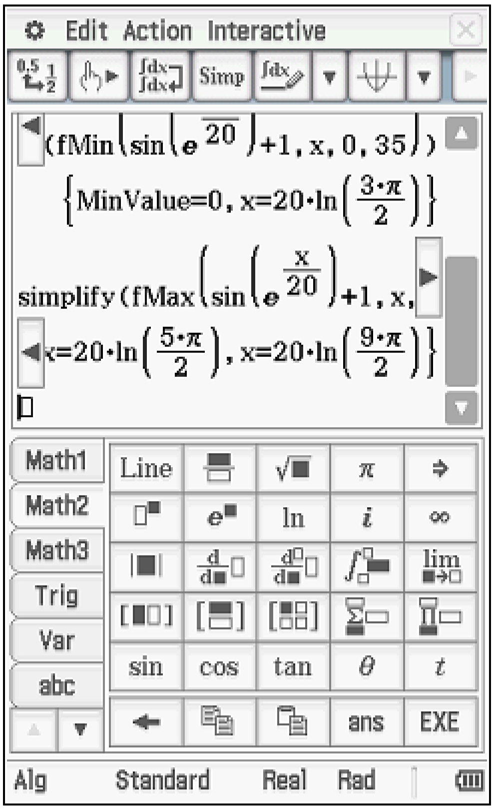
- 1 mark for the correct answer

Question 2d.ii.

Worked solution







So, this maximum height of 2 m occurs at

$$x = 20\log_e\left(\frac{\pi}{2}\right),\ x = 20\log_e\left(\frac{5\pi}{2}\right),\ x = 20\log_e\left(\frac{9\pi}{2}\right).$$

Mark allocation: 2 marks

- 1 mark awarded for giving only one correct value
- 2 marks awarded for all three correct values

Tip

- *Remember: an exact value is required here! Be sure to be in standard mode and to help with getting a simplified answer, put the simplify command in your CAS calculation.*

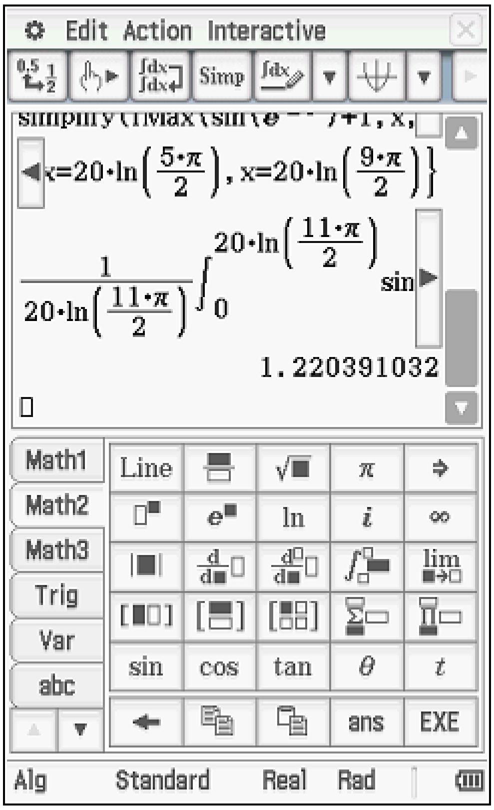
Question 2e.

Worked solution

This horizontal cross-section height is the average value of the function.

$$\text{Average height} = \frac{1}{20\log_e\left(\frac{11\pi}{2}\right)}\int_0^{20\log_e(\frac{11\pi}{2})}\left(\sin(e^{\frac{x}{20}})+1\right)dx$$

Using CAS, this is:



So, the average height is 1.22 metres.

Mark allocation: 2 marks

- 1 mark for writing the average value $\frac{1}{20\log_e\left(\frac{11\pi}{2}\right)}\int_0^{20\log_e(\frac{11\pi}{2})}\left(\sin(e^{\frac{x}{20}})+1\right)dx$
- 1 mark for the correct answer of 1.22

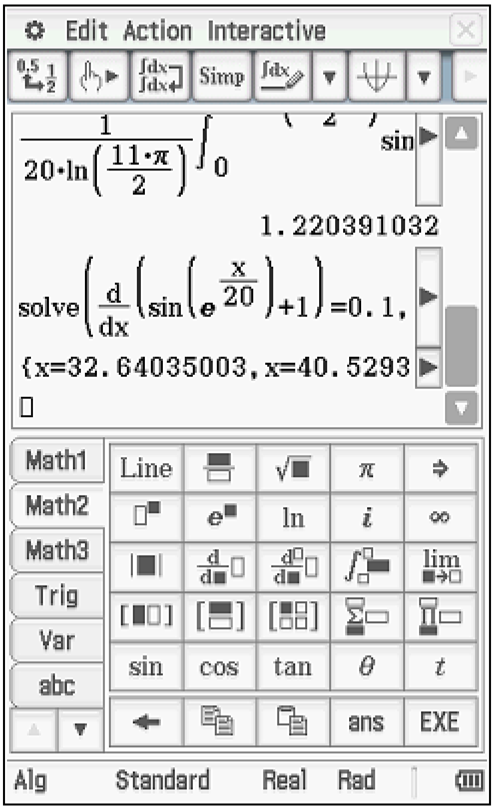
Question 2f.

Worked solution

The gradient of the pole is –10, which means the gradient of the curve is $\frac{1}{10}$.

So, we need to find $\left\{ x : f'(x) = \frac{1}{10}, \text{ for } x \in [30, 40] \right\}$.

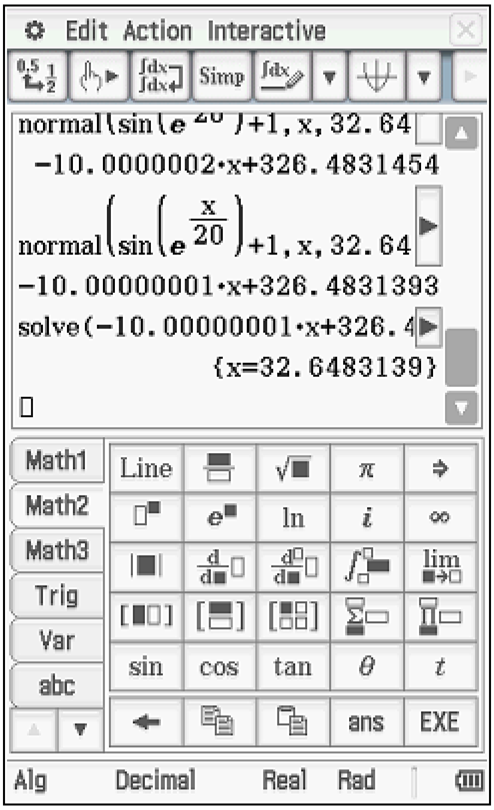
Using CAS gives:



The pole is inserted at $x = 32.64035$ metres.

Using CAS, we must find the equation of the normal to the curve at this point.

Equation of normal is $y = -10x + 326.48314$.



The pole reaches the base level at $y = 0$.

So, $-10x + 326.48314 = 0$

$$x = 32.6483$$

$$\therefore\ x = 32.648 \text{ m}$$

Mark allocation: 3 marks

- 1 mark for setting $f'(x) = \dfrac{1}{10}$
- 1 mark for finding the equation of the normal
- 1 mark for the correct answer

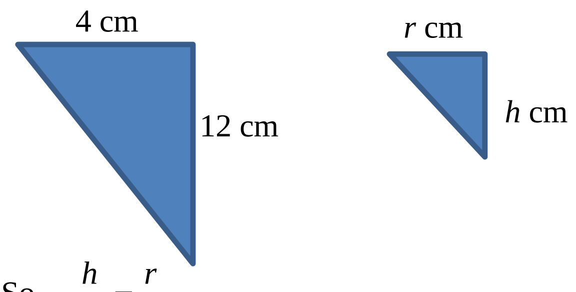
Tip

- *Remember: when asked to write an answer correct to 3 decimal places, always work to more decimal places and then round at the last step.*

Question 3a.i

Worked solution

Using similar triangles:



So, $\frac{h}{12} = \frac{r}{4}$

$$h = \frac{12r}{4}$$
$$= 3r$$

Mark allocation: 1 mark

- 1 mark for using similar triangles method

Question 3a.ii.

Worked solution

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi \frac{h^2}{9} h$$
$$V = \frac{\pi h^3}{27}$$

Mark allocation: 1 mark

- 1 mark for substituting correctly into the formula to get $V = \frac{\pi h^3}{27}$

Question 3b.

Worked solution

$$\frac{dh}{dt}=\frac{dV}{dt}\times\frac{dh}{dV}$$

$$\frac{dV}{dt}=\frac{4\pi}{9}$$

$$V=\frac{\pi h^3}{27} \;\Rightarrow\; \frac{dV}{dh}=\frac{\pi h^2}{9}$$

$$\therefore \frac{dh}{dV}=\frac{9}{\pi h^2}$$

$$\frac{dh}{dt}=\frac{dV}{dt}\times\frac{dh}{dV}$$

$$=\frac{4\pi}{9}\times\frac{9}{\pi h^2}=\frac{4}{h^2}$$

Mark allocation: 2 marks

- 1 mark for the rate equation $\frac{dh}{dt}=\frac{dV}{dt}\times\frac{dh}{dV}$
- 1 mark for finding $\frac{dh}{dV}=\frac{9}{\pi h^2}$

Question 3c.i.

Worked solution

When $h=2$, $\frac{dh}{dt}=\frac{4}{(2)^2}=1\,\text{cm/min}$.

Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *Remember to give the units.*

Question 3c.ii.

Worked solution

Let $\frac{dh}{dt} = \frac{1}{2}$.

$$\frac{4}{h^2} = \frac{1}{2}$$

$$h^2 = 8$$

$$\therefore h = 2\sqrt{2} \text{ cm}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3d.i.

Worked solution

$$\frac{dh}{dt} = \frac{4}{h^2}$$

$$\therefore \frac{dt}{dh} = \frac{h^2}{4}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3d.ii.

Worked solution

$$t = \int \frac{h^2}{4}\, dh$$

$$t = \frac{h^3}{12} + c$$

When $t = 0$, $h = 0$, so $c = 0$.

$$t = \frac{h^3}{12}$$

$$\therefore h = \sqrt[3]{12t}$$

Mark allocation: 1 mark

- 1 mark for the correct answer, which must include + c

Question 3e.i.

Worked solution

$d = 36 - 1.5t$

Mark allocation: 1 mark

- 1 mark for the correct expression

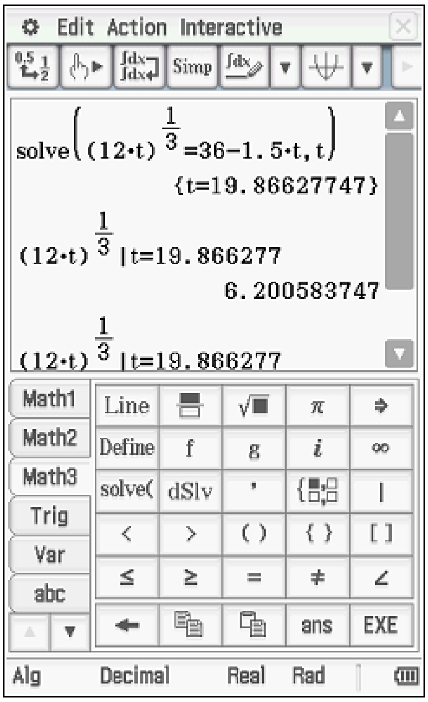
Quesiton 3e.ii.

Worked solution

Let $h = d$, giving:

$\sqrt[3]{12t} = 36 - 1.5t$

Using CAS to solve for *t*, we get:



$t = 19.87$ min; i.e. 9:20 a.m.

Mark allocation: 2 marks

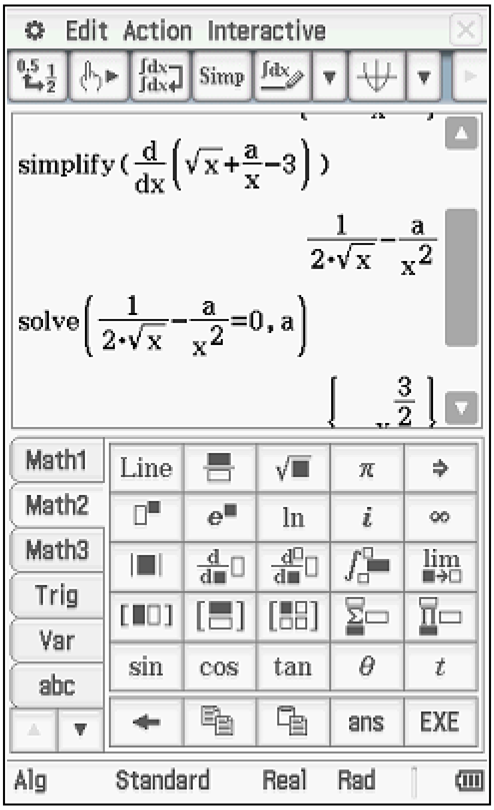
- 1 mark for equating the two equations correctly
- 1 mark for the correct answer

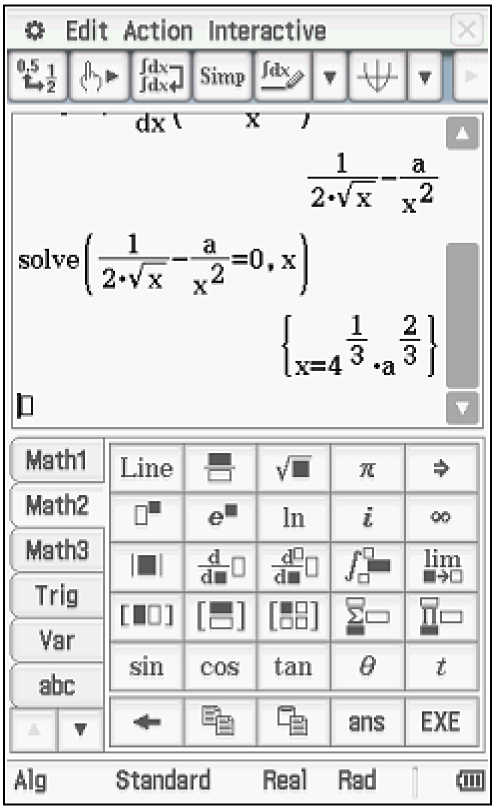
Question 4a.

Worked solution

The function f_a is strictly increasing for $f'(x) \geq 0$.

Use CAS to find $f'(x) = 0$.





So, the function is increasing for $x \in [4^{\frac{1}{3}} a^{\frac{2}{3}}, \infty)$.

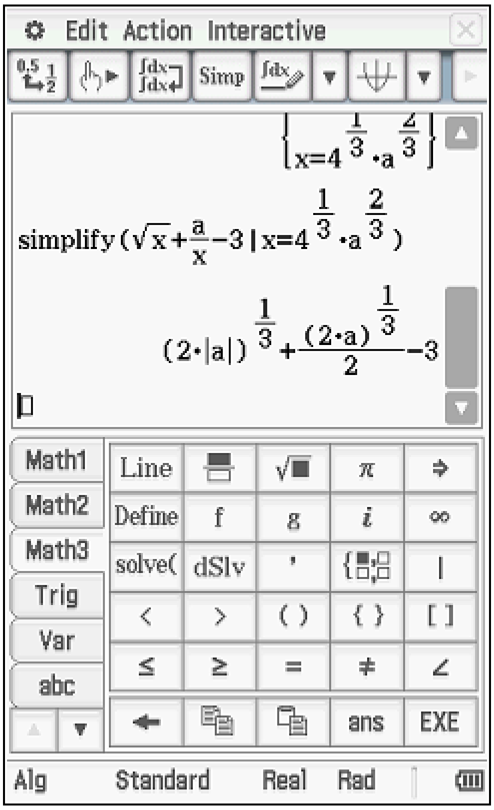
Mark allocation: 2 marks

- 1 mark for letting $f'(x) = 0$
- 1 mark for the correct answer

Question 4b.

Worked solution

The minimum occurs when $f'(x) = 0$.



So, when $x = 4^{\frac{1}{3}}a^{\frac{2}{3}}$, $f(x) = \dfrac{3(2a)^{\frac{1}{3}}}{2} - 3$.

Mark allocation: 2 marks

- 1 mark for $x = 4^{\frac{1}{3}}a^{\frac{2}{3}}$
- 1 mark for $f(x) = \dfrac{3(2a)^{\frac{1}{3}}}{2} - 3$

Question 4c.

Worked solution

For $f_a(x)=\frac{a}{x}+\sqrt{x}-3$ and $f'(x)=\frac{1}{2\sqrt{x}}-\frac{a}{x^2}$ at x = 4,

$f(4)=\frac{a}{4}-1$ and $f'(4)=\frac{1}{4}-\frac{a}{16}$.

So, the equation of the tangent is:

$$y-y_1=m(x-x_1)$$

$$y-\left(\frac{a}{4}-1\right)=\left(\frac{1}{4}-\frac{a}{16}\right)(x-4)$$

$$y=\left(-\frac{a}{16}+\frac{1}{4}\right)x-1+\frac{a}{4}+\frac{a}{4}-1$$

$$y=\left(-\frac{a}{16}+\frac{1}{4}\right)x+\frac{a}{2}-2$$

$$y=-\left(\frac{a}{16}-\frac{1}{4}\right)x+\frac{a}{2}-2$$

$$y=-\frac{(a-4)}{16}x+\frac{(a-4)}{2}$$

Mark allocation: 3 marks

- 1 mark for $x=4,\ f(4)=\frac{a}{4}-1$
- 1 mark for $f'(4)=\frac{1}{4}-\frac{a}{16}$
- 1 mark for the correct tangent line equation

Question 4d.

Worked solution

The y-intercept of $f_a(x)$ is $y = \frac{a-4}{2}$.

The y-intercept of $g_a(x) = f_a(x) + b$ is $b + \frac{a-4}{2}$.

Let the y-intercept equal zero, giving:

$$b + \frac{a-4}{2} = 0$$
$$b = \frac{4-a}{2}$$

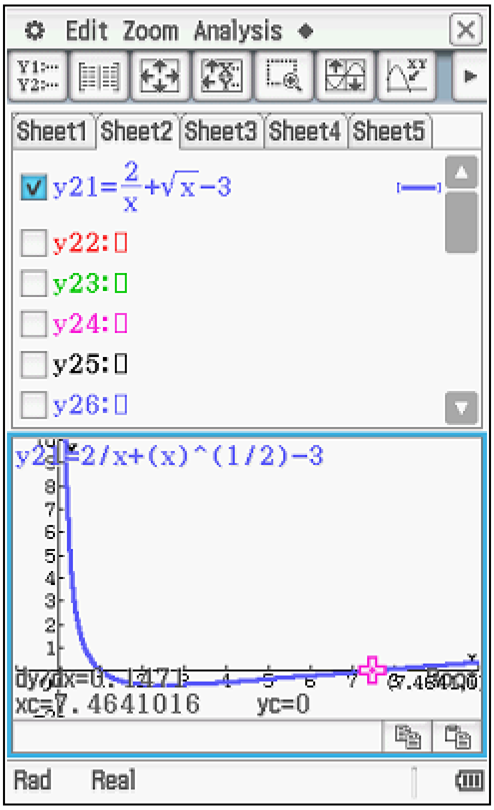
Mark allocation: 2 marks

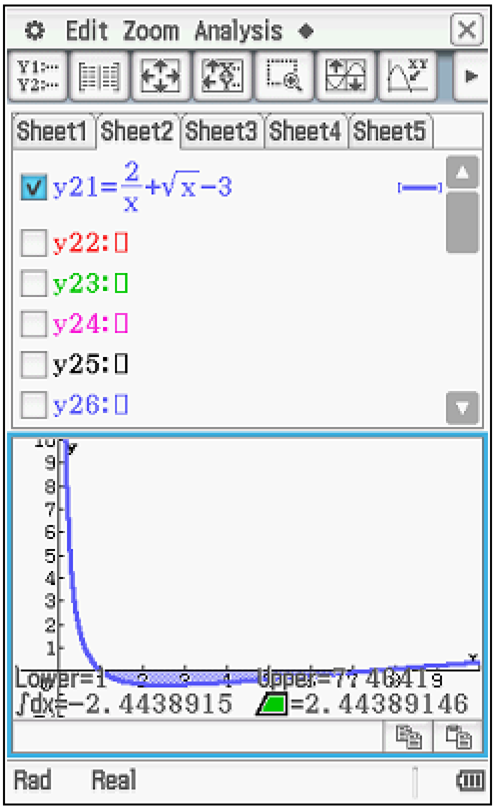
- 1 mark for finding the y-intercept
- 1 mark for $b = \frac{4-a}{2}$

Question 4e.i.

Worked solution

Using CAS, the x-intercepts occur at $x = 1$ and $x = 7.4641$.





So, the area is equal to $\left|\int_{1}^{7.4641}\left(\frac{2}{x}+\sqrt{x}-3\right)dx\right|$.

Using CAS, this is 2.444.

Mark allocation: 2 marks

- 1 mark for writing area is equal to $\left|\int_{1}^{7.4641}\left(\frac{2}{x}+\sqrt{x}-3\right)dx\right|$
- 1 mark for the correct answer

Tip

- *This question is worth 2 marks, so be careful to show a 'thinking step'. In this case, include the integral that allows the area to be calculated.*

Question 4e.ii.

Worked solution

Given that $h_a(x) = f_a(2x)$, the graph of $y = h_a(x)$ is formed by a dilation of factor 0.5 in the x-direction. This means that the area formed is half the size; i.e. the area that is bounded by the x-axis and the graph of $y = h_a(x)$ for $a = 2$ is

$0.5 \times 2.44389 = 1.2219 = 1.222$

Mark allocation: 2 marks

- 1 mark for 0.5×2.44389
- 1 mark for the correct answer

Tip

- *This is a 'hence' question and therefore you must show that you have used the previous result.*

Question 5a.i.

Worked solution

$0.7^4 = 0.2401$

Mark allocation: 1 mark

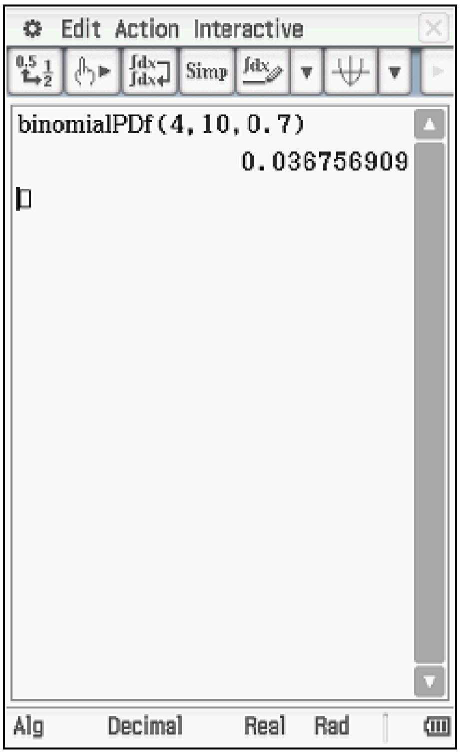
- 1 mark for the correct answer

Question 5a.ii.

Worked solution

$X \sim Bi(n = 10,\ p = 0.7)$

$\Pr(X = 4) = 0.0368$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5a.iii

Worked solution

$$\Pr(GGGGMMMMMM \mid X = 4)$$
$$= \frac{\Pr(GGGGMMMMMM)}{\Pr(X = 4)}$$
$$= \frac{0.7^4 0.3^6}{0.036756909}$$
$$= 0.0048$$

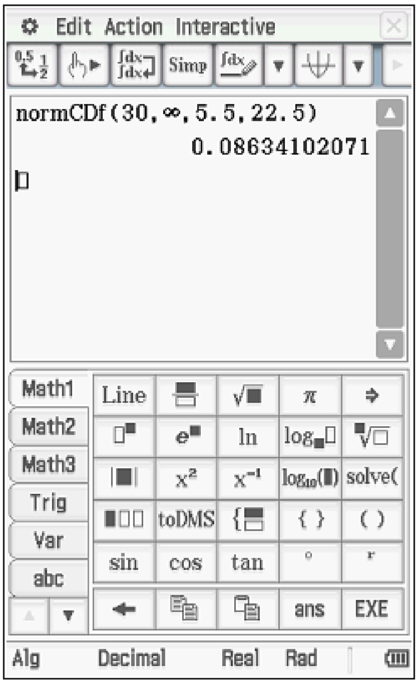
Mark allocation: 2 marks

- 1 mark for $\Pr(GGGGMMMMMM \mid X = 4)$
- 1 mark for the correct answer

Question 5b.

Worked solution

Using CAS:



$X \sim N(\mu = 22.5, \sigma = 5.5)$

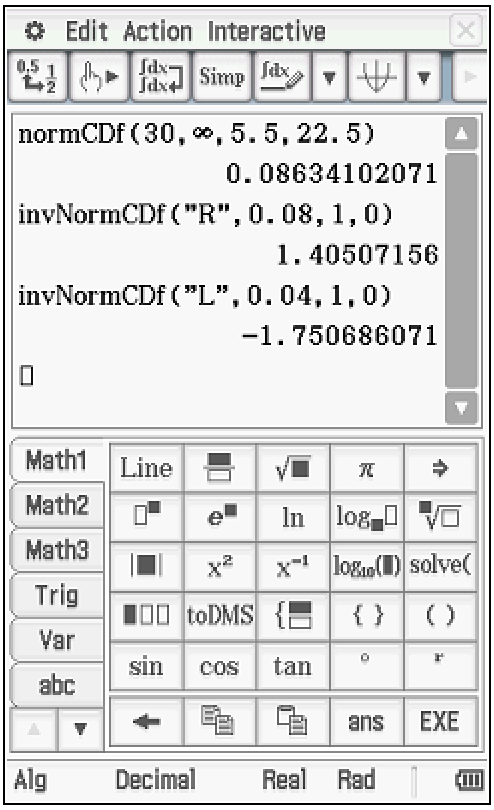
$\Pr(X > 30) = 0.0863$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5c.i.

Worked solution



$X_A \sim N(\mu, \sigma)$

$\Pr(X_A > 35) = 0.08$ and $\Pr(X_A < 5) = 0.04$

$\Pr(Z > \frac{35-\mu}{\sigma}) = 0.08$ and $\Pr(Z < \frac{5-\mu}{\sigma}) = 0.04$

Using the inverse normal distribution:

$\frac{35-\mu}{\sigma} = 1.4051$ and $\frac{5-\mu}{\sigma} = -1.7507$

$\mu + 1.4051\sigma = 35$ and $\mu - 1.7507\sigma = 5$

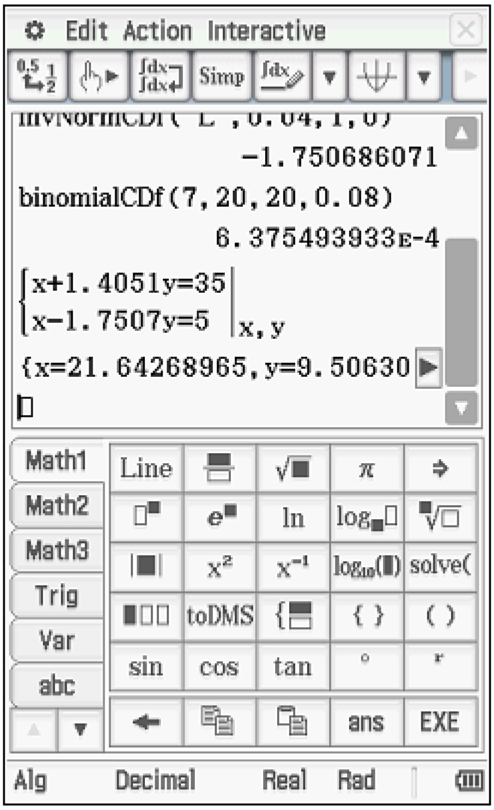
Mark allocation: 2 marks

- 1 mark for $\Pr(X_A > 35) = 0.08$ and $\Pr(X_A < 5) = 0.04$
- 1 mark for leading to the result $\frac{35-\mu}{\sigma} = 1.4051$ and $\frac{5-\mu}{\sigma} = -1.7507$

Question 5c.ii.

Worked solution

$\mu = 21.643$ and $\sigma = 9.506$.



Mark allocation: 2 marks

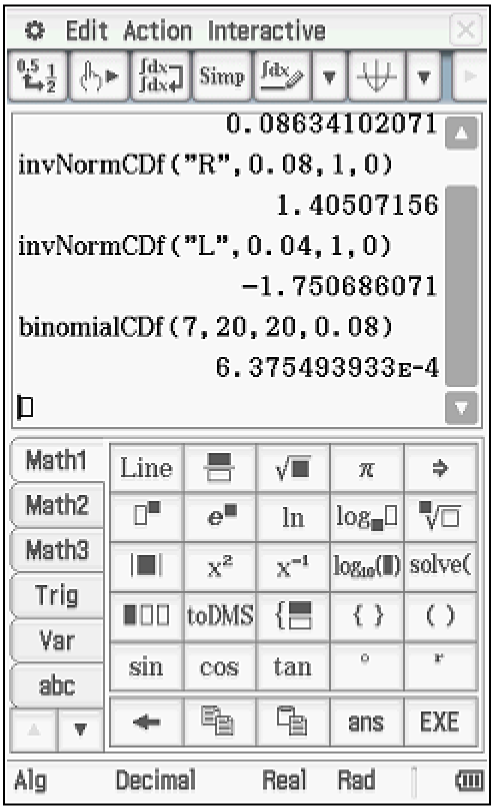
- 1 mark for finding the mean
- 1 mark for finding the standard deviation

Question 5d.

Worked solution

$X \sim Bi(n = 20,\ p = 0.08)$

$\Pr(X \geq 7) = 0.00064$



Mark allocation: 2 marks

- 1 mark for identifying the binomial distribution $X \sim Bi(n = 20,\ p = 0.08)$
- 1 mark for the answer $\Pr(X \geq 7) = 0.00064$

Question 5e.

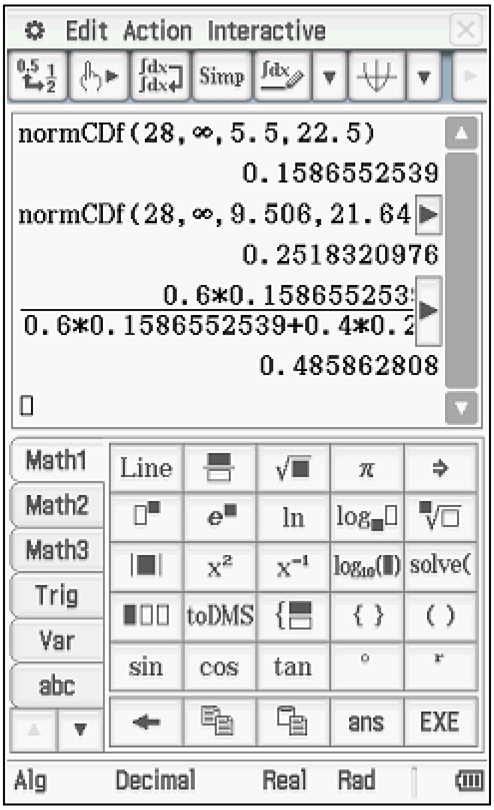
Worked solution

$$\Pr(\text{home goal} \mid \text{goal is more than } 28\,\text{m}) = \frac{\Pr(\text{home goal and goal is more than } 28\,\text{m})}{\Pr(\text{goal is more than } 28\,\text{m})}$$

$$= \frac{0.6\Pr(X_{\text{H}} > 28)}{0.6\Pr(X_{\text{H}} > 28) + 0.4\Pr(X_{\text{A}} > 28)}$$

$$= \frac{0.6 \times 0.158655}{0.6 \times 0.158655 + 0.4 \times 0.251832}$$

$$= 0.4859$$



Mark allocation: 2 marks

- 1 mark for understanding the conditional probability
  $$\Pr(\text{home goal} \mid \text{goal is more than } 28\,\text{m}) = \frac{\Pr(\text{home goal and goal is more than } 28\,\text{m})}{\Pr(\text{goal is more than } 28\,\text{m})}$$
- 1 mark for the correct answer