2015 VCAA Math Methods CAS Exam 2 Solutions

CAS should be used whenever possible to speed up the solution process.

SECTION 1

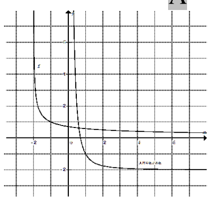
1	2	3	4	5	6	7	8	9	10	11
A	A	C	B	E	C	C	D	B	D	A

12	13	14	15	16	17	18	19	20	21	22
E	E	D	B	D	D	A	C	B	D	B

Q1 Period $= \frac{2\pi}{3}$, the range is $[-2-3, 2-3]$, i.e. $[-5, -1]$ **A**

Q2 **A**

$x = \frac{1}{\sqrt{y+2}}$, $y = \frac{1}{x^2} - 2$, .: $f^{-1}(x) = \frac{1}{x^2} - 2$



Q3 **C**

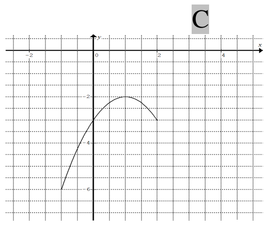
Q4 $y = x^2$, $m_t = \frac{dy}{dx} = 2x$. At $(2,4)$, $m_t = 2(2) = 4$.

Consider $(3,8)$ and $(2,4)$, $m = \frac{8-4}{3-2} = 4 = m_t$ **B**

Q5 Reflection in the line $y = x$ **E**

Q6 $P(3) = 3^3 - a.3^2 - 4(3) + 4 = 10$, .: $a = 1$ **C**

Q7 **C**



Q8 Area under the straight line $= \frac{1}{2}p^2 = \frac{25}{8}$, .: $p = \frac{5}{2}$ **D**

Q9 $\frac{1}{6}(a-2) = 1$, .: $a = 8$, $E(X) = \int_2^8 \frac{x}{6}\,dx = \left[\frac{x^2}{12}\right]_2^8 = 5$ **B**

Q10 $np = 2$, $npq = \frac{4}{3}$, .: $n = 6$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$

$\Pr(X = 1) = {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5$ **D**

Q11 $\sqrt{x^3+1} = \sqrt{8\left(\frac{x}{2}\right)^3 + 1}$ **A**

Q12 $\Pr(\text{at least one}) = 1 - \Pr(\text{none}) = 1 - \frac{1}{{}^8C_3} = \frac{55}{56}$ **E**

Q13

$\int_0^1 ae^x\,dx + ae.1 = 1$, $\left[ae^x\right]_0^1 + ae = 1$, $2ae - a = 1$, $a = \frac{1}{2e-1}$ **E**

Q14 $p + 2p + 3p + 4p + 5p = 1$, .: $p = \frac{1}{15}$

Mean of $X = 1\times\frac{1}{15} + 2\times\frac{2}{15} + 3\times\frac{3}{15} + 4\times\frac{4}{15} + 5\times\frac{5}{15} = \frac{11}{3}$ **D**

Q15 $\int_0^5 (2g(x) + ax)\,dx = \int_0^5 2g(x)\,dx + \int_0^5 ax\,dx$

$= 2\int_0^5 g(x)\,dx + \left[\frac{ax^2}{2}\right]_0^5 = 2\times 20 + \frac{25a}{2} = 90$, .: $a = 4$ **B**

Q16 $f'(x) = \int g(x)\,dx$, $max^{m-1} = \frac{bx^{n+1}}{n+1}$

.: $\frac{b}{a} = m(n+1)x^{m-n-2}$.: $m - n - 2 = 0$.: $\frac{b}{a} = m(n+1)$ is an integer since *m* and *n* are positive integers. **D**

Q17 By CAS $y = x^3 - 3x^2$ has a local max at $(0,0)$ and a local min at $(2,-4)$. To have three distinct *x*-intercepts, $y = x^3 - 3x^2$ must translate upwards by less than 4 units. Hence $c \in (0,4)$. **D**

Q18 $f(x) = x^2$ satisfies $|f(x+y) - f(x-y)| = 4\sqrt{f(x)f(y)}$.

L.H.S. $= \left|(x+y)^2 - (x-y)^2\right| = |4xy| = 4|xy|$

R.H.S. $= 4\sqrt{x^2y^2} = 4\sqrt{(xy)^2} = 4|xy| =$ L.H.S. **A**

Q19 Let $F(t) = \int\left(\sqrt{t^2+4}\right)dt$

.: $f(x) = \int_0^x\left(\sqrt{t^2+4}\right)dt = [F(t)]_0^x = F(x) - F(0)$

.: $f'(x) = F'(x) = \sqrt{x^2+4}$, $f'(-2) = \sqrt{(-2)^2+4} = 2\sqrt{2}$ **C**

Q20 $f(x-1) = x^2 - 2x + 3$

Replace x with $x+1$: $f(x+1-1) = (x+1)^2 - 2(x+1) + 3$

.: $f(x) = x^2 + 2$ **B**

Q21 Let $ax^2 = mx + c$, $ax^2 - mx - c = 0$

No intersections: $\Delta < 0$, $(-m)^2 - 4a(-c) < 0$, $m^2 + 4ac < 0$

If $a > 0$, $c < -\frac{m^2}{4a}$; if $a < 0$, $c > -\frac{m^2}{4a}$ **D**

Q22 Let $f(x) = -a|x|$ where $a \in R^+$.

.: $g(-f(x)) = g(a|x|) = g(a|-x|) \geq 0$ and $g(-f(x))$ is even. **B**

SECTION 2

Q1a $f(x)=\frac{1}{5}(x-2)^2(5-x)$

$f'(x)=\frac{2}{5}(x-2)(5-x)-\frac{1}{5}(x-2)^2=\frac{3}{5}(x-2)(4-x)$

Q1bi At $P\left(1,\frac{4}{5}\right)$, $m=f'(1)=-\frac{9}{5}$

$y-\frac{4}{5}=-\frac{9}{5}(x-1)$, $9x+5y=13$ or $y=-\frac{9}{5}x+\frac{13}{5}$

Q1bii $9x+5y=13$ When $x=0$, $y=\frac{13}{5}$, .: $S\left(0,\frac{13}{5}\right)$

When $y=0$, $x=\frac{13}{9}$, .: $Q\left(\frac{13}{9},0\right)$

Q1c Distance $PS=\sqrt{(0-1)^2+\left(\frac{13}{5}-\frac{4}{5}\right)^2}=\frac{\sqrt{106}}{5}$

Q1d Use CAS to solve simultaneous equations $y=\frac{1}{5}(x-2)^2(5-x)$ and $9x+5y=13$ to find the second point of intersection at $x=7$.

Area of the shaded region $=\int_1^7\left(\frac{1}{5}(x-2)^2(5-x)-\left(-\frac{9}{5}x+\frac{13}{5}\right)\right)dx$

$=21.6$ square units by CAS

Q2a $y=60-\frac{3}{80}x^2$, $\frac{dy}{dx}=-\frac{3}{40}x$

At $A(-40,0)$, $m=-\frac{3}{40}(-40)=3$, $\tan\theta=3$, $\theta\approx 72°$

Q2b $y=\frac{x^3}{25600}-\frac{3x}{16}+35$, $\frac{dy}{dx}=\frac{3x^2}{25600}-\frac{3}{16}$

The turning point of $\frac{dy}{dx}$ is $\left(0,-\frac{3}{16}\right)$.

.: the max downward slope is $-\frac{3}{16}$.

Q2c Vertical distance $D=60-\frac{3}{80}x^2-\left(\frac{x^3}{25600}-\frac{3x}{16}+35\right)$

Let $\frac{dD}{dx}=0$. $-\frac{3}{40}x-\frac{3x^2}{25600}+\frac{3}{16}=0$

.: $u=x\approx 2.49$ m (2.49031)

and $v=\frac{2.49031^3}{25600}-\frac{3\times 2.49031}{16}+35\approx 34.53$ m

Q2d $P(-2.49031,w)$,

$w=\frac{(-2.49031)^3}{25600}-\frac{3(-2.49031)}{16}+35\approx 35.47$ m

$D_{MN}=60-\frac{3}{80}(2.49031)^2-\left(\frac{2.49031^3}{25600}-\frac{3\times 2.49031}{16}+35\right)\approx 25.23$ m

$D_{PQ}=60-\frac{3}{80}({}^{-}2.49031)^2-\left(\frac{({}^{-}2.49031)^3}{25600}-\frac{3({}^{-}2.49031)}{16}+35\right)\approx 24.30$ m

Q2e Let $D=60-\frac{3}{80}x^2-\left(\frac{x^3}{25600}-\frac{3x}{16}+35\right)=0$

By CAS, $x_E=-23.71$ and $x_F=28.00$

Q2f Area of the shaded region

$=\int_{-23.71}^{28.00}\left(60-\frac{3}{80}x^2-\left(\frac{x^3}{25600}-\frac{3x}{16}+35\right)\right)dx\approx 870\text{ m}^2$ by CAS

Q3ai $\Pr(X>7)=\int_7^8\frac{3}{4}(x-6)^2(8-x)dx=0.6875=\frac{11}{16}$ by CAS

Q3aii Binomial distribution: $n=3$, $p=\frac{11}{16}$

$\Pr(X=1)={}^3C_1\left(\frac{11}{16}\right)^1\left(\frac{5}{16}\right)^2=\frac{825}{4096}$

Q3b Mean $=\int_6^8\frac{3}{4}(x-6)^2(8-x)x\,dx=7.2$ cm

Q3c Normal distribution: $\mu=74$, $\sigma=9$

$\Pr(X<85\mid X>74)=\frac{\Pr(74<X<85)}{\Pr(X>74)}\approx 0.778$

Q3di Binomial distribution: $n=3$, $p=0.03$, $q=0.97$

$\Pr(X\ge 1)=1-\Pr(X=0)=1-0.97^4\approx 0.1147$

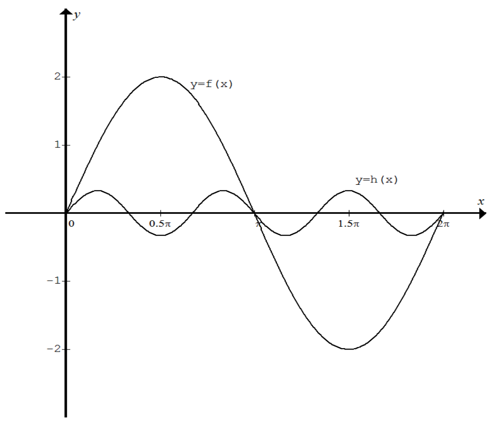
Q3dii Binomial distribution: n?, $p=0.03$, $q=0.97$

$1-\Pr(X=0)=1-0.97^n>0.5$

.: $0.97^n<0.5$, $n>22.75$, .: $n=23$

Q4a Area of shaded region $=2\times\int_0^{\pi}2\sin(x)dx=4\int_0^{\pi}\sin(x)dx$, $a=4$

Q4b



Q4c Dilate from the x-axis by a factor of $\frac{1}{6}$, then dilate from the y-axis by a factor of $\frac{1}{3}$.

Q4di If n is even, the calculation is similar to part a.

Area of shaded region $= 2\times\int_0^{\pi} m\sin(x)dx = 2m\int_0^{\pi}\sin(x)dx$

$= 2m[-\cos(x)]_0^{\pi} = 4m = 4m + \frac{0}{n^2}$

If n is odd, area of shaded region

$= 2\left(\int_0^{\pi} m\sin(x)dx - \int_0^{\frac{\pi}{n}}\frac{1}{n}\sin(nx)dx\right) = 4m + \frac{^{-}4}{n^2}$

Q5ai $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, $0 \le t \le 5$

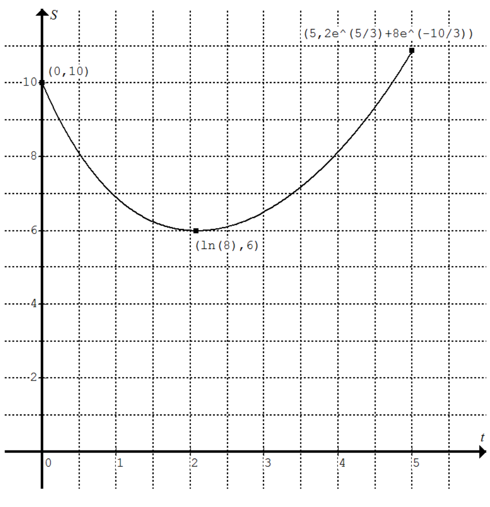
$S(0) = 10$, $S(5) = 2e^{\frac{5}{3}} + 8e^{\frac{-10}{3}}$

Q5aii $\frac{dS}{dt} = \frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}}$ Let $\frac{dS}{dt} = 0$ to find $S_{\min}$.

.: $\frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}} = 0$, $\left(\frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}}\right)e^{\frac{2t}{3}} = 0$

.: $e^t = 8$, $t = \log_e 8$, .: $c = 8$ and $S_{\min} = 2e^{\frac{\log_e 8}{3}} + 8e^{\frac{-2\log_e 8}{3}} = 6$

Q5aiii



Q5aiv Average rate of change $= \frac{6-10}{\log_e 8 - 0} = -\frac{4}{\log_e 8}$

Q5b $V(t) = de^{\frac{t}{3}} + (10-d)e^{\frac{-2t}{3}}$, $0 \le t \le 5$ and $0 < d < 10$

$\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

$V_{\min}$ occurs when $t = \log_e 9$ (i.e. $e^t = 9$) and $\frac{dV}{dt} = 0$.

.: $\frac{d}{3}\left(9^{\frac{1}{3}}\right) - \frac{2(10-d)}{3}\left(9^{-\frac{2}{3}}\right) = 0$.: $d = \frac{20}{11}$

Note: When d decreases (or increases), the turning point (local min) of $V(t)$ occurs at increased (or decreased) t values. It can also occur outside the interval $[0, 5]$ and .: the minimum value (not necessarily the turning point) of $V(t)$ occurs at one of the endpoints.

Q5ci $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

Turning point at $t = 0$, $\frac{d}{3} - \frac{2(10-d)}{3} = 0$, .: $d = \frac{20}{3}$

.: $V_{\min}$ occurs at $t = 0$ when $d \in \left[\frac{20}{3}, 10\right)$

Q5cii $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

Turning point at $t = 5$, $\frac{d}{3}e^{\frac{5}{3}} - \frac{2(10-d)}{3}e^{\frac{-10}{3}} = 0$, .: $d = \frac{20}{e^5+2}$

.: $V_{\min}$ occurs at $t = 5$ when $d \in \left(0, \frac{20}{e^5+2}\right]$.

Q5d $V(t) = de^{\frac{t}{3}} + (10-d)e^{\frac{-2t}{3}}$, $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

(a, m) is a local minimum, .: $\frac{d}{3}e^{\frac{a}{3}} - \frac{2(10-d)}{3}e^{\frac{-2a}{3}} = 0$ … (1)

and $m = de^{\frac{a}{3}} + (10-d)e^{\frac{-2a}{3}} = \frac{k}{2}d^{\frac{2}{3}}(10-d)^{\frac{1}{3}}$ … (2)

From (1), $e^a = \frac{2(10-d)}{d}$

.: $e^{\frac{a}{3}} = \left(\frac{2(10-d)}{d}\right)^{\frac{1}{3}}$ … (3) and $e^{-\frac{2a}{3}} = \left(\frac{2(10-d)}{d}\right)^{-\frac{2}{3}}$ … (4)

Substitute (3) and (4) in (2):

$m = d\frac{2^{\frac{1}{3}}(10-d)^{\frac{1}{3}}}{d^{\frac{1}{3}}} + (10-d)\frac{2^{-\frac{2}{3}}(10-d)^{-\frac{2}{3}}}{d^{-\frac{2}{3}}} = \frac{k}{2}d^{\frac{2}{3}}(10-d)^{\frac{1}{3}}$

.: $2^{\frac{1}{3}} + \frac{1}{2^{\frac{2}{3}}} = \frac{k}{2}$, .: $k = 2\times 2^{\frac{1}{3}} + 2^{\frac{1}{3}} = 2^{\frac{1}{3}}(2+1) = 3\times 2^{\frac{1}{3}}$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors