

Year 2016
VCE
Mathematical Methods
Trial Examination 1
Solutions



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9018 5376
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

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Question 1

a. $y = \sqrt{16-x^2} = u^{\frac{1}{2}}$ where $u = 16-x^2$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \quad \frac{du}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{2x}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16-x^2}}$$

A1

b. $f(x) = \log_e(\tan(2x))$

$f(x) = y = \log_e(u)$ where $u = \tan(2x)$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = \frac{2}{\cos^2(2x)}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$f'(x) = \frac{1}{u} \times \frac{2}{\cos^2(2x)}$$

$$f'(x) = \frac{1}{\tan(2x)} \times \frac{2}{\cos^2(2x)}$$

M1

$$f'(x) = \frac{1}{\frac{\sin(2x)}{\cos(2x)}} \times \frac{2}{\cos^2(2x)}$$

$$f'(x) = \frac{2}{\sin(2x)\cos(2x)}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{2}{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)} = \frac{2}{\frac{\sqrt{3}}{2} \times \frac{1}{2}} = \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{3}$$

A1

Question 2

(1) $(7-k)x - 6y = 10$

(2) $4x + 3ky = k + 6$

(1) $\Rightarrow 6y = (7-k)x - 10$

(2) $3ky = -4x + (k+6)$

$$y = \frac{(7-k)x}{6} - \frac{5}{3}$$

$$y = -\frac{4x}{3k} + \frac{k+6}{3k}$$

equating gradients, when the lines are parallel

$$\frac{7-k}{6} = -\frac{4}{3k} \Rightarrow 3k(7-k) = -24$$

$$21k - 3k^2 = -24$$

$$3k^2 - 21k - 24 = 0$$

$$3(k^2 - 7k - 8) = 0$$

M1

$$3(k-8)(k+1) = 0$$

There is a unique solution when $k \in \mathbb{R} \setminus \{-1, 8\}$

A1

When $k = 8$ the equations become $-x - 6y = 10$
 $4x + 24y = 14$ these lines are parallel

with different y -intercepts, therefore there is no solution when $k = 8$

When $k = -1$ the equations become $8x - 6y = 10$
 $4x - 3y = 5$ these lines are both the same line,

therefore we have an infinite number of solutions when $k = -1$

A1

Alternatively

(1) $(7-k)x - 6y = 10$

(2) $4x + 3ky = k + 6$

$$\Delta = \begin{vmatrix} 7-k & -6 \\ 4 & 3k \end{vmatrix} = 3k(7-k) + 24 = 21k - 3k^2 + 24$$

M1

$$\Delta = -3(k^2 - 7k - 8) = -3(k-8)(k+1) \dots \text{ as before}$$

Question 3

a. $y = \frac{8}{(x-2)^2} - 2$ crosses the x -axis when $y = 0$

$\Rightarrow (x+2)^2 = 4 \Rightarrow x+2 = \pm 2 \Rightarrow x = 0, 4$

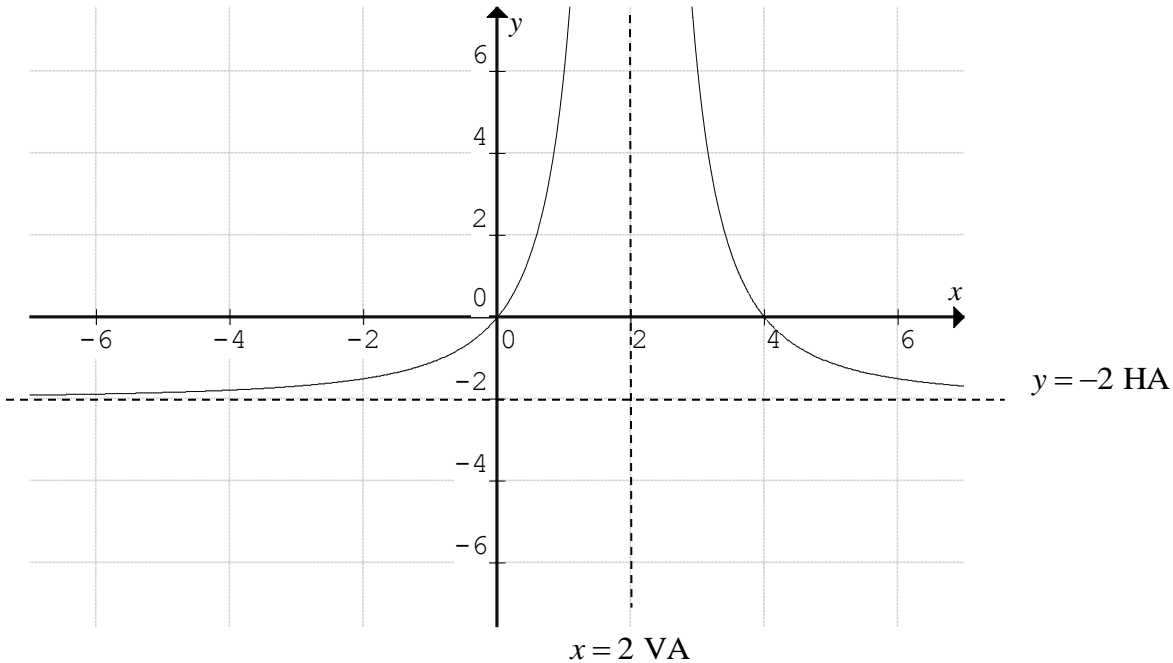
$x = 2$ is a vertical asymptote, $y = -2$ is a horizontal asymptote

A1

correct graph, shape, asymptotes axial intercepts,

$(0,0), (4,0)$ domain $x \in R \setminus \{2\}$ range $(-2, \infty)$

G1



b. the required area is $A_1 - A_2$ where

$$A_1 = \int_3^4 \left(\frac{8}{(x-2)^2} - 2 \right) dx$$

$$A_2 = \int_4^5 \left(\frac{8}{(x-2)^2} - 2 \right) dx$$

$$A_1 = \left[\frac{-8}{x-2} - 2x \right]_3^4$$

$$A_2 = \left[\frac{-8}{x-2} - 2x \right]_4^5$$

$$A_1 = (-4 - 8) - (-8 - 6)$$

$$A_2 = \left(-\frac{8}{3} - 10 \right) - (-4 - 8) = -\frac{2}{3}$$

M1

$$A_1 = 2$$

The area is $2\frac{2}{3}$ units²

A1

Question 4 $f(x) = \log_4(x)$ $g(x) = \sqrt{4x^2 - 1}$

a. $f(g(x)) = f(\sqrt{4x^2 - 1}) = \log_4(\sqrt{4x^2 - 1})$ A1

for maximal domain, we require

$$4x^2 - 1 > 0 \Rightarrow 4x^2 > 1 \Rightarrow x^2 > \frac{1}{4}$$

$$x > \frac{1}{2} \text{ or } x < -\frac{1}{2} \quad \text{domain } \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$
 A1

b. $g(f(x)) = g(\log_4(x)) = \sqrt{4(\log_4(x))^2 - 1}$ A1

for maximal domain, we require

$$4(\log_4(x))^2 - 1 \geq 0 \Rightarrow (\log_4(x))^2 \geq \frac{1}{4}$$

$$\log_4(x) \geq \frac{1}{2} \quad \text{or} \quad \log_4(x) \leq -\frac{1}{2} \quad \text{but } x > 0$$

$$x \geq 4^{\frac{1}{2}} = 2 \quad \text{or} \quad 0 < x \leq 4^{-\frac{1}{2}} = \frac{1}{2} \quad \text{domain } \left(0, \frac{1}{2}\right] \cup [2, \infty)$$
 A1

Question 5

a. $2\cos^2(x) - \cos(x) - 1 = 0$
 $(2\cos(x) + 1)(\cos(x) - 1) = 0$
 $\cos(x) = -\frac{1}{2} \qquad \qquad \qquad \cos(x) = 1$ M1

$$x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right) \qquad \qquad \qquad x = 2n\pi$$

$$x = 2n\pi \pm \frac{2\pi}{3} = \frac{2\pi}{3}(3n \pm 1), 2n\pi \quad \text{where } n \in \mathbb{Z}$$
 A1

b. $2\cos^2(x) - \cos(x) - 1 = 0, \quad 0 \leq x \leq 2\pi$
 $\cos(x) = -\frac{1}{2} \qquad \qquad \qquad \cos(x) = 1$ M1

$$x = \frac{2\pi}{3}, \pi + \frac{\pi}{3} \qquad \qquad \qquad x = 0, 2\pi$$

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi \quad (\text{or } n = 0, 1 \text{ from a.})$$
 A1

Question 6

a. $f : [0,4] \rightarrow R$, $f(x) = 3 \sin\left(\frac{\pi x}{2}\right)$ has amplitude 3 and period $\frac{2\pi}{\frac{\pi}{2}} = 4$

the graph has one cycle over its domain

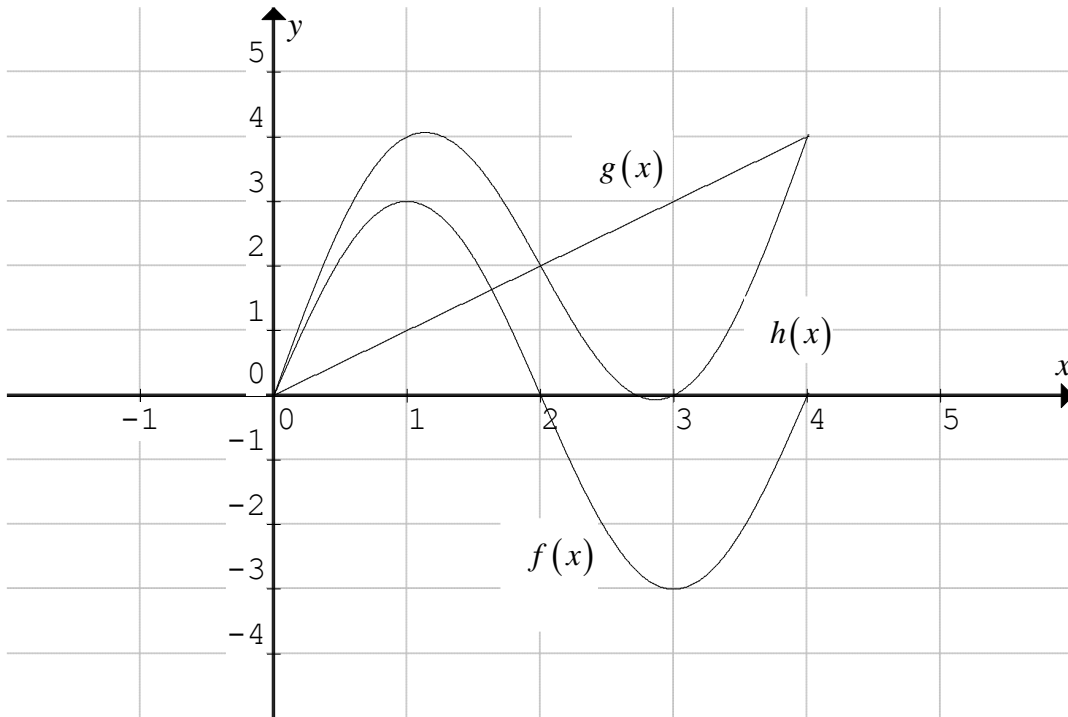
$g : [0,4] \rightarrow R$, $g(x) = x$, $h : [0,4] \rightarrow R$, $h(x) = 3 \sin\left(\frac{\pi x}{2}\right) + x = f(x) + g(x)$

by addition of ordinates

$h(x)$ must pass through $(0,0)$, $(1,4)$, $(2,2)$, $(3,0)$, $(4,4)$

correct graphs and endpoints

A1
G2



b. $h' : (0,4) \rightarrow R$, $h'(x) = \frac{3\pi}{2} \cos\left(\frac{\pi x}{2}\right) + 1$

A1

must have correct domain, open interval, as the gradient function is not defined at the end-points.

Question 7

Since the probabilities sum to one. $\sum \Pr(X = x) = 1$

$$\frac{1}{2e^k} + \frac{e^k}{4} = 1 \quad \text{let } u = e^k$$

M1

$$\frac{1}{2u} + \frac{u}{4} = 1 \quad \times \text{ by } 4u$$

$$2 + u^2 = 4u$$

$$u^2 - 4u = -2$$

$$u^2 - 4u + 4 = -2 + 4$$

completing the square

M1

$$(u - 2)^2 = 2$$

$$u - 2 = \pm\sqrt{2}$$

$$u = e^k = 2 \pm \sqrt{2}$$

$$k = \log_e(2 \pm \sqrt{2}) \text{ both answers acceptable}$$

A1

Question 8

$$\left(p - 2\sqrt{\frac{p(1-p)}{n}}, p + 2\sqrt{\frac{p(1-p)}{n}} \right) = (0.7, 0.9)$$

$$(1) \quad p - 2\sqrt{\frac{p(1-p)}{n}} = 0.7$$

M1

$$(2) \quad p + 2\sqrt{\frac{p(1-p)}{n}} = 0.9$$

$$(1) + (2) \Rightarrow 2p = 1.6$$

$$p = 0.8$$

A1

$$(2) - (1) \Rightarrow 4\sqrt{\frac{p(1-p)}{n}} = 0.2 \Rightarrow \sqrt{\frac{0.8 \times 0.2}{n}} = 0.05$$

$$\sqrt{\frac{0.16}{n}} = 0.05 \Rightarrow \frac{0.4}{\sqrt{n}} = 0.05$$

$$\sqrt{n} = \frac{0.4}{0.05} = 8$$

$$n = 64$$

A1

Question 9

a. using the product rule

$$\begin{aligned}\frac{d}{dx}(xe^{-2x}) &= x \frac{d}{dx}(e^{-2x}) + e^{-2x} \frac{d}{dx}(x) \\ &= -2xe^{-2x} + e^{-2x}\end{aligned}\quad \text{A1}$$

b. $f(x) = xe^{-2x}$

$$f'(x) = e^{-2x}(1-2x) \text{ for turning points } f'(x) = 0$$

$$\Rightarrow x = \frac{1}{2} \quad f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1}$$

$$\left(\frac{1}{2}, \frac{1}{2e}\right) \quad \text{A1}$$

c. The area is $A = \int_0^1 xe^{-2x} dx$

$$\text{Since } \frac{d}{dx}(xe^{-2x}) = -2xe^{-2x} + e^{-2x}$$

$$xe^{-2x} = -2 \int xe^{-2x} dx + \int e^{-2x} dx \quad \text{M1}$$

$$\begin{aligned}2 \int xe^{-2x} dx &= \int e^{-2x} dx - xe^{-2x} \\ &= -\frac{1}{2}e^{-2x} - xe^{-2x} = -\frac{e^{-2x}}{2}(2x+1)\end{aligned}\quad \text{A1}$$

$$\int xe^{-2x} dx = -\frac{e^{-2x}}{4}(2x+1)$$

$$A = \int_0^1 xe^{-2x} dx$$

$$A = \left[-\frac{e^{-2x}}{4}(2x+1) \right]_0^1$$

$$A = -\frac{3e^{-2}}{4} + \frac{1}{4} = \frac{1}{4}(1-3e^{-2}) \text{ units}^2 \quad \text{A1}$$

Question 10

a. $f(x) = 9 - x^2$, $f'(x) = -2x$

At $P(p, f(p))$, $f(p) = 9 - p^2$, $f'(p) = -2p$

Tangent $y - (9 - p^2) = -2p(x - p) = -2px + 2p^2$

$y = -2px + 9 + p^2$

A1

b. crosses the x -axis when $y = 0 \Rightarrow -2px + 9 + p^2 = 0 \Rightarrow 2px = 9 + p^2$

$x_S = \frac{9 + p^2}{2p}$ $S\left(\frac{9 + p^2}{2p}, 0\right)$

A1

crosses the y -axis when $x = 0 \Rightarrow y = 9 + p^2$

$y_R = 9 + p^2$ $R(0, 9 + p^2)$

A1

c. $A(p) = \frac{1}{2} \times (9 + p^2) \times \frac{9 + p^2}{2p} = \frac{(9 + p^2)^2}{4p}$ using the quotient rule

$u = (9 + p^2)^2$ $v = 4p$

$\frac{du}{dp} = 4p(9 + p^2)$ $\frac{dv}{dp} = 4$

M1

$\frac{dA}{dp} = \frac{4p(9 + p^2) \times 4p - 4(9 + p^2)^2}{(4p)^2} = \frac{4(9 + p^2)[4p^2 - (9 + p^2)]}{16p^2}$

$= \frac{3(p^2 + 9)(p^2 - 3)}{4p^2}$

A1

For a maximum or a minimum

we require $\frac{dA}{dp} = 0 \Rightarrow p^2 = 3$ but $1 \leq p \leq 3$ so $p = \sqrt{3}$

$A(\sqrt{3}) = \frac{(9 + 3)^2}{4\sqrt{3}} = \frac{12^2}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 12\sqrt{3}$

The minimum value of the area occurs when $p = \sqrt{3}$ and is $12\sqrt{3}$

A1

d. Investigate the end-points of the function

Since $1 \leq p \leq 3$ $A(1) = \frac{10^2}{4} = 25$ and $A(3) = \frac{18^2}{4 \times 3} = 27$

The maximum value of the area occurs when $p = 3$ and is 27

A1

END OF SUGGESTED SOLUTIONS