Year 2016

VCE

Mathematical Methods

Trial Examination 1

Solutions

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- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

Question 1

a. $y = \sqrt{16 - x^2} = u^{\frac{1}{2}}$ where $u = 16 - x^2$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$ $\quad\quad \frac{du}{dx} = -2x$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -\frac{2x}{2\sqrt{u}}$

$\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}$ A1

b. $f(x) = \log_e(\tan(2x))$

$f(x) = y = \log_e(u)$ where $u = \tan(2x)$

$\frac{dy}{du} = \frac{1}{u}$ $\quad\quad \frac{du}{dx} = \frac{2}{\cos^2(2x)}$

$f'(x) = \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$f'(x) = \frac{1}{u} \times \frac{2}{\cos^2(2x)}$

$f'(x) = \frac{1}{\tan(2x)} \times \frac{2}{\cos^2(2x)}$ M1

$f'(x) = \frac{1}{\frac{\sin(2x)}{\cos(2x)}} \times \frac{2}{\cos^2(2x)}$

$f'(x) = \frac{2}{\sin(2x)\cos(2x)}$

$f'\left(\frac{\pi}{6}\right) = \frac{2}{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)} = \frac{2}{\frac{\sqrt{3}}{2} \times \frac{1}{2}} = \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$f'\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{3}$ A1

Question 2

$(1) \quad (7-k)x - 6y = 10$

$(2) \quad 4x + 3ky = k + 6$

$(1) \Rightarrow \quad 6y = (7-k)x - 10$ $\qquad$ $(2) \; 3ky = -4x + (k+6)$

$y = \dfrac{(7-k)x}{6} - \dfrac{5}{3}$ $\qquad$ $y = -\dfrac{4x}{3k} + \dfrac{k+6}{3k}$

equating gradients, when the lines are parallel

$\dfrac{7-k}{6} = -\dfrac{4}{3k} \;\Rightarrow\; 3k(7-k) = -24$

$21k - 3k^2 = -24$

$3k^2 - 21k - 24 = 0$

$3(k^2 - 7k - 8) = 0$ M1

$3(k-8)(k+1) = 0$

There is a unique solution when $k \in R \setminus \{-1, 8\}$ A1

When $k = 8$ the equations become $\begin{aligned} -x - 6y &= 10 \\ 4x + 24y &= 14 \end{aligned}$ these lines are parallel with different y-intercepts, therefore there is no solution when $k = 8$

When $k = -1$ the equations become $\begin{aligned} 8x - 6y &= 10 \\ 4x - 3y &= 5 \end{aligned}$ these lines are both the same line, therefore we have an infinite number of solutions when $k = -1$ A1

Alternatively

$(1) \quad (7-k)x - 6y = 10$

$(2) \quad 4x + 3ky = k + 6$

$\Delta = \begin{vmatrix} 7-k & -6 \\ 4 & 3k \end{vmatrix} = 3k(7-k) + 24 = 21k - 3k^2 + 24$ M1

$\Delta = -3(k^2 - 7k - 8) = -3(k-8)(k+1)\ldots$ as before

Question 3

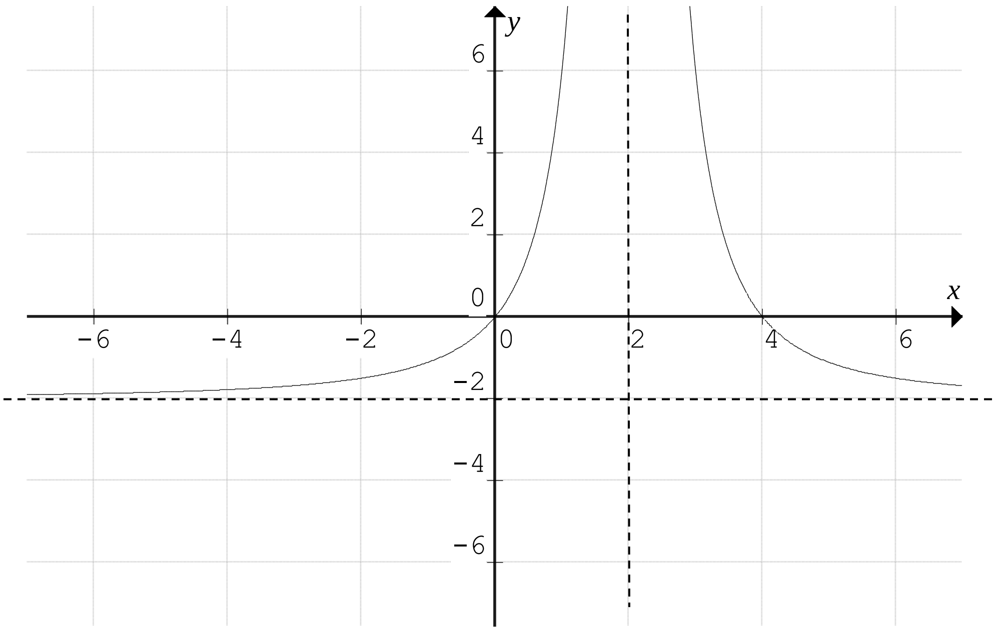
a. $y=\dfrac{8}{(x-2)^2}-2$ crosses the x-axis when $y=0$

$\Rightarrow\ (x+2)^2=4\ \Rightarrow x+2=\pm 2\ \Rightarrow x=0,\ 4$

$x=2$ is a vertical asymptote, $y=-2$ is a horizontal asymptote A1

correct graph, shape, asymptotes axial intercepts,

$(0,0)$, $(4,0)$ domain $x\in R\setminus\{2\}$ range $(-2,\infty)$ G1



$y=-2$ HA

$x=2$ VA

b. the required area is A_1-A_2 where

$A_1=\displaystyle\int_3^4\left(\frac{8}{(x-2)^2}-2\right)dx$ $\qquad A_2=\displaystyle\int_4^5\left(\frac{8}{(x-2)^2}-2\right)dx$

$A_1=\left[\dfrac{-8}{x-2}-2x\right]_3^4$ $\qquad A_2=\left[\dfrac{-8}{x-2}-2x\right]_4^5$

$A_1=(-4-8)-(-8-6)$ $\qquad A_2=\left(-\dfrac{8}{3}-10\right)-(-4-8)=-\dfrac{2}{3}$ M1

$A_1=2$

The area is $2\dfrac{2}{3}$ units^2 A1

Question 4 $f(x)=\log_4(x) \quad g(x)=\sqrt{4x^2-1}$

a. $f(g(x))=f\left(\sqrt{4x^2-1}\right)=\log_4\left(\sqrt{4x^2-1}\right)$ A1

for maximal domain, we require

$4x^2-1>0 \;\Rightarrow\; 4x^2>1 \;\Rightarrow\; x^2>\frac{1}{4}$

$x>\frac{1}{2}$ or $x<-\frac{1}{2}$ domain $\left(-\infty\,,-\frac{1}{2}\right)\cup\left(\frac{1}{2}\,,\infty\right)$ A1

b. $g(f(x))=g(\log_4(x))=\sqrt{4(\log_4(x))^2-1}$ A1

for maximal domain, we require

$4(\log_4(x))^2-1\ge 0 \;\Rightarrow\; (\log_4(x))^2\ge\frac{1}{4}$

$\log_4(x)\ge\frac{1}{2}$ or $\log_4(x)\le-\frac{1}{2}$ but $x>0$

$x\ge 4^{\frac{1}{2}}=2$ or $0<x\le 4^{-\frac{1}{2}}=\frac{1}{2}$ domain $\left(0\,,\frac{1}{2}\right]\cup[2\,,\infty)$ A1

Question 5

a. $2\cos^2(x)-\cos(x)-1=0$

$(2\cos(x)+1)(\cos(x)-1)=0$

$\cos(x)=-\frac{1}{2}$ $\qquad \cos(x)=1$ M1

$x=2n\pi\pm\cos^{-1}\left(-\frac{1}{2}\right)$ $\qquad x=2n\pi$

$x=2n\pi\pm\frac{2\pi}{3}=\frac{2\pi}{3}(3n\pm1)\,,\;2n\pi$ where $n\in Z$ A1

b. $2\cos^2(x)-\cos(x)-1=0\,,\;0\le x\le 2\pi$

$\cos(x)=-\frac{1}{2}$ $\qquad \cos(x)=1$

$x=\frac{2\pi}{3}\,,\;\pi+\frac{\pi}{3}$ $\qquad x=0\,,\;2\pi$ M1

$x=0\,,\;\frac{2\pi}{3}\,,\;\frac{4\pi}{3}\,,\;2\pi$ (or $n=0,1$ from a.) A1

Question 6

a. $f:[0,4]\to R$, $f(x)=3\sin\left(\frac{\pi x}{2}\right)$ has amplitude 3 and period $\frac{2\pi}{\frac{\pi}{2}}=4$

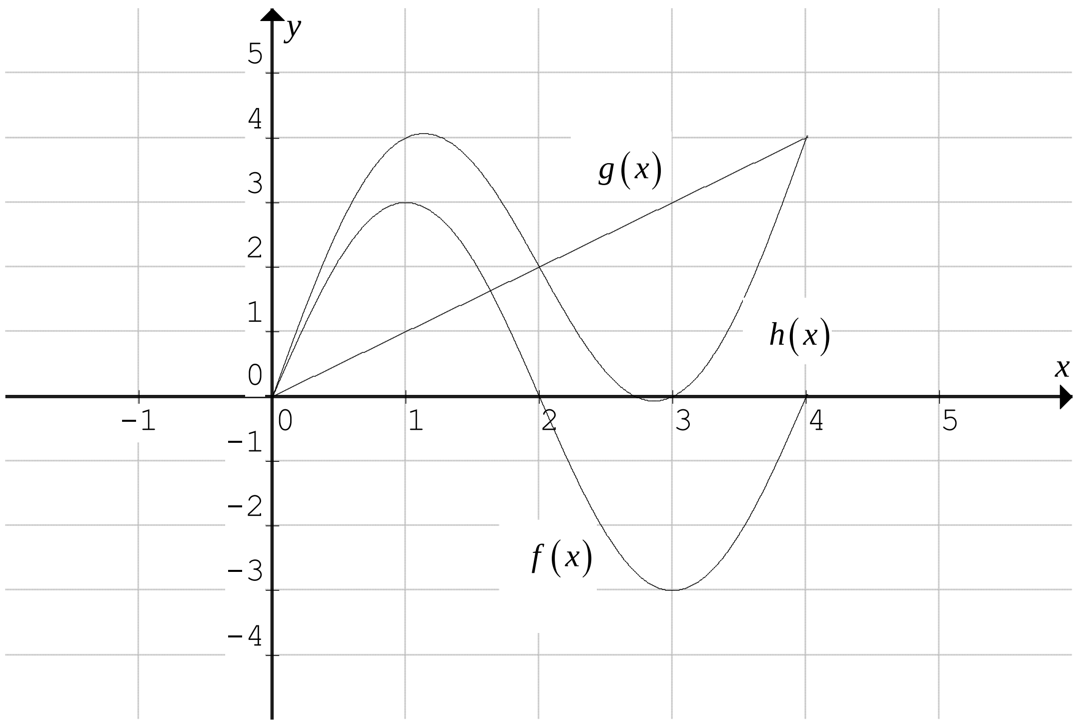
the graph has one cycle over its domain

$g:[0,4]\to R$, $g(x)=x$, $h:[0,4]\to R$, $h(x)=3\sin\left(\frac{\pi x}{2}\right)+x=f(x)+g(x)$

by addition of ordinates

$h(x)$ must pass through $(0,0)$, $(1,4)$, $(2,2)$, $(3,0)$, $(4,4)$ A1

correct graphs and endpoints G2



b. $h':(0,4)\to R$, $h'(x)=\frac{3\pi}{2}\cos\left(\frac{\pi x}{2}\right)+1$ A1

must have correct domain, open interval, as the gradient function is not defined at the end-points.

Question 7

Since the probabilities sum to one. $\sum \Pr(X = x) = 1$

$\frac{1}{2e^k} + \frac{e^k}{4} = 1$ let $u = e^k$ M1

$\frac{1}{2u} + \frac{u}{4} = 1 \quad \times$ by $4u$

$2 + u^2 = 4u$

$u^2 - 4u = -2$

$u^2 - 4u + 4 = -2 + 4$ completing the square M1

$(u-2)^2 = 2$

$u - 2 = \pm\sqrt{2}$

$u = e^k = 2 \pm \sqrt{2}$

$k = \log_e\left(2 \pm \sqrt{2}\right)$ both answers acceptable A1

Question 8

$$\left(p - 2\sqrt{\frac{p(1-p)}{n}} \,,\; p + 2\sqrt{\frac{p(1-p)}{n}} \right) = (0.7, 0.9)$$

$(1) \quad p - 2\sqrt{\frac{p(1-p)}{n}} = 0.7$ M1

$(2) \quad p + 2\sqrt{\frac{p(1-p)}{n}} = 0.9$

$(1) + (2) \Rightarrow 2p = 1.6$

$p = 0.8$ A1

$(2) - (1) \Rightarrow 4\sqrt{\frac{p(1-p)}{n}} = 0.2 \quad \Rightarrow \quad \sqrt{\frac{0.8 \times 0.2}{n}} = 0.05$

$\sqrt{\frac{0.16}{n}} = 0.05 \quad \Rightarrow \quad \frac{0.4}{\sqrt{n}} = 0.05$

$\sqrt{n} = \frac{0.4}{0.05} = 8$

$n = 64$ A1

Question 9

a. using the product rule

$$\frac{d}{dx}\left(xe^{-2x}\right)=x\frac{d}{dx}\left(e^{-2x}\right)+e^{-2x}\frac{d}{dx}(x)$$

$$=-2xe^{-2x}+e^{-2x}$$ A1

b. $f(x)=xe^{-2x}$

$f'(x)=e^{-2x}(1-2x)$ for turning points $f'(x)=0$

$\Rightarrow\ x=\frac{1}{2}\qquad f\left(\frac{1}{2}\right)=\frac{1}{2}e^{-1}$

$\left(\frac{1}{2},\ \frac{1}{2e}\right)$ A1

c. The area is $A=\int_0^1 xe^{-2x}\,dx$

Since $\frac{d}{dx}\left(xe^{-2x}\right)=-2xe^{-2x}+e^{-2x}$

$xe^{-2x}=-2\int xe^{-2x}\,dx+\int e^{-2x}\,dx$ M1

$2\int xe^{-2x}\,dx=\int e^{-2x}\,dx-xe^{-2x}$

$$=-\frac{1}{2}e^{-2x}-xe^{-2x}=-\frac{e^{-2x}}{2}(2x+1)$$ A1

$\int xe^{-2x}\,dx=-\frac{e^{-2x}}{4}(2x+1)$

$A=\int_0^1 xe^{-2x}\,dx$

$A=\left[-\frac{e^{-2x}}{4}(2x+1)\right]_0^1$

$A=-\frac{3e^{-2}}{4}+\frac{1}{4}=\frac{1}{4}\left(1-3e^{-2}\right)$ units^2 A1

Question 10

a. $f(x)=9-x^2$, $f'(x)=-2x$

At $P(p, f(p))$, $f(p)=9-p^2$, $f'(p)=-2p$

Tangent $y-(9-p^2)=-2p(x-p)=-2px+2p^2$

$y=-2px+9+p^2$ A1

b. crosses the x-axis when $y=0 \Rightarrow -2px+9+p^2=0 \Rightarrow 2px=9+p^2$

$x_S=\dfrac{9+p^2}{2p}$ $S\left(\dfrac{9+p^2}{2p},0\right)$ A1

crosses the y-axis when $x=0 \Rightarrow y=9+p^2$

$y_R=9+p^2$ $R(0,9+p^2)$ A1

c. $A(p)=\dfrac{1}{2}\times(9+p^2)\times\dfrac{9+p^2}{2p}=\dfrac{(9+p^2)^2}{4p}$ using the quotient rule

$u=(9+p^2)^2$ $v=4p$

$\dfrac{du}{dp}=4p(9+p^2)$ $\dfrac{dv}{dp}=4$ M1

$$\frac{dA}{dp}=\frac{4p(9+p^2)\times4p-4(9+p^2)^2}{(4p)^2}=\frac{4(9+p^2)\left[4p^2-(9+p^2)\right]}{16p^2}$$

$$=\frac{3(p^2+9)(p^2-3)}{4p^2}$$ A1

For a maximum or a minimum

we require $\dfrac{dA}{dp}=0 \Rightarrow p^2=3$ but $1\le p\le3$ so $p=\sqrt{3}$

$$A(\sqrt{3})=\frac{(9+3)^2}{4\sqrt{3}}=\frac{12^2}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=12\sqrt{3}$$

The minimum value of the area occurs when $p=\sqrt{3}$ and is $12\sqrt{3}$ A1

d. Investigate the end-points of the function

Since $1\le p\le3$ $A(1)=\dfrac{10^2}{4}=25$ and $A(3)=\dfrac{18^2}{4\times3}=27$

The maximum value of the area occurs when $p=3$ and is 27 A1

END OF SUGGESTED SOLUTIONS