The Mathematical Association of Victoria

Trial Examination 2016

MATHEMATICAL METHODS

Written Examination 1 - SOLUTIONS

Question 1

a. $f(x) = \log_e\left(\cos(4x)\right)$

$f'(x) = \dfrac{-4\sin(4x)}{\cos(4x)}$ **1M**

$= -4\tan(4x)$ **1A**

b.i. $x^3 - 3x^2 + 3x - 1$

$= (x-1)^3$ **1A**

b.ii. $\int\left(\dfrac{1}{(1-x)(x^3-3x^2+3x-1)}\right)dx$

$= -\int\left(\dfrac{1}{(x-1)^4}\right)dx$

$= -\int\left((x-1)^{-4}\right)dx$

$= \dfrac{1}{3(x-1)^3} + c$ **1A**

Question 2

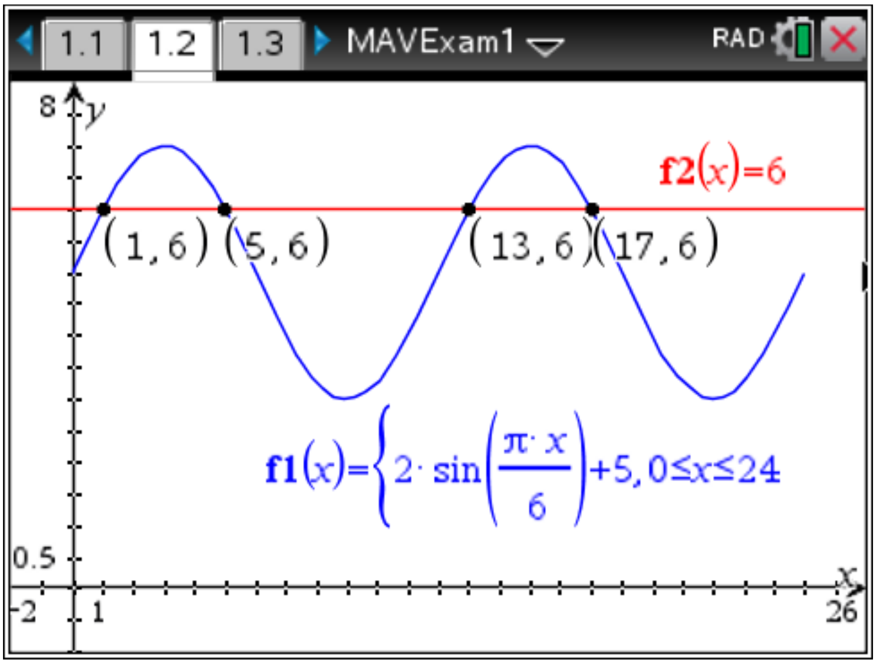
Let $2\sin\left(\dfrac{\pi t}{6}\right) + 5 = 6$

$\sin\left(\dfrac{\pi t}{6}\right) = \dfrac{1}{2}$

$\dfrac{\pi t}{6} = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \ldots$ **1A**

$t = 1, 5, 13, 17$ **1A**

$\left\{t : 1 < t < 5\right\} \cup \left\{t : 13 < t < 17\right\}$ **1A**



Question 3

a. $g(x)=2x^5-10x^4+20x^3-20x^2+10x+2$

$g(x)=A(x+B)^5+C=Ax^5+5Ax^4B+.....+AB^5+C$

Equating coefficients

$A=2$ **1A**

$5AB=-10$, $AB^5+C=2$

$10B=-10$ $B=-1$ **1A**

$-2+C=2$, $C=4$ **1A**

OR

$g(x)=2\left(x^5-5x^4+10x^3-10x^2+5x+1\right)$

$=2\left((x-1)^5+2\right)$

$=2(x-1)^5+4$

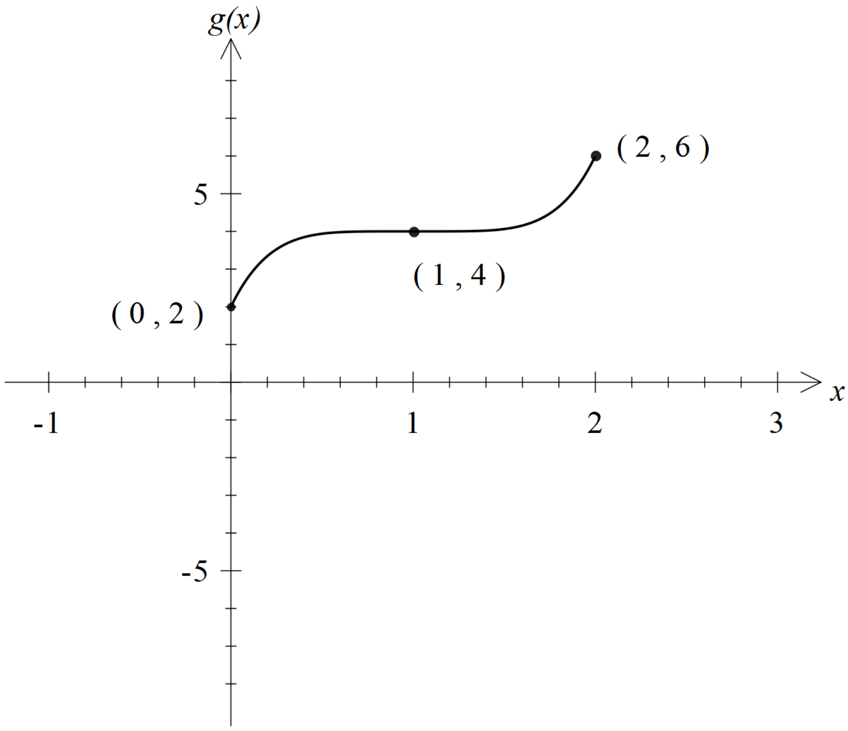
$A=2$ **1A**

$B=-1$ **1A**

$C=4$ **1A**

b. shape and stationary point of inflection **1A**

coordinates of endpoints **1A**



Question 4

a. $\dfrac{9}{e^{2x}} - \dfrac{6}{e^{x}} + 2e^{x} = 5$

LHS = $\dfrac{9}{e^{2\log_e(3)}} - \dfrac{6}{e^{\log_e(3)}} + 2e^{\log_e(3)}$

$= \dfrac{9}{9} - \dfrac{6}{3} + 6$

= 5 = RHS **1M Show that**

b. $\dfrac{9}{e^{2x}} - \dfrac{6}{e^{x}} + 2e^{x} = 5$

By inspection, let $x = 0$

LHS = $\dfrac{9}{e^{0}} - \dfrac{6}{e^{0}} + 2e^{0}$

$= 9 - 6 + 2$

= 5 as required **1M**

$x = 0$ or $x = \log_e(3)$ **1A**

OR

$9 - 6e^{x} + 2e^{3x} = 5e^{2x}$

$2e^{3x} - 5e^{2x} - 6e^{x} + 9 = 0$ **1M**

Let $a = e^{x}$

Let $f(x) = 2a^3 - 5a^2 - 6a + 9 = 0$

$f(1) = 0$, $a - 1$ is a factor

First term in the quadratic factor has to be $2a^2$.

Last term has to be -9.

$-2a^2 + ? = -5a^2$, $? = -3a^2$

The middle term is $-3a$

$(a-1)(2a^2 - 3a - 9) = 0$

$(a-1)(2a+3)(a-3) = 0$

$a = 1$ or $a = -\frac{3}{2}$ or $a = 3$

$e^x = 1,\ e^x \neq -\frac{3}{2},\ e^x = 3$

$x = 0$ or $x = \log_e(3)$ **1A**

Question 5

a. $f(x) = xe^{2x}$

$f'(x) = 2xe^{2x} + e^{2x}$ **1A**

b. $\int\left(2xe^{2x} + e^{2x}\right)dx = xe^{2x} + c$ **1M**

$\int\left(2xe^{2x}\right)dx = xe^{2x} - \int\left(e^{2x}\right)dx + c$

$\int\left(xe^{2x}\right)dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1$

$\text{Average Value} = \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(xe^{2x}\right)dx$ **1M**

$= 2\int_0^{\frac{1}{2}}\left(xe^{2x}\right)dx$

$= \left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2}e - \frac{1}{2}e\right) - \left(0 - \frac{1}{2}\right)$

$= \frac{1}{2}$ **1A**

Question 6

$10 = A\log_e(2-B)$.....(1)

$20 = A\log_e(8-B)$.....(2) **1M**

Divide (2) by (1)

$$2 = \frac{\log_e(8-B)}{\log_e(2-B)}$$

$$2\log_e(2-B) = \log_e(8-B)$$

$$\log_e\left((2-B)^2\right) = \log_e(8-B)$$

$$(2-B)^2 = 8-B$$

$$4-4B+B^2 = 8-B$$

$$B^2-3B-4=0$$

$$(B-4)(B+1)=0$$

$B=-1,\ B \neq 4$ as $B<2$ **1A**

Substitute $B=-1$ into (1)

$A = \dfrac{10}{\log_e(3)}$ **1A**

Question 7

a.

$$-2(m-1)x + my = -m+4$$

$$mx - 3y = 2m+1$$

Using ratios for an infinite number of solutions or no solution gives

$\dfrac{-2(m-1)}{m} = \dfrac{m}{-3}$ **1M**

OR

The gradients will be the same for an infinite number of solutions or no solution
Rearranging the equations gives

$$y = \frac{2(m-1)}{m}x - 1 + \frac{4}{m}$$

$$y = \frac{mx}{3} - \frac{2m}{3} - \frac{1}{3}$$

$\dfrac{2(m-1)}{m} = \dfrac{m}{3}$ **1M**

Solve for m

$$6(m-1) = m^2$$

$$m^2-6m+6=0$$

Using the quadratic formula gives

$m = \frac{6 \pm \sqrt{36 - 24}}{2}$

$m = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2}$

$m = 3 \pm \sqrt{3}$

So for unique solutions we need $m \in R \setminus \left\{3 \pm \sqrt{3}\right\}$ **1A**

b. If $m = -1$, $y = 4x - 5$ and $y = -x^2 + 2x - 6$

Point of intersection $-x^2 + 2x - 6 = 4x - 5$ **1M**

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$ only one unique solution, hence the line is a tangent to the parabola. **1A**

(At $x = -1$, the gradient of the line and the parabola are both 4.)

Question 8

a. length = $14-2x$

width = $11-2x$ **1A**

Using Pythagoras

The diagonal

$= \sqrt{(11-2x)^2 + (14-2x)^2}$

$= \sqrt{121-44x+4x^2+4x^2-56x+196}$ **1M Show that**

$= \sqrt{8x^2-100x+317}$

b. $\sqrt{8x^2-100x+317} = 15$

$8x^2-100x+317 = 225$ **1A**

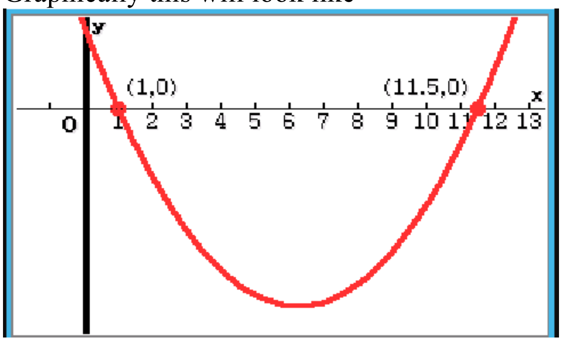
$8x^2-100x+92 = 0$

$4(2x^2-25x+23) = 0$

$4(2x-23)(x-1) = 0$

Giving $x = \frac{23}{2}, x = 1$ **1A**

Graphically this will look like



So the inequation $4(2x-23)(x-1) \ge 0$

Has solutions $x \le 1, x \ge \frac{23}{2}$

But implicit domain is $x \in \left(0, \frac{11}{2}\right)$

So solutions are $x \in (0,1]$ **1A**

c. Area $= (11-2x)(14-2x),\ x \in \left(0, \frac{11}{2}\right)$

So area < 11×14

Area < 154 sq m.,

It is not possible to have an area of 155 square metres and above. **1M**

Question 9

a. $\hat{P} = 0.8,\ n = 4$

$\text{sd}\left(\hat{P}\right) = \sqrt{\dfrac{pq}{n}} = \sqrt{\dfrac{0.8 \times 0.2}{4}} = 0.2$ **1A**

$\Pr\left(0.6 \le \hat{P} \le 1\right) = \Pr\left(2.4 \le X \le 4\right)$

$= \Pr\left(3 \le X \le 4\right)$ as X is an integer

$= \binom{4}{3}(0.8)^3(0.2)^1 + \binom{4}{4}(0.8)^4(0.2)^0$ **1A**

$= \binom{4}{1}(0.8)^3(0.2) + 0.8^4$

$= 2 \times (0.8)^4$

$= 0.8192$ or $\dfrac{512}{625}$ **1A**

b. $Y : Bi(4, 0.2)$

$\Pr\left(Y \le 1)\right) = \Pr(Y = 0) + \Pr(Y = 1)$

$= (0.8)^4 + \binom{4}{1}(0.2)(0.8)^3$

$= 0.8192$ or $\dfrac{512}{625}$ **1A**

Pr(first 2 shots were not goals) = $2(0.8)^3(0.2) + (0.8)^4,\ G'G'G'G + G'G'GG' + G'G'G'G'$

Pr(first two shots were not goals | there was no more than 1 goal)

$= \dfrac{2(0.8)^3(0.2) + (0.8)^4}{(0.8)^4 + 4(0.8)^3(0.2)}$

$= \dfrac{(0.8)^3(1.2)}{(0.8)^3(1.6)}$ or $\dfrac{0.6144}{0.8192}$ or $\dfrac{\frac{384}{625}}{\frac{512}{625}}$

$= \dfrac{3}{4}$ **1A**

c. $\Pr(W \ge 1) > 0.9$

$1 - \Pr(W = 0) > 0.9$ **1M**

$1 - 0.8^n > 0.9$

$0.8^n < 0.1$ **1A**

$n > \dfrac{\log_{10}(0.1)}{\log_{10}(0.8)}$

$n > \dfrac{-1}{\log_{10}(8) - \log_{10}(10)}$

$n > \dfrac{-1}{-0.097}$ (≈ 10.3 or is slightly greater than 10)

$n = 11$ **1A**