The Mathematical Association of Victoria

Trial Examination 2016

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology(calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer – based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 22 pages
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale

SECTION A

Question 1

The range and period of the graph of $f : R \to R, f(x) = -2 - 3\cos\left(\frac{x}{4} + 1\right)$ are respectively

A. $[-5,1]$ and 8π

B. $[-3,3]$ and 8π

C. $[-1,5]$ and $\frac{\pi}{2}$

D. $[-5,1]$ and $\frac{\pi}{2}$

E. R and 8π

Question 2

The line with equation $y = k$, where k is a real constant, will intersect the curve

$g : [-3\pi, \pi] \to R, g(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 1$ nine times if

A. $k > 4$

B. $k < -2$

C. $k = 1$

D. $-2 < k < 1$

E. $1 < k < 4$

Question 3

Consider the functions *f* with rule $f(x) = \sin(2x) + 1$ and *g* with rule $g(x) = \tan(\pi x)$ over their maximal domains. The domain of $f(g(x))$ is

A. R

B. $[0, 2]$

C. $R \setminus \left\{\frac{1}{2} + k\right\}, k \in R$

D. $R \setminus \left\{\frac{\pi}{2} + \pi k\right\}, k \in R$

E. $R \setminus \left\{\frac{1}{2} + k\right\}, k \in Z$

SECTION A - continued
TURN OVER

Question 4

If $\sin(x) = -\dfrac{2}{15}$ and $\pi \le x \le 2\pi$ then $\sin\left(\dfrac{3\pi}{2} - x\right)$ could equal

A. $-\dfrac{2}{11}$

B. $-\dfrac{11}{15}$

C. $-\dfrac{\sqrt{221}}{15}$

D. $\dfrac{15}{\sqrt{221}}$

E. $\dfrac{2}{\sqrt{221}}$

Question 5

The graph with equation $y = x^2$ is transformed to its image equation $y = 2x^2 + 3x - 1$.

A possible rule for the transformation $T : R^2 \to R^2$ is

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{17}{8} \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ -\frac{17}{8} \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{17}{8} \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ -\frac{17}{8} \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{17}{8} \end{bmatrix}$

Question 6

If $f(x) = x^{\frac{3}{2}}$ then which one of the following is true for **all values of *x* and *y***?

A. $f(xy) = f(x)f(y)$

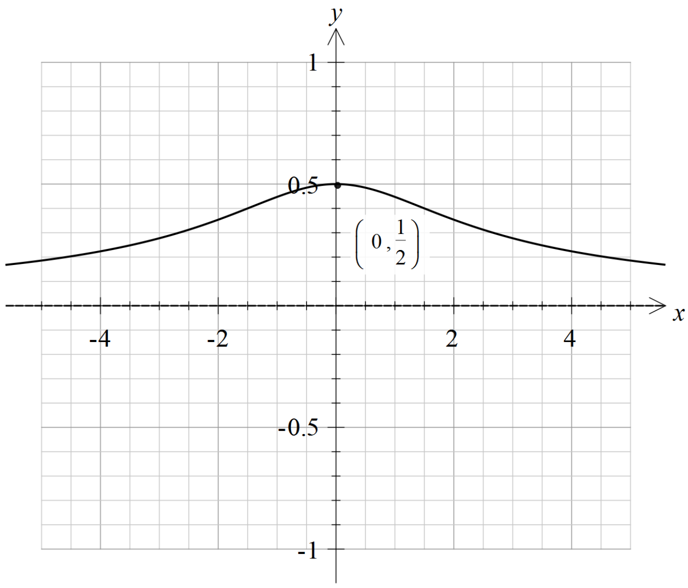
B. $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

C. $f(x+y) = f(x) + f(y)$

D. $f\left(x^2y^2\right) = f\left(x^2\right)f\left(y^2\right)$

E. $f(x-y) = f(x) - f(y)$

Question 7



A possible equation for the graph shown above is

A. $y = \frac{1}{4 - x^2}$

B. $y = \frac{1}{4 + x^2}$

C. $y = \frac{1}{2 + x^2}$

D. $y = \frac{1}{\sqrt{2 + x^2}}$

E. $y = \frac{1}{\sqrt{4 + x^2}}$

SECTION A - continued
TURN OVER

Question 8

The inverse f^{-1} of the function $f:(1,\infty)\to R, f(x)=\dfrac{3}{(x-1)^2}+2$ is

A. $f^{-1}:(1,\infty)\to R, f^{-1}(x)=\pm\sqrt{\dfrac{3}{x-2}}+1$

B. $f^{-1}:(2,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{3}{x-2}}+1$

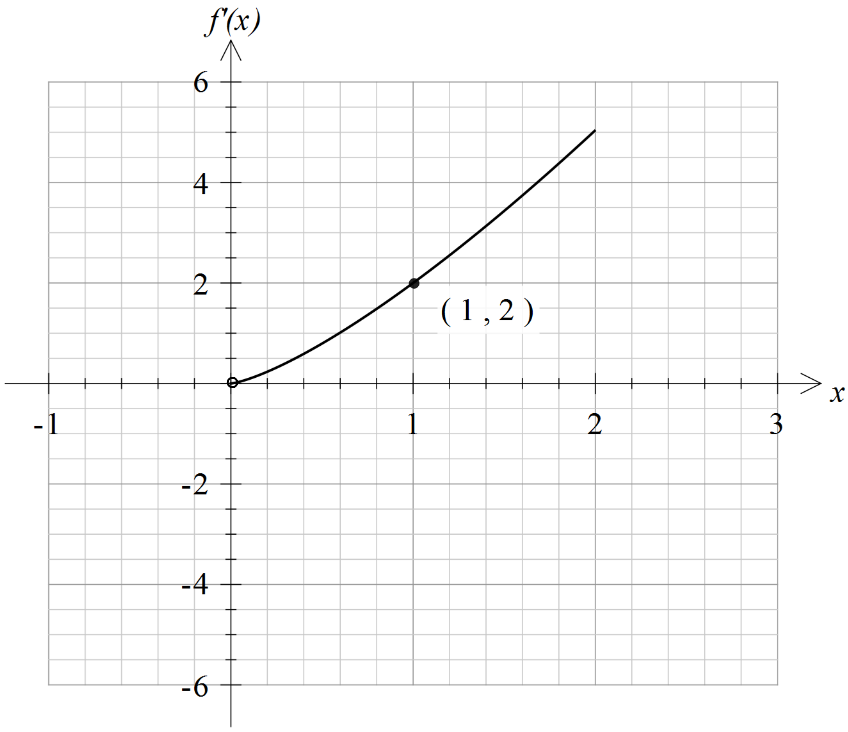
C. $f^{-1}:(2,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{3}{2-x}}+1$

D. $f^{-1}:(1,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{3}{x-2}}+1$

E. $f^{-1}:(3,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{1}{x-3}}+2$

Question 9

The graph of $y=f'(x)$ is shown below.



Which one of the following statements must be **correct**?

A. The graph has rule $f'(x)=2\sqrt{x}$.

B. The gradient of the graph of f at $x = 1$ is 2.

C. The domain of the graph of f is $[0,\infty)$.

D. The graph has rule $f'(x)=2x^2$.

E. The graph of f goes through the point $\left(4,\dfrac{128}{5}\right)$.

SECTION A - continued

Question 10

If the graph of f with rule $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + h$, where $a \neq 0$, has a stationary point of inflection at $(-3,0)$ and a turning point at (2,0), then $\frac{h}{a}$ equals

A. 108
B. −108
C. −72
D. 72
E. −6

Question 11

The average rate of change for the function with rule $f(x) = -2x^{\frac{7}{2}}$ from $y = -4$ to $y = -1$ is closest to

A. −4.3
B. −7
C. −11.5
D. −84.7
E. −7.5

Question 12

If $\log_a(b) = c$ and $\log_c(a) = b$, where a, b and c are positive real constants, then which one of the following is **false**?

A. $\log_b(a) = \frac{1}{c}$

B. $\log_b(c) = \frac{1}{bc}$

C. $\log_a(c) = \frac{1}{b}$

D. $\log_a\left(\frac{b}{c}\right) = b + c$

E. $\log_c(b) = bc$

Question 13

If $f'(x) = 2x^3 - 2x$ and $f(2) = \frac{9}{2}$ then $\{x : f(x) = 0\}$ equals

A. ± 1
B. $0, \pm 1$
C. $0, \pm\sqrt{2}$
D. $\pm\sqrt{2}$
E. 1

SECTION A - continued
TURN OVER

Question 14

If $\int_2^3 g(x)dx = -3$ then $2\int_3^2 \left(1-2g(x)\right)dx$ equals

A. 4
B. 10
C. 14
D. -8
E. -14

Question 15

If left endpoint rectangles with widths 0.5 are used to find the approximate area bounded by the graph with equation $f(x) = e^{2x}$, the *x*-axis and the ordinates $x = 0$ and $x = 2$, then this area will equal

A. $\frac{1}{2}\left(1+e+e^2+e^3\right)$ and will be an overestimate of the actual area
B. $\frac{1}{2}\left(1+e+e^2+e^3\right)$ and will be an underestimate of the actual area
C. $\frac{1}{2}\left(e+e^2+e^3+e^4\right)$ and will be an overestimate of the actual area
D. $\frac{1}{2}\left(e+e^2+e^3+e^4\right)$ and will be an underestimate of the actual area
E. $1+e+e^2+e^3$ and will be an underestimate of the actual area

Question 16

x	0	1	2	3
$\Pr(X = x)$	a	a^2	a^3	a^4

For the probability distribution shown in the above table the mean and variance are respectively closest to

A. 0 and 0.945
B. 1.5 and 0.945
C. 2 and 0.945
D. 0.766 and 0.893
E. 0.766 and 1.480

SECTION A - continued

Question 17

Rowing training is at 5:00 am every morning. The probability that Rod goes to training each morning is 0.3 and the probability that Joe goes to training each morning is p. The probability that neither of them go is p^2. If these event are independent, then the probability that they will both go to rowing training on a particular morning is

A. 0.7
B. 0
C. $\dfrac{\sqrt{329}-7}{20}$
D. $\dfrac{3(\sqrt{329}-7)}{200}$
E. $-\dfrac{3(\sqrt{329}+7)}{200}$

Question 18

If X is the random variable with probability density function f and rule

$$f(x)=\begin{cases}\log_e(x) & 1 \le x \le a \\ 0 & \text{elsewhere}\end{cases}, \text{ where } a \text{ is a real constant.}$$

then Var(X) is

A. e
B. $\dfrac{e^2+1}{4}$
C. $-\dfrac{e^4}{16}+\dfrac{2e^3}{9}-\dfrac{e^2}{8}+\dfrac{7}{144}$
D. $\dfrac{\sqrt{-9e^4+32e^3-18e^2+7}}{12}$
E. $\dfrac{2e^3+1}{9}$

Question 19

A random sample of 2000 people in a particular town were asked if they were in favour of a new shopping complex. It is known that 20% of the people in the town are in favour of the complex. Using the Normal Approximation to the Binomial Distribution, the approximate probability that between 380 and 550 people in the sample were in favour of the complex is closest to

A. 0.874
B. 0.132
C. 0.868
D. 0.862
E. 0.205

SECTION A - continued
TURN OVER

Question 20

The 90% confidence interval for the proportion p of secondary students who would like to use their mobile phones during class time was found to be $(0.5, b)$, where b is a real constant, when a random sample of 50 secondary students was taken.

This means

A. $\Pr(0.5 < p < b) = 0.90$

B. if 200 such samples were taken, 180 of the confidence intervals would be expected to contain p.

C. $0.5 < p < b$

D. $\hat{p} = 0.6$

E. $b = 0.7$

END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (14 marks)

The shape of the vertical cross section of an ancient musical instrument is

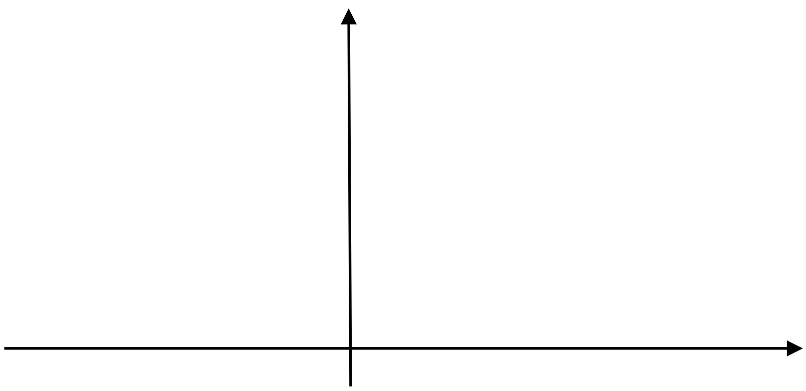
$$f:[-8,8]\to R, f(x)=ax^{\frac{2}{3}}.$$

The x-axis represents the lowest point of the instrument and f is the vertical distance, in cm, of the edge of the instrument above the sharp point.

It is found that the musical instrument goes through the point $\left(-8,\frac{16}{3}\right)$.

a. Show that the value of a is $\frac{4}{3}$. 1 mark

b. Sketch the graph of $y = f(x)$, labelling the coordinates of the cusp and endpoints. 2 marks



SECTION B – Question 1 – continued
TURN OVER

A musical mathematician, Miriam, wants to transform the equation of the sides to see if she gets a better sound.

c. Her first attempt is to graph the equation $f_1 : [-8,8] \to R, f_1(x) = f(-x)$. Why there is no difference? 1 mark

d. Her second attempt is to transform the graph of $y = f(x)$ to f_2 using the following transformations:

- a dilation of a factor of 2 units from the x-axis and then
- a translation of 3 units in the positive direction of the y-axis

What is the image equation $f_2(x)$? 1 mark

e. Her third attempt is to transform the graph of $y = f(x)$ to f_3 using the following transformations in the given order:

- a reflection in the x-axis
- a reflection in the y-axis
- a translation of 1 unit in the positive direction of the y-axis
- a translation of 3 units in the positive direction of the x-axis.

What is the image equation $f_3(x)$? 2 marks

Miriam gets quite ambitious and uses a transformation matrix to find a new image equation for $f(x)=\frac{4}{3}x^{\frac{2}{3}}$.

f. Use $T:R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to find the image equation for $f(x)=\frac{4}{3}x^{\frac{2}{3}}$.

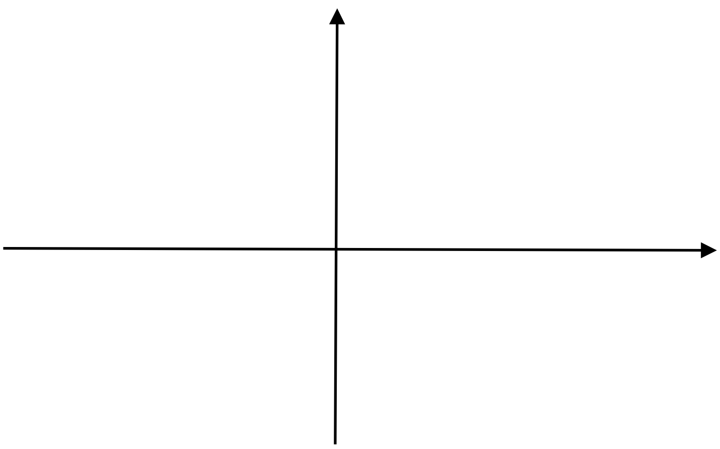
3 marks

g. Show that the domain for the image graph in **part f.** will be $[-15,17]$.

1 mark

h. Miriam is happy with this new shape. Sketch a graph of the image found in **part f.**, labelling the endpoints and y-axis intercept with their coordinates.

3 marks



SECTION B – continued

Question 2 (15 marks)

The rise and fall of lap swimmer Stephen's heartbeat, while swimming laps using a freestyle stroke, can be modelled by the equation

$$P(t) = 95 + 16\sin\left(\frac{\pi}{15}(t - 7.5)\right)$$

where $P(t)$ is Stephen's pulse, in beats per minute, at time t minutes after the start of his exercise session. Stephen swims for an hour at a time.

a. Show that Stephen's pulse at the start of his exercise session is 79 beats per minute**.** 1 mark

b. State the period and amplitude of the graph of $y = P(t)$. 2 marks

Period = ____________________________

Amplitude = ____________________________

c. Show that the derivative function $P'(t)$ can be written as $P'(t) = \frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right)$. 2 marks

d. Hence state the times, after the start of his hour long exercise session, where Stephen reaches his maximum pulse rate. 2 marks

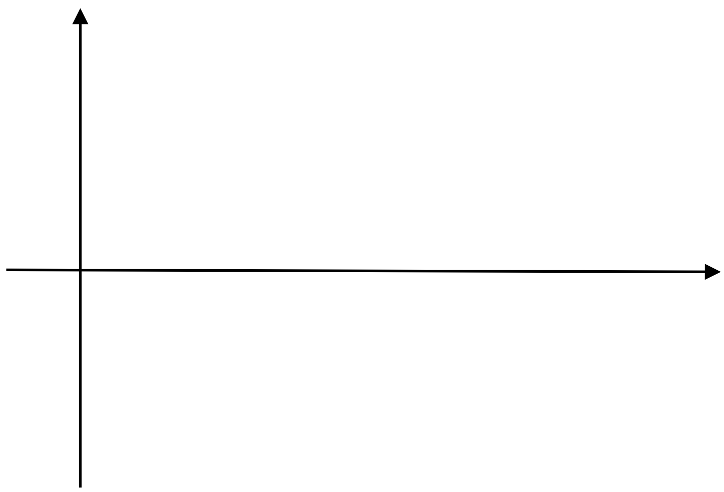
Stephen is not happy with his swimming session.

He swims in a new way and his pulse rate, $P_N(t)$, can now be modelled by the function

$$P_N : [0,60] \rightarrow R, P_N(t) = 0.01(t+85) \times P(t)$$

where $P_N(t)$ is Stephen's pulse, in beats per minute, at time t minutes after the start of his new exercise session.

e. Sketch the graph of $P_N : [0,60] \rightarrow R, P_N(t) = 0.01(t+85) \times P(t)$, labelling the coordinates of any local maximum and minimum points and endpoints, correct to the nearest whole number. 3 marks



f. Find the maximum heart rate, rounded to the nearest whole number, over the 1 hour exercise session. 1 mark

g. When is his pulse rate changing the fastest? Give your answer in minutes correct to one decimal place. 2 marks

SECTION B – Question 2 – continued
TURN OVER

Stephen's twin sister Sarah is also a swimmer. She likes to swim following the same model as Stephen's original model

$$P:[0,60] \rightarrow R, P(t) = 95 + 16\sin\left(\frac{\pi}{15}(t - 7.5)\right)$$

where $P(t)$ is Sarah's pulse, in beats per minute, at time t minutes after the start of her exercise session.

Sarah swims in the lane next to Stephen, while Stephen swims according to the model $P_N(t)$.

h. Solve the equation $0.01(t+85)=1$ and hence find the values of t for $t \in [0,60]$ where Sarah's pulse can be measured to be lower than her twin brother Stephen's. 2 marks

Question 3 (17 marks)

The gestation period (the time the fetus is inside the womb) for pandas is approximately normally distributed with a mean of 135 days and standard deviation of 10 days.

a. What percentage of pandas have a gestation period greater than 150 days? Give your answer to the nearest whole percent. 1 mark

b. If 80% of pandas have a gestation period greater than x days, what is the value of x to the nearest whole number? 1 mark

SECTION B – Question 3– continued
TURN OVER

It is known that 70% of people in a particular city are in favour of the pandas being kept in zoos so that they can reproduce in a safe environment. A random sample of 500 people was taken from this city and asked whether they were in favour of the pandas being kept in zoos.

c. What is the probability, correct to four decimal places, that more than 360 people in the survey are in favour of the pandas being kept in zoos? Do not use the Normal Approximation. 2 marks

d. What is the answer to **part c.** if the Normal Approximation is used. 2 marks

e. Find the probability the sample proportion, $\hat{P}$, lies within two standard deviations of the population proportion. Give your answer correct to four decimal places. Do not use the Normal Approximation. 3 marks

Another random sample of 500 people was taken from the city and they asked whether they were in favour of a tax being introduced to pay for animals in captivity. 400 said they were **not** in favour of the tax.

f. Using the Normal Approximation find an approximate 95% confidence interval for the proportion of people, p, in the city who are **not** in favour of the tax. Give your answers correct to four decimal places. 1 mark

SECTION B – Question 3 – continued

Pandas like playing with balls. A zoo keeper has 6 red balls and 5 yellow balls in a bag. She takes a random sample of four balls to give to the pandas.

g. Complete the following probability distribution table for the proportion of yellow balls in the sample. 2 marks

Proportion of yellow balls $\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr\left(\hat{P}=\hat{p}\right)$			$\frac{5}{11}$	$\frac{2}{11}$	$\frac{1}{66}$

h. Given that the proportion of yellow balls is more than $\frac{1}{2}$, what is the probability that they will all be yellow? 2 marks

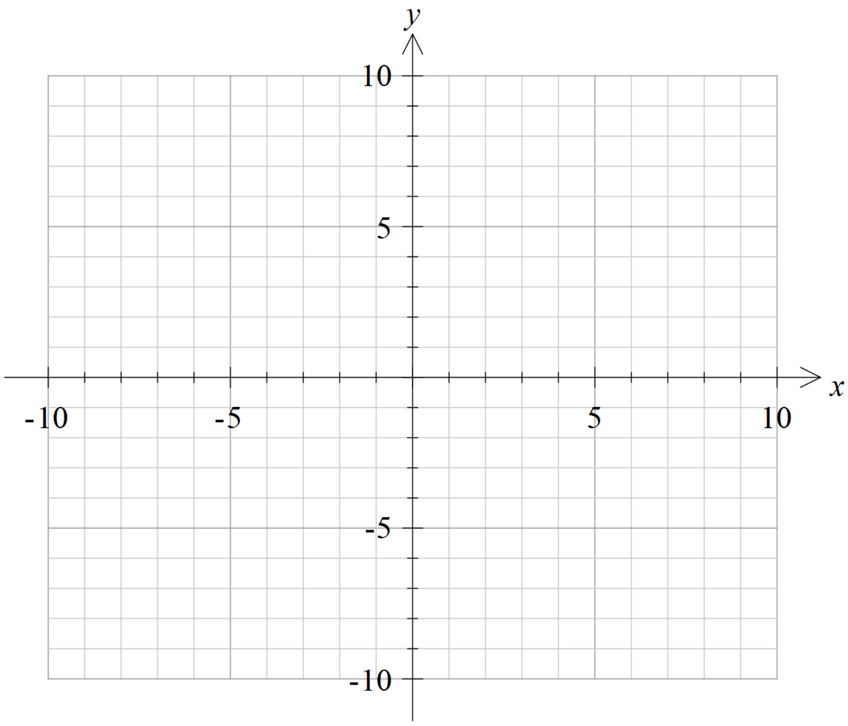
It is known that 10% of a certain species has a gestation period of more than 200 days and 5% of the species less than 175 days. The gestation period is normally distributed.

i. Find the mean and standard deviation of the gestation period for this species. Give your answers to the nearest whole number. 3 marks

Question 4 (14 marks)

Let $f: R \setminus \{1\} \to R, f(x) = \dfrac{x-4}{x-1}$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label any asymptotes with their equations and axial intercepts with their coordinates. 3 marks



SECTION B – Question 4 – continued

b. Find $f^{-1}(x)$. Comment on the result. 1 mark

Let $g: R\setminus\{-n\} \to R, g(x) = \dfrac{x+m}{x+n}$, where *m* and *n* are non-zero real constants.

c. For what values of *m* and *n* will $g = g^{-1}$? 2 marks

Let $h: R\setminus\left\{-\dfrac{b}{n}\right\} \to R, h(x) = \dfrac{ax+m}{bx+n}$, where a, b, m and n are non-zero real constants.

d. Show that $a = -n$ if $h = h^{-1}$. 1 mark

SECTION B – Question 4– continued
TURN OVER

e. If $h_1 : R\backslash\left\{-\frac{n}{4}\right\} \rightarrow R, h_1(x) = \frac{3x+2}{4x+n}$ and $h_1 = h_1^{-1}$ find n. 1 mark

Consider the function $w: R \rightarrow R, w(x) = -(2x-1)^3 + \frac{1}{2}$.

f. Find the coordinates of the points of intersection of w and w^{-1}. 3 marks

g. Find the area bounded by the curves of w and w^{-1}. 2 marks

h. Consider the family of functions $w_k : R \rightarrow R, w_k(x) = -(rx-1)^3 + s$, where $k \in Z$ and s and r are non-zero real constants. Find a relationship between r and s if the curves of w_k and w_k^{-1} have three points of intersection. 1 mark

END OF QUESTION AND ANSWER BOOK