MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

(TSSM's *2011 trial exam updated for the current study design)*

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: E

Explanation:
If $k = 2$: $(2-1)x + 4y = 8$ and $3x - (-2+2)y = 2 + 1$
$x + 4y = 8$ and $3x = 3$
$x = 1$ and $y = \frac{7}{4}$

Therefore it will have a unique solution.

Question 2

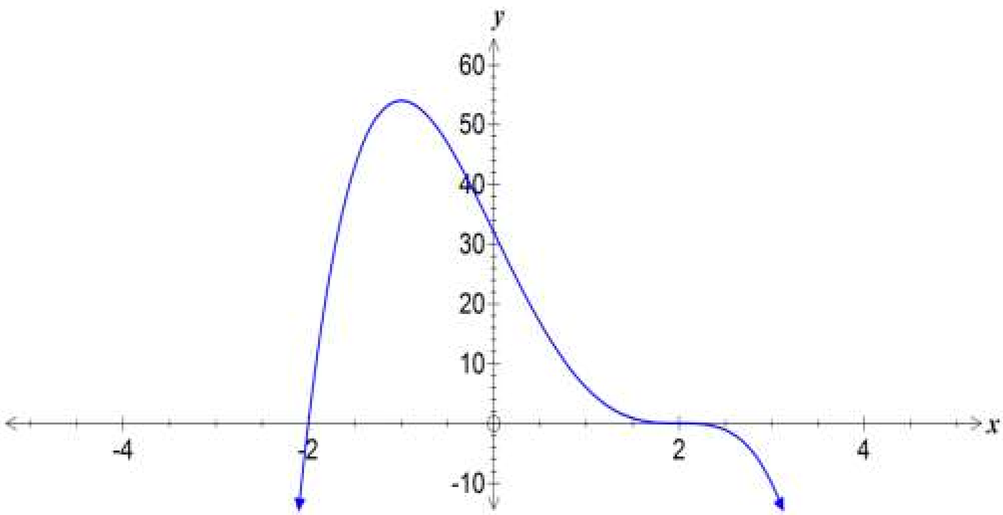
Answer: C

Explanation:
$f(x-y) = (x-y)^3$
$f(x) - f(y) = x^3 - y^3$
$f(x-y) \neq f(x) - f(y)$

Question 3

Answer: B

Explanation:
Sketch graph:



Question 4

Answer: C

Explanation:
$\int_1^4 (2f(x)+6)dx$
$= 2\int_1^4 f(x)dx + \int_1^4 6\ dx$
$= 2\int_1^4 f(x)dx + 18$

Question 5

Answer: B

Explanation:
$m_{\text{tangent}} = -3$
$y = -3(x-3) + 4 - 6 = -3x + 7$

Question 6

Answer: B

Explanation:
$Pr(Z < -z) = Pr(Z > z) = 0.75$

Use CAS Probability menu: inverse normal. Change the area to 0.25 because the TI-nspire calculators use $Z < c$ and not $Z > c$.
$c = -0.6745$

Question 7

Answer: B

Explanation:
At $x = 2$, the gradient function changes from positive to negative, therefore local max on $f(x)$

Question 8

Answer: A

Explanation:
fog exists if ran $g \subseteq$ dom f , therefore $D = x < -1$ or $x > 3$.
(For $fog\ to$ exists, the range of g: $(1, \infty)$)

Question 9

Answer: C

Explanation:
Use CAS: $f(x) = \int (2-x)(2x+1)^2 dx = -x^4 + \frac{4}{3}x^3 + \frac{7}{2}x^2 + 2x$

Question 10

Answer: B

Explanation:
Solve $\int_k^{\pi} \frac{2}{\pi} \cos^2(x) dx = 0.3$ on CAS
$k = 2.63$

Question 11

Answer: B

Explanation:
$\int_0^6 f\left(\frac{1}{3}x\right) + 1dx = \int_0^6 f\left(\frac{x}{3}\right) dx + \int_0^6 1\, dx = 12 + [x]_0^6 = 12 + 6 = 18$

Question 12

Answer: C

Explanation:
$Pr(-1 < Z < 2) = Pr(\mu - \sigma < X < \mu + 2\sigma)$.
$\mu + 2\sigma = 4.9$
$\mu - \sigma = 4$

Question 13

Answer: D

Explanation:
Use CAS: $\frac{d}{dx}(e^{-4x}\sin(x-2))|x = 2$

Question 14

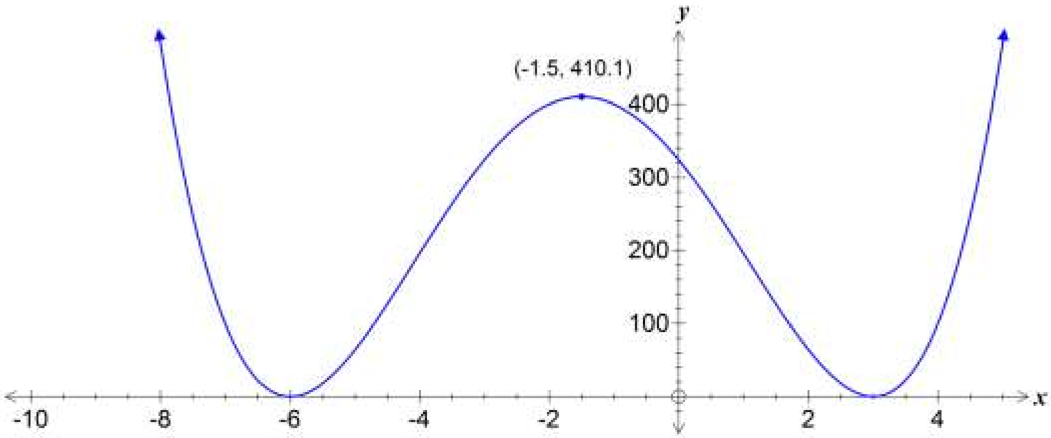
Answer: C

Explanation:
Use CAS and solve $\cos(2x) = -\frac{\sqrt{3}}{2}$, then expand to get $\pm\frac{5\pi}{12} + k\pi,\ \ k \epsilon Z$

Question 15

Answer: E

Explanation:
Use CAS to sketch graph and find local maximum, then $(-6, -1.5) \cup (3, \infty)$.



Question 16

Answer: E

Explanation:
$y = -5e^{2(x+2)} + 2$
$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Question 17

Answer: B

Explanation:
Use CAS: Binomial pdf $(20, 0.45, 5)$, $\quad \Pr(X = 5) = 0.0365$

Question 18

Answer: A

Explanation:
Period $= \frac{\pi}{b} = 5\pi$, asymptote $x = \frac{\pi}{2b} = \frac{5\pi}{2}$

Question 19

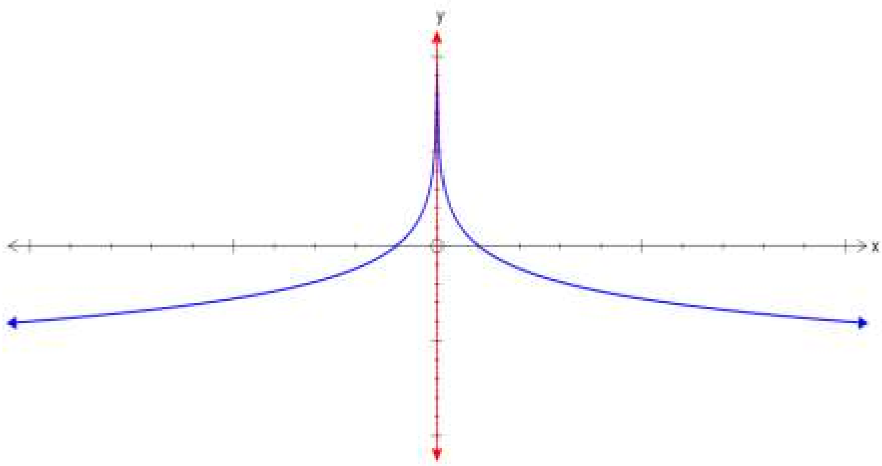
Answer: C

Explanation:
$l_1 = 3 - e^{1-1} = 3 - 1 = 2,\ l_2 = 3 - e^{2-1} = 3 - e$
$A = 2 + 3 - e = 5 - e$

Question 20

Answer: D

Explanation:
Use CAS to sketch the graph.



There is a vertical line at $x = 0$.

Question 21

Answer: C

Explanation:
You have to use the points of intersection and for each section it is the top graph minus the bottom graph.

Question 22

Answer: E

Explanation:
The graph has a vertical asymptote at $x = a$ and the function lies to the right of the asymptote, sketch the graph.

SECTION 2: Analysis Questions

Question 1

a. $f(x) = \frac{2}{x-3} - 4 = 2(x-3)^{-1} - 4$

Use chain rule: $m_{\text{tangent}} = f'(x) = -2(x-3)^{-2} \times 1 = \frac{-2}{(x-3)^2}$

$m_{\text{normal}} = \frac{(x-3)^2}{2}$

M1+A1

2 marks

b. $m = 2, \quad (1, -5)$

$y - y_1 = m(x - x_1)$

$y + 5 = 2(x - 1)$

$g(x) = 2x - 7$

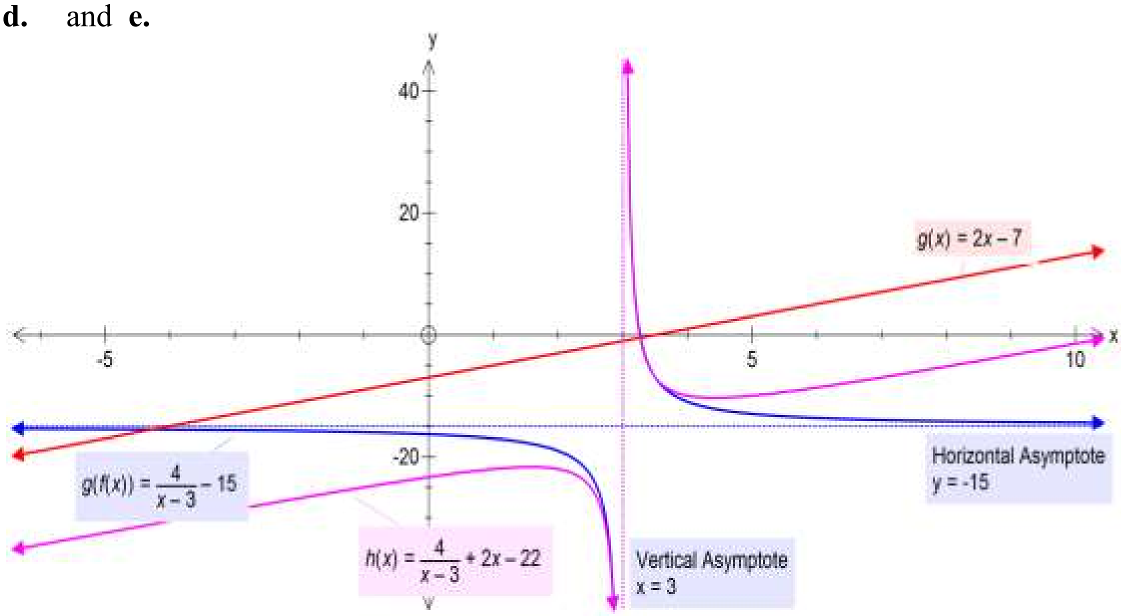
M1+A1

2 marks

c. $g(f(x)) = g\left(\frac{2}{x-3} - 4\right) = 2\left(\frac{2}{x-3} - 4\right) - 7 = \frac{4}{x-3} - 15$, the domain is the set of all values in the domain of f (the domain of the inner function). Domain: $x \in R\backslash\{3\}$. M2+A1

3 marks

d. and **e.**



A2

A2

2 + 2 marks

f. Reflection in the x −axis, dilation factor of 2 away from the x −axis, dilation factor of $\frac{1}{2}$ away from the y −axis, translation of 2 units to the right and a translation of 1 unit up.

equation: $y = -2\left(\frac{2}{2x-4-3} - 4\right) + 1 = 9 - \frac{4}{2x-7}$

M2+A2
4 marks

Question 2

a. $np = 212.5$
$np(1-p) = 31.875$
Solve for n & p
$n = 250$
$p = 0.85$

M1+A1
2 marks

b. $Binomial\ CDF\ (5, 0.9, 3, 5)$
$= 0.991$

A1
1 mark

c.
$\Pr(at\ least\ one\ lasts\ less\ than\ 200\ hours) > 0.95$
$\Pr(zero\ last\ less\ than\ 200\ hours) < 0.05$
$\binom{n}{0}(0.1)^0(0.9)^n < 0.05$
$n > 28.433$
$n = 29$

M2+A1
3 marks

d. normcdf$(200, \infty, 196, 4.2) = 0.17$

A1
1 mark

e. $\Pr(X > m) = 0.42$
$\Pr(X < m) = 0.58$
$m = 196.85$

M1+A1
2 marks

f. $Pr(X \geq 2) \geq 0.8$

$Pr(X = 0) + Pr(X = 1) \leq 0.2$

$0.6^n + \binom{n}{1} \times 0.4 \times (0.6)^{n-1} \leq 0.2$

$n = 6.4$

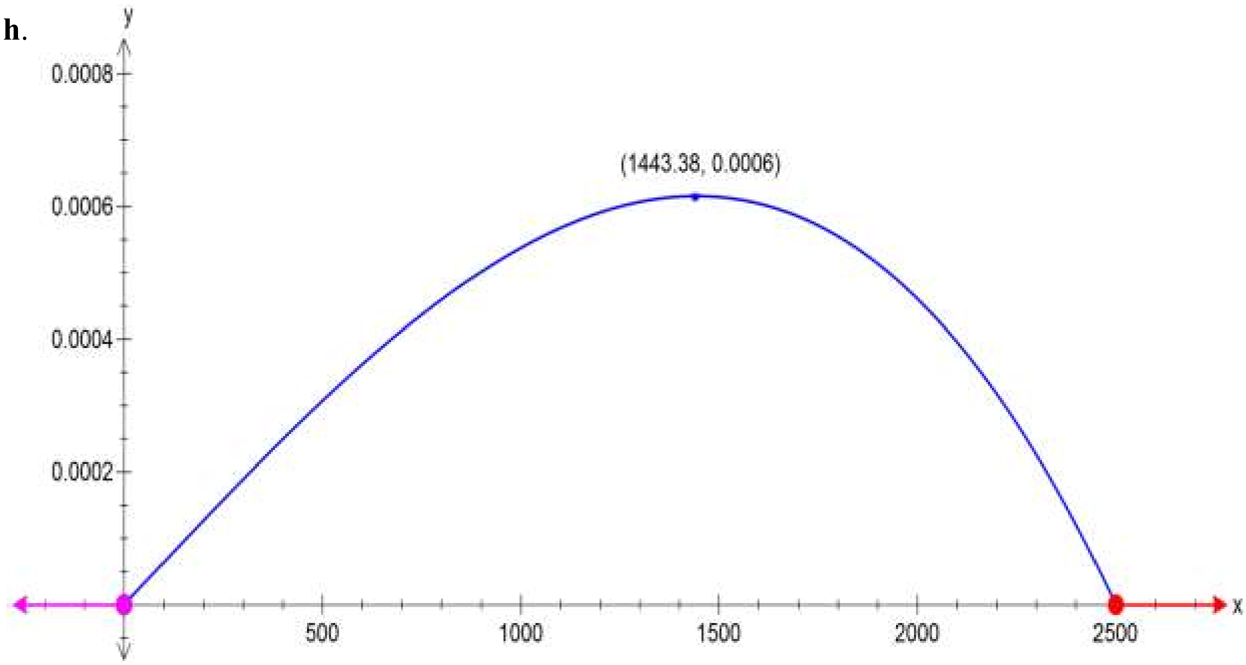
At least 7 coloured light globes needs to be sold.

M2+A1
3 marks

g. $\int_0^{2500} \frac{1}{9.765625 \times 10^{12}} \left(6.25 \times 10^6 t - t^3\right) dt = 1$ and $f(t) \geq 0$

M1+A1
2 marks

h.



M1+A1
2 marks

i. $\mu = \int_0^{2500} \frac{t}{9.765625 \times 10^{12}} \left(6.25 \times 10^6 t - t^3\right) dt = \frac{4000}{3}$ hours

A1
1 mark

j. $\Pr\left(X > \frac{4000}{3} | X \geq 1000\right) = \frac{\Pr(X > \frac{4000}{3})}{\Pr(X \geq 1000)} = \frac{0.51202}{0.7056} = 0.7257$

M1+A1
2 marks

Question 3

a. average value of gradient $= \frac{1}{1.5+1.5}\int_{-1.5}^{1.5}\left((x+1)^2(5-4x)\right)dx = 2.75$

M1+A1
2 marks

b. $f(x) = \int f'(x)\,dx$
$= \int -4x^3 - 3x^2 + 6x + 5\,dx$
$= -x^4 - x^3 + 3x^2 + 5x + c$
$f(2) = -16 - 8 + 12 + 10 + c = -2$
$c = 0$

M1+A1
2 marks

c. $(x+1)^2(5-4x) = 0$
$x = -1$ point of inflection – gradient is positive before and after the point.
$x = \frac{5}{4}$ local maximum – gradient changes from positive before the point to negative after the point.

M2+A2
4 marks

d. Area $= -\int_{-1}^{0} f(x)\,dx + \int_{0}^{1.91964} f(x)\,dx - \int_{1.91964}^{2} f(x)\,dx = 9.21.$
(separate integral using the x −ints).

M1+A1
2 marks

Question 4

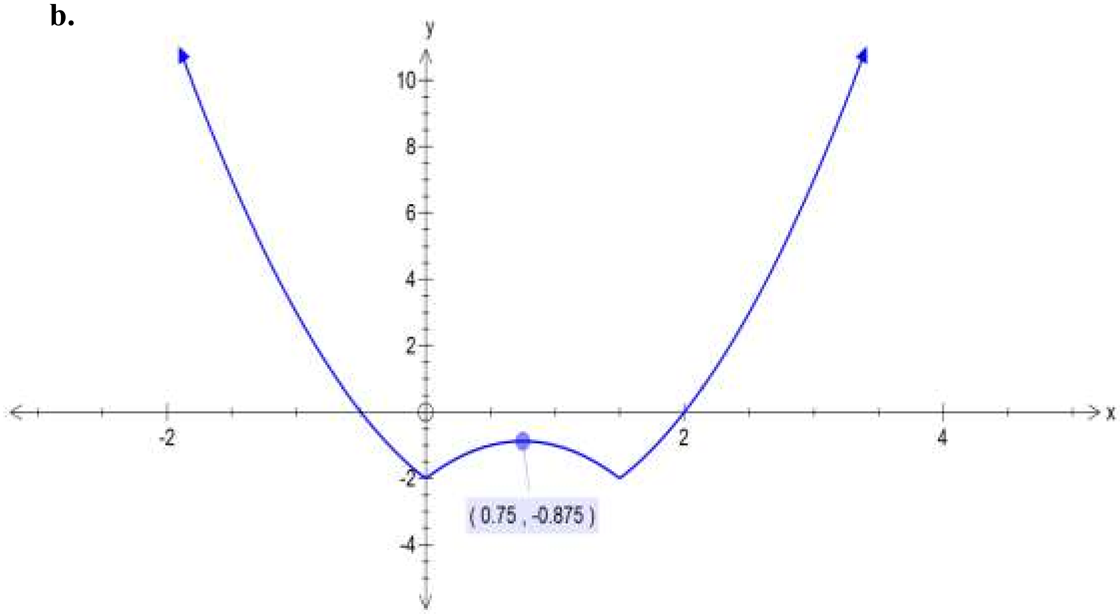
a.

$$f(x) = \begin{cases} 2x^2 - 3x - 2, & x \le 0 \ \cup \ x \ge \frac{3}{2} \\ -2x^2 + 3x - 2, & 0 < x < \frac{3}{2} \end{cases}$$

$2x^2 - 3x - 2 = 0$ or $-(2x^2 - 3x) - 2 = 0 \Rightarrow$ no solutions.
$(x-2)(2x+1) = 0$
$x = 2$ or $x = -\frac{1}{2}$

M1+A2
3 marks

b.



A2
2 marks

Question 5

a. MS $= x$

$\Delta ANC ||| \Delta AMS$

$\frac{AN}{NC} = \frac{AM}{MS}$

$\frac{200}{100} = \frac{AM}{x}$

$AM = 2x$ mm

Area $= 2x(200 - 2x) = 400x - 4x^2$ mm^2

--- OR ---

Let N be the origin (0,0) then A is the point $(0,200)$, C is the point $(100,0)$

And line AC has equation $y = 200 - 2x$.

Then point S has co-ordinates $(x, 200 - x)$

And the required area is $2x \times (200 - x) = 400x - 4x^2$ mm^2

M2+A2
4 marks

b. $x \in (0, 100)$

$\frac{dA}{dx} = 400 - 8x = 0$

$x = 50$ mm

M1+A1
2 marks

c. $A = 100 \times 100 = 10\ 000\ \text{mm}^2$

A1
1 mark