MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

*(*TSSM's *2015 trial exam updated for the current study design)*

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: A

Explanation:

Solve the two equations on CAS.

Question 2

Answer: C

Explanation:

It is negative cubic so either C or D. Check the x-intercept.

Question 3

Answer: E

Explanation:

Define the functions on CAS and find $f(g(x))$

Question 4

Answer: D

Explanation:

$f(x) = 2\left(\sqrt{x} + \frac{1}{2}\right)$
$g(x) = 2 \times \frac{1}{2}\left(\sqrt{x} + \frac{1}{2}\right)$

Question 5

Answer: C

Explanation:

$Domain: 4 - x \geq 0 \ \ gives\ x \leq 4$ and the graph is above the x-axis.

Question 6

Answer: A

Explanation:

$Av\ ROC = \frac{f(8)-f(2)}{8-2}$

Question 7

Answer: C

Explanation:

Note the shaded end-points.

Question 8

Answer: C

Explanation:

$f\big(g(x)\big) = \frac{3}{x+5},\ x \neq -2$

Question 9

Answer: E

Explanation:

Eliminate incorrect options

Question 10

Answer: D

Explanation:

$Amp = 2,\ Period = \frac{2\pi}{\frac{1}{5}}.$

Question 11

Answer: E

Explanation:

$\frac{dy}{dx}\ \ at\ x = 4$ on CAS.

Question 12

Answer: B

Explanation:

$A_1 = A_2$

Question 13

Answer: B

Explanation:

$normalline(f(x), x = 0)$ on CAS.

Question 14

Answer: C

Explanation:

$\left(f(x)\right)^2 \times \left(f(y)\right)^2 = e^{2x} \times e^{2y} = e^{2x+2y} = f(2x+2y)$

Question 15

Answer: A

Explanation:
$\frac{1}{k}\int_0^k x^3 dx = 9 \ \ gives \ \ k = 6^{\frac{2}{3}}$ on CAS.

Question 16

Answer: B

Explanation:

$binompdf\left(10, \frac{1}{5}, 6\right)$

Question 17

Answer: C

Explanation:

$normcdf(165{,}170{,}165{,}7.62)$.

Question 18

Answer: A

Explanation:

$binomcdf(6{,}0.2{,}5{,}6)$ on CAS.

Question 19

Answer: D

Explanation:

$50th$ percentile means she is on average, due to the symmetry of the normal distribution

Question 20

Answer: C

Explanation:

Sketch on CAS and read the maximum value.

Question 21

Answer: C

Explanation:

$k = 0.2, \ E(X) = 3.9$

Question 22

Answer: B

Explanation:

$\frac{\pi}{n} = 3$ gives $n = \frac{\pi}{3}$

SECTION 2: Analysis Questions

Question 1

a. $r = l\ sin\alpha,\ h = l\ cos\alpha$

A2
2 marks

b. $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(lsin\alpha)^2(lcos\alpha) = \frac{\pi}{3}l^3 \sin^2\alpha\ cos\alpha$

M1
1 mark

c. $V'(\alpha) = \frac{\pi}{3}l^3\left(\sin^2\alpha \times -sin\alpha + \ cos\alpha \times 2sin\alpha\ cos\alpha\right) = 0$

$sin\alpha\left(-\sin^2\alpha + 2\cos^2\alpha\right) = 0$

$sin\alpha = 0,\ \tan^2\alpha = 2$

$\alpha = 0,\ \alpha = \pm\tan^{-1}\sqrt{2}$

$\alpha = \tan^{-1}\sqrt{2},\quad V(\alpha) = \frac{2\sqrt{3}}{27}\pi l^3$

$\left(\tan^{-1}\sqrt{2},\ \frac{2\sqrt{3}}{27}\pi l^3\right)$

Alternate form: $\left(\cos^{-1}\frac{\sqrt{3}}{3},\ \frac{2\sqrt{3}}{27}\pi l^3\right)$ also correct

M3+A1
4 marks

d. $\alpha = \tan^{-1}\sqrt{2}$ is a point of maximum volume.

$Max\ volume = \frac{2\sqrt{3}}{27}\pi \times 6^3 = 16\sqrt{3}\pi\ cm^3.$

M1+A1
2 marks

Question 2

a. Period $= \frac{2\pi}{\frac{\pi}{2.2}} = 4.4$ years and Amplitude $= 300$

A2
2 marks

b. Min $= 200$, Max $= 800$

A2
2 marks

c. Solve $\mathrm{P(t)} = 800$ over $[0, 5]$
t = 0.7. After 8.4 months

M1+A1
2 marks

d. Sketch the graph on CAS and read the domain when $P < 300$
$2.3 < t < 3.5 \quad and \quad 6.7 < t < 7.9$

M1+A2
3 marks

e. Strictly increasing for $t \in [0\,, 0.7] \cup [2.9\,, 5]$
Note that we include endpoints for strictly increasing intervals.

A3
3 marks

Question 3

a. Sketch on CAS and read the max: 0.45 μg/mL

A1
1 mark

b. 3.5 minutes

A1
1 mark

c. $C(10) = 0.32$ μg/mL

M1+A1
2 marks

d. $\frac{C(5)-C\left(\frac{3}{2}\right)}{5-\frac{3}{2}} = 0.0115\,\frac{\mu\mathrm{g}}{\mathrm{mL}}$/minute

M1+A1
2 marks

e. Solve $\frac{dC}{dt} < 0$ on CAS
t > 3.53 minutes

M1+A1
2 marks

f. $\frac{dC_1}{dt} = 0$ at $t = 120 \ldots\ldots\ldots..(1)$

$C_1(120) = 120 \ldots\ldots\ldots\ldots\ldots..(2)$

Solve the above equations on CAS to get $a = e$ and $b = \frac{1}{120}$

M2+A1
3 marks

Question 4

a. $f(x) = x^2 + bx + \frac{b^2}{4} + 3 - \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2 + 3 - \frac{b^2}{4}$

$\frac{b}{2} = 5$ gives $b = 10 \quad (b > 0)$

M1+A1
2 marks

b. Translation of $+\ 5$ units parallel to the $x -$ axis

Translation of $+\ 22$ units parallel to the $y -$ axis

A2
2 marks

c. Range of g: $[0, \infty)$

Domain of f: R

Range of g is a subset of domain of f, hence $f(g(x))$ exists.

$f(g(x)) = (x^2 + 5)^2 - 22$

Or $f(g(x)) = x^4 + 10x^2 + 3$

M1+A2
3 marks

d. $tangentline(h(x), x, k)$

$y = (4k^3 + 20k)x + (-3k^4 - 10k^2 + 3)$

M1+A1
2 marks

e. $Area = \int_0^3 ((x^2 + 5)^2 - 22)\, dx$

A2
2 marks

Question 5

a.

i. Let $X \sim N(7.5, 2.5^2)$

$\Pr(X < 11) = 0.9192$ (using CAS: $normcdf(-\infty, 11, 7.5, 2.5)$)

ii. $\Pr(5.5 < X < 10.5) = 0.6731$ (using CAS: $normcdf(5.5, 10.5, 7.5, 2.5)$)

M1+A1
2 marks

b. $\Pr(D < d) = 0.1$

$d = 4.3$ km

M1+A1

2 marks

c. $n = 6, \quad p = \Pr(X \geq 6.8) = 0.6103, \; r = 4$

Let $Y \sim \text{Bi}(6, 0.6103)$

$\Pr(Y = 4) = 0.3160$ (using $binompdf(6, 0.6103, 4)$)

M2+A1
3 marks

d. $\Pr(x > 5) = 0.65 \rightarrow \Pr(X < 5) = 0.35$

$\frac{5 - 6.4}{\sigma} = -0.3853$

$\sigma = 3.63$

M2+A1
3 marks

e. $\hat{p} = 0.8$ and $n = 200$

$\sqrt{\frac{\hat{p}(1-\hat{p}\ )}{n}} = 0.028284$

$(0.8 - 1.96 \times 0.028284, 0.8 + 1.96 \times 0.028284)$

$(0.745, \; 0.855)$

M2+A1
3 marks

f. $M = 0.02, \hat{p} = 0.8$

$0.02 = 1.96\sqrt{\frac{0.8 \times 0.2}{n}}$

$n = 1536.64$

Thus we need a sample size of 1537 people.

M1+A1
2 marks