VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
Year

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Day Date

Reading time: *. to *.** (15 minutes)**
Writing time: *. to *.** (2 hours)**

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 20 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Version 2 – April 2016

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The point P (4, –3) lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. (–4, 1)

B. (–4, 3)

C. (0, –3)

D. (4, –6)

E. (–4, –1)

Question 2

The linear function $f: D \to R, f(x) = 4 - x$ has range [–2, 6).

The domain D of the function is

A. [–2, 6)

B. (–2, 2]

C. R

D. (–2, 6]

E. [–6, 2]

Question 3

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\dfrac{2}{\pi}$

B. 2

C. $\dfrac{1}{2}$

D. $\dfrac{1}{4}$

E. 2π

Question 4

If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. -4

B. -2

C. -1

D. 1

E. 2

Question 5

The function $g: [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.
The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Question 6

If $\int_1^4 f(x)\,dx = 6$, then $\int_1^4 (5 - 2f(x))\,dx$ is equal to

A. 3

B. 4

C. 5

D. 6

E. 16

Question 7

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.
If A and B are independent, then the value of p is

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

E. $\frac{3}{5}$

Question 8

Which one of the following functions satisfies the functional equation $f\left(f(x)\right) = x$ for every real number x?

A. $f(x) = 2x$

B. $f(x) = x^2$

C. $f(x) = 2\sqrt{x}$

D. $f(x) = x - 2$

E. $f(x) = 2 - x$

Question 9

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A. $\frac{20}{81}$

B. $\frac{5}{18}$

C. $\frac{4}{9}$

D. $\frac{40}{81}$

E. $\frac{5}{9}$

Question 10

The transformation $T : R^2 \to R^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

maps the line with equation $x - 2y = 3$ onto the line with equation

A. $x + y = 0$

B. $x + 4y = 0$

C. $-x - y = 4$

D. $x + 4y = -6$

E. $x - 2y = 1$

Question 11

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

A. 0

B. $\frac{1}{a}$

C. 1

D. a

E. $-\frac{1}{a}$

Question 12

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$

B. $a = -3$

C. both $a = 3$ and $a = -3$

D. $a \in R\backslash\{3\}$

E. $a \in R\backslash[-3, 3]$

Question 13

It is known that 26% of the 19-year-olds in a region do not have a driver's licence.

If a random sample of ten 19-year-olds from the region is taken, the probability, correct to four decimal places, that more than half of them will not have a driver's licence is

A. 0.0239

B. 0.0904

C. 0.2600

D. 0.9096

E. 0.9761

Question 14

An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council.

An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A. $\left(0.76 - \sqrt{\frac{0.76 \times 0.24}{574}},\ 0.76 + \sqrt{\frac{0.76 \times 0.24}{574}}\right)$

B. $\left(0.76 - 1.65\sqrt{\frac{0.76 \times 0.24}{574}},\ 0.76 + 1.65\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

C. $\left(0.76 - 2.58\sqrt{\frac{0.76 \times 0.24}{574}},\ 0.76 + 2.58\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

D. $\left(436 - 1.96\sqrt{0.76 \times 0.24 \times 574},\ 436 + 1.96\sqrt{0.76 \times 0.24 \times 574}\right)$

E. $\left(0.76 - 2\sqrt{0.76 \times 0.24 \times 574},\ 0.76 + 2\sqrt{0.76 \times 0.24 \times 574}\right)$

Question 15

The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \frac{b^2}{4a}$

B. $c < \frac{b^2}{4a}$

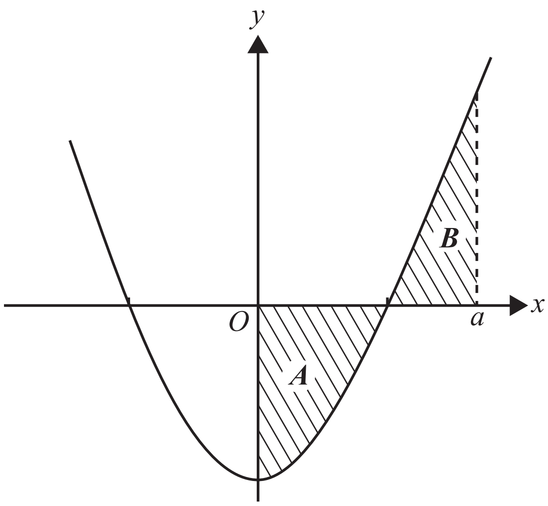
C. $c < 4b^2a$

D. $c > \frac{b^2}{3a}$

E. $c < \frac{b^2}{3a}$

Question 16

A part of the graph of $g: R \to R, g(x) = x^2 - 4$ is shown below.



The area of the region labelled A is the same as the area of the region labelled B.

The exact value of a is

A. 0

B. 6

C. $\sqrt{6}$

D. 12

E. $2\sqrt{3}$

Question 17

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

C. $w \geq 25$

D. $w < -7$ or $w > 25$

E. $w > 1$

Question 18

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at (2, 0) and (–3, 0). The maximum value of the derivative function is 10.

The value of x for which the graph of $y = f(x)$ has a local maximum is

A. –2

B. 2

C. –3

D. 3

E. $-\frac{1}{2}$

Question 19

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days.

B. 13 days.

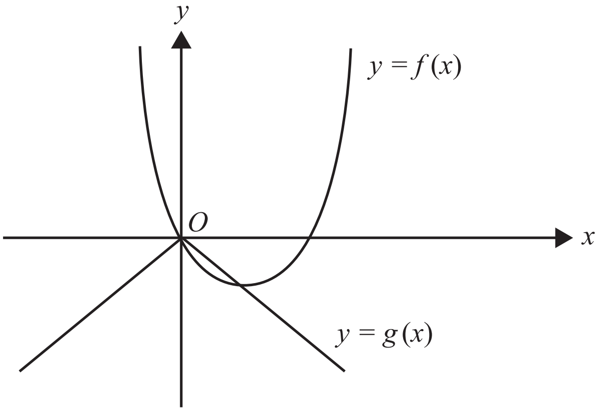
C. 17 days.

D. 21 days.

E. 37 days.

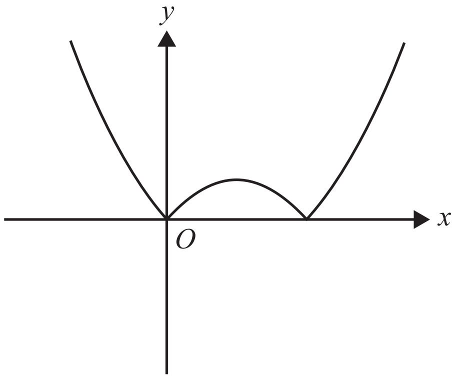
Question 20

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.

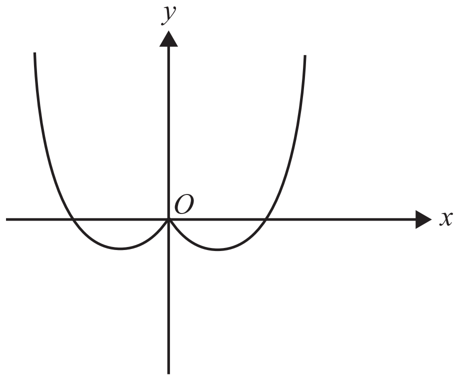


The graph of $y = f\left(g(x)\right)$ is best represented by

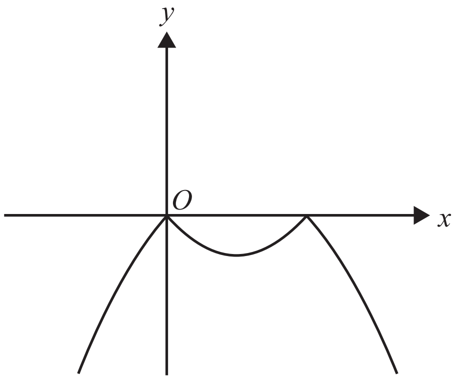
A.



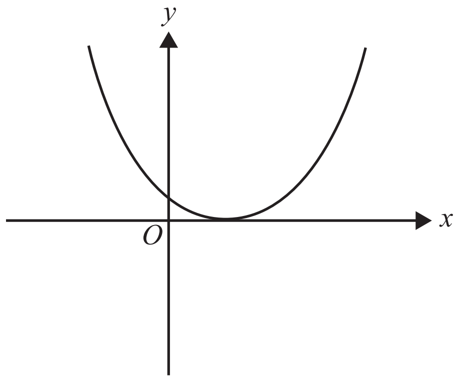
B.



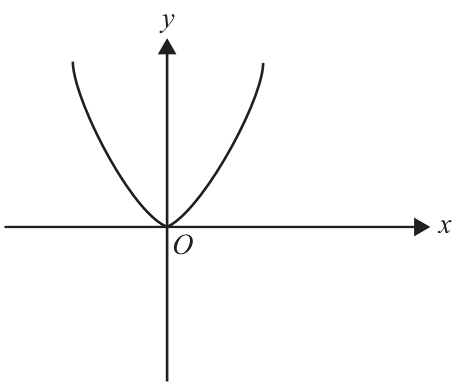
C.



D.



E.



END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (7 marks)

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n. 2 marks

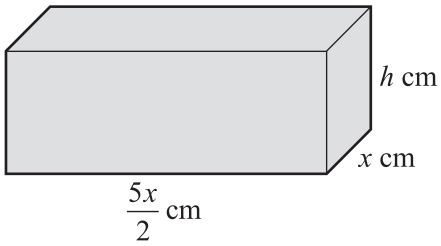
b. Find the maximum and minimum populations of wombats in this location. 2 marks

c. Find $n(10)$. 1 mark

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$. 2 marks

Question 2 (10 marks)

A solid block in the shape of a rectangular prism has a base of width x centimetres. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 cm^2.

a. Show that if the height of the block is h centimetres, $h = \dfrac{6480 - 5x^2}{7x}$. 2 marks

b. The volume, V cubic centimetres, of the block is given by $V(x) = \dfrac{5x(6480 - 5x^2)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x. 3 marks

c. Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers. 3 marks

d. Find the exact values of x and h if the block is to have maximum volume. 2 marks

Question 3 (20 marks)

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a. It has been found that the probability that any member of FullyFit will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i. Find $\Pr(X \geq 10)$ correct to four decimal places. 2 marks

ii. Find $\Pr(X \geq 15 \mid X \geq 10)$ correct to three decimal places. 3 marks

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete **S** in less than three minutes.

iii. Find the expected value and variance of $\hat{P}$. 3 marks

iv. Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation. 3 marks

v. Find $\Pr\left(\hat{P} \geq \frac{3}{4} \mid \hat{P} \geq \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation. 2 marks

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months? 2 marks

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete another exercise routine, **T**, is a continuous random variable W with a probability density function g, as defined below.

$$g(w) = \begin{cases} \dfrac{(w-3)^3 + 64}{256} & 1 \le w \le 3 \\ \dfrac{w+29}{128} & 3 < w \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $E(W)$ correct to four decimal places. 2 marks

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **T**? Give your answer to the nearest integer. 2 marks

d. From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6

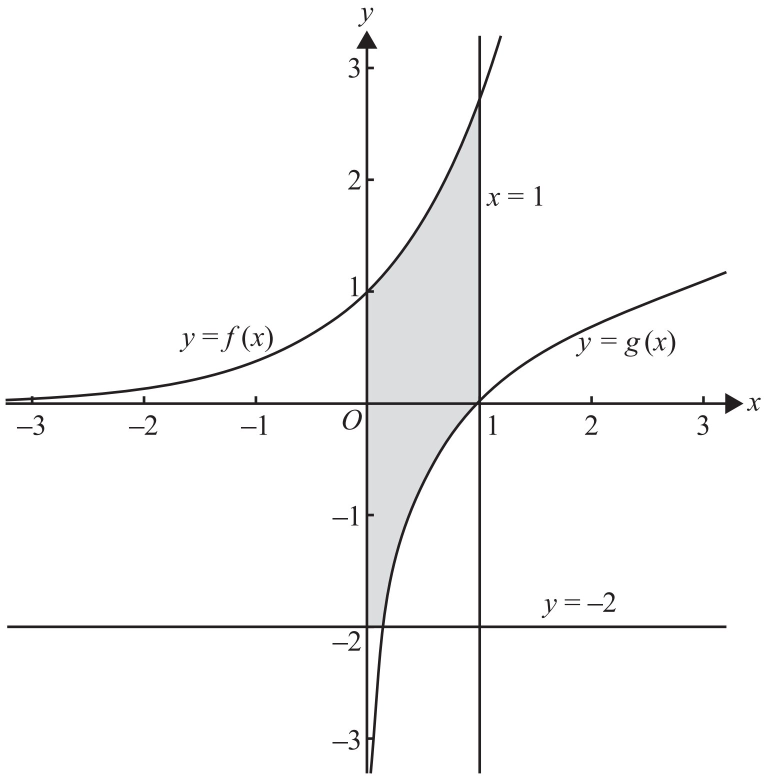
Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places.

1 mark

Question 4 (8 marks)

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres. Two of the boundaries of the mine site are in the shape of the graphs of the following functions.

$f: R \to R, f(x) = e^x$ $\qquad$ $g: R^+ \to R, g(x) = \log_e(x)$



a. **i.** Evaluate $\int_{-2}^{0} f(x)dx$. 1 mark

ii. Hence, or otherwise, find the area of the region bounded by the graph of g, the x-axis and y-axis, and the line $y = -2$. 1 mark

iii. Find the total area of the shaded region. 1 mark

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function $k: (-\infty, a) \rightarrow R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k. 2 marks

ii. Hence, find the set of values of a for which the graphs of g and k have two distinct points of intersection. 1 mark

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection.

Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$.

Find the value of a in this case. 2 marks

Question 5 (15 marks)

Let $f: R \to R,\ f(x) = (x-3)(x-1)(x^2+3)$ and $g: R \to R,\ g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)\left((x+b)^2+c\right)$. 2 marks

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 2 marks

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept 1 mark

ii. two positive x-axis intercepts. 1 mark

d. Find the value of n for which the equation $g(x) = n$ has one solution. 1 mark

e. At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^3$. 2 marks

ii. Find u and v if $u + v = 1$. 1 mark

f. **i.** Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$. 2 marks

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$. 3 marks

END OF QUESTION AND ANSWER BOOK