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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2017

SECTION A – Multiple-choice answers

1.	A	9.	B	17.	D
2.	D	10.	E	18.	C
3.	E	11.	E	19.	C
4.	C	12.	A	20.	A
5.	B	13.	D		
6.	E	14.	A		
7.	B	15.	B		
8.	E	16.	D		

SECTION A – Multiple-choice solutions

Question 1

$$f(x) = 2\sin\left(\frac{\pi x}{3}\right) - 1$$

$$\begin{aligned}\text{period} &= 2\pi \div \frac{\pi}{3}\\ &= 2\pi \times \frac{3}{\pi}\\ &= 6\end{aligned}$$

For $y = \sin\left(\frac{\pi x}{3}\right)$, range $= [-1,1]$

For $y = 2\sin\left(\frac{\pi x}{3}\right)$, range $= [-2,2]$

For $y = 2\sin\left(\frac{\pi x}{3}\right) - 1$, range $= [-3,1]$

So $r_f = [-3,1]$

The answer is A.

Question 2

$f(x) = \log_e(x-1)$

For the function f to be defined,

$$\begin{aligned}x - 1 &> 0\\ x &> 1\end{aligned}$$

Note that the maximal domain of f is $(1, \infty)$ which is not offered in the options.

But if f were to have a restricted domain, then it could be $(1,2)$. For all the other options, f is not defined. So $D = (1,2)$

The answer is D.

Question 3

The period of the graph is $\frac{\pi}{2}$.

For $y = \tan(nx)$, period $= \frac{\pi}{n}$

So $\quad \frac{\pi}{n} = \frac{\pi}{2}$

and $\quad n = 2$

So we reject options A, B and C. The graph shown is that of a tan graph that has been reflected in the *x* or *y*-axis. Option E offers the case where there has been a reflection in the *x*-axis. So the required rule for *f* is $f(x) = -\tan(2x)$

The answer is E.

Question 4

$$\text{average rate of change} = \frac{g(8) - g(0)}{8 - 0}$$
$$= \frac{7}{2}$$

Note, if you put B as your answer you didn't bracket the numerator of the fraction above when entering it in your CAS. If you obtained E, then you found the average value, not the average rate of change.

The answer is C.

Question 5

Function *f* is strictly decreasing over an interval if $x_2 > x_1$ implies that $f(x_2) < f(x_1)$.

This is the case for the interval $x \in [1,3]$.

Note that we have not been asked for the interval where the gradient of *f* is negative. Such an interval would not include the endpoints $x = 1$ nor $x = 3$.

The answer is B.

Question 6

$h(x) = \frac{3}{x-2} - 1$

Let $y = \frac{3}{x-2} - 1$

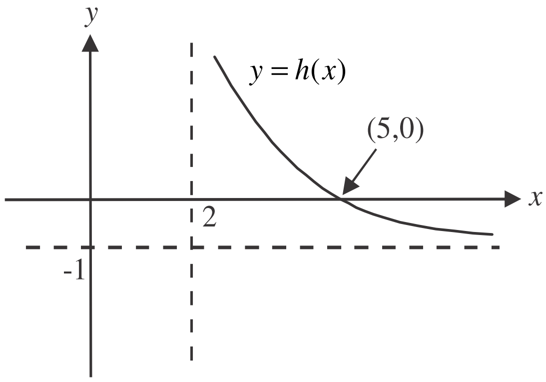
Swap *x* and *y* for inverse.

$x = \frac{3}{y-2} - 1$

Solve for *y* using CAS.

$y = \frac{2x+5}{x+1}$

$= \frac{3}{x+1} + 2$



So $\quad h^{-1}(x) = \frac{3}{x+1} + 2$

Also, $\quad d_h = (2, \infty) \quad$ so $\quad r_h = (-1, \infty) \quad$ (from the graph)

therefore $d_{h^{-1}} = (-1, \infty)$

We have $h^{-1} : (-1, \infty) \rightarrow R,\ h^{-1}(x) = \frac{3}{x+1} + 2$.

The answer is E.

Question 7

Start by finding $\text{Var}(X)$.

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - \{\text{E}(X)\}^2 \\ &= 9.8 - (1.6)^2 \\ &= 7.24 \\ \text{sd}(X) &= \sqrt{7.24} \\ &= 2.6907...\end{aligned}$$

The closest answer is 2.69.
The answer is B.

Question 8

$$\begin{aligned}y &= x^3 - ax \\ \frac{dy}{dx} &= 3x^2 - a\end{aligned}$$

Stationary points occur when $\frac{dy}{dx} = 0$.

$$\begin{aligned}3x^2 - a &= 0 \\ 3x^2 &= a \\ x &= \pm\sqrt{\frac{a}{3}}\end{aligned}$$

So $\sqrt{\frac{a}{3}} = 2$

$$\begin{aligned}\frac{a}{3} &= 4 \\ a &= 12\end{aligned}$$

Alternatively, since $x = \pm 2$

$$x^2 = 4$$

and since $3x^2 = a$

$$12 = a$$

The answer is E.

Question 9

The right endpoint of f occurs at the point $(2, f(2))$ i.e. $(2,1)$.
So the horizontal line running through this right endpoint is $y = 1$.

$$\begin{aligned}\text{area} &= \int_0^2 (f(x) - 1)dx \\ &= \int_0^2 (4 - x^2)dx\end{aligned}$$

The answer is B.

Question 10

$f(x)+f(2x+1)=3f(x)$

For option A, $f(x)=\frac{1}{x}$

$$\frac{1}{x}+\frac{1}{2x+1}=\frac{3}{x}. \text{ NOT TRUE.}$$

For option B, $f(x)=e^x$

$$e^x+e^{2x+1}=3e^x \text{ NOT TRUE.}$$

For option C, $f(x)=\sqrt{x}$

$$\sqrt{x}+\sqrt{2x+1}=3\sqrt{x} \text{ NOT TRUE.}$$

For option D, $f(x)=x^2$

$$x^2+(2x+1)^2=3x^2 \text{ NOT TRUE.}$$

Option E must be true i.e. $f(x)=x+1$

$$x+1+2x+1+1=3(x+1) \text{ TRUE}$$

The answer is E.

Question 11

$$f(x)=\log_e(x)-1$$

Let $y=\log_e(x)-1$

After a reflection in the y-axis, the rule becomes $y=\log_e(-x)-1$.

After a dilation from the x-axis by a factor of $\frac{1}{3}$, the rule becomes

$$\frac{y}{1/3}=\log_e(-x)-1$$

$$y=\frac{1}{3}\log_e(-x)-\frac{1}{3}$$

So $g(x)=\frac{1}{3}\log_e(-x)-\frac{1}{3}$.

The answer is E.

Question 12

$$2x-ay=a+5$$
$$ax-8y=-2$$

This system of equations can be expressed as the matrix equation

$$\begin{bmatrix}2 & -a\\ a & -8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}a+5\\ -2\end{bmatrix}$$

There are no solutions or infinitely many solutions

when $2\times{}^-8-{}^-a^2=0$

$$-16+a^2=0$$
$$(a-4)(a+4)=0$$
$$a=\pm4$$

When $a=4$, we have $2x-4y=9 \quad -(A)$

$4x-8y=-2 \quad -(B)$

(A)$\times2$ $\quad 4x-8y=18$

There are no solutions when $a=4$.

When $a=-4$, we have $2x+4y=1 \quad -(A)$

$-4x-8y=-2 \quad -(B)$

(A)$\times{}^-2$ $\quad -4x-8y=-2$

When $a=-4$ there are infinitely many solutions.

The answer is A.

Question 13

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times a \times (a-2)^2$$
$$\frac{dA}{da} = \frac{(a-2)(3a-2)}{2} \quad \text{using CAS}$$
$$\frac{dA}{da} = 0 \text{ for min/max.}$$
$$\frac{(a-2)(3a-2)}{2} = 0$$
$$a = 2 \quad \text{or } a = \frac{2}{3}$$
So $a = \frac{2}{3}$ since $a < 2$.

$$\text{When } a = \frac{2}{3}, A = \frac{1}{2} \times \frac{2}{3} \times \left(\frac{2}{3} - 2\right)^2$$
$$= \frac{1}{3} \times \left(\frac{-4}{3}\right)^2$$
$$= \frac{16}{27}$$
The answer is D.

Question 14

$$h(x) = \int_1^x \log_e(2t)\,dt$$
$$= x\log_e(x) + x(\log_e(2) - 1) - \log_e(2) + 1 \quad \text{(using CAS)}$$
So $h'(x) = \log_e(2x)$ (using CAS)
$$h'\left(\frac{1}{2}\right) = \log_e(1)$$
$$= 0$$
The answer is A.

Question 15

$$\mathrm{E}(\hat{P}) = p$$
$$= 0.4$$
$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{0.4(1-0.4)}{600}}$$
$$= 0.02$$
The answer is B.

Question 16

$p = 0.05,\ n = 200$

$\Pr\left(\hat{P} \le \frac{7}{200}\right) = \Pr(X \le 7), \qquad \hat{P} = \frac{X}{n}$

$= 0.213304...$

using binomCdf(200, 0.05, 0,7)

The closest answer is 0.2133.

Note that if you gave 0.1652, you used a normal approximation which the question asked you not to do.

The answer is D.

Question 17

Since f defines a probability density function, $\int_0^1 \frac{e^x}{2} dx + \int_1^a e\, dx = 1$.

Solve this equation for a using CAS.

$$a = \frac{(e+3)\, e^{-1}}{2}$$

which can be written as

$$a = \frac{e+3}{2e}$$

The answer is D.

Question 18

Method 1 – using a Karnaugh map

	ready	not ready	
OS	**5**	**20**	25
OS'	15	15	30
	20	**35**	55

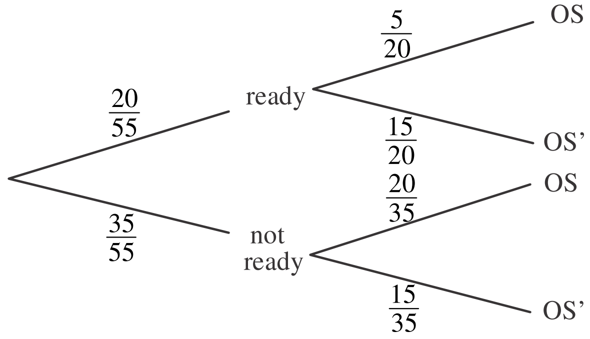
The entries in bold are what we are given in the question and the others we can work out.

$\Pr(\text{ready} \mid \text{OS}) = \frac{5}{25}$

$= \frac{1}{5}$

The answer is C.

<u>Method 2</u> – using a tree diagram



$$\Pr(\text{ready}|\text{OS}) = \frac{\Pr(\text{ready} \cap \text{OS})}{\Pr(\text{OS})}$$

$$= \frac{20}{55} \times \frac{5}{20} \div \left(\frac{20}{55} \times \frac{5}{20} + \frac{35}{55} \times \frac{20}{35} \right)$$

$$= \frac{1}{11} \div \left(\frac{1}{11} + \frac{4}{11} \right)$$

$$= \frac{1}{11} \times \frac{11}{5}$$

$$= \frac{1}{5}$$

The answer is C.

Question 19

$n = 400 \quad p = 0.08$

For the normal approximation, $\mu = 0.08$ and $\text{sd} = \sqrt{\dfrac{0.08(1-0.08)}{400}}$

$$= 0.01356...$$

So $X \sim N(0.08, 0.1356...^2)$

$\Pr(0.05 < X < 0.1) = 0.916319...$ (using CAS)

The closest answer is 0.9163.

The answer is C.

Question 20

The graph of f is

- reflected in the y-axis
- dilated by a factor of 2 from the x-axis
- translated 3 units down to become the graph of g.

i.e. $g(x) = 2f(-x) - 3$

So $\displaystyle\int_{-4}^{-1} g(x)dx = \int_{-4}^{-1} 2f(-x)dx - \int_{-4}^{-1} (3)dx$

$$= 2\int_{1}^{4} f(x)dx - [3x]_{-4}^{-1}$$

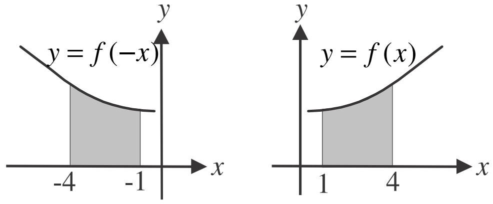
$$= 2 \times 3 - (-3 - {}^{-}12)$$

$$= 6 - 9$$

$$= -3$$

The answer is A.

As an example:



Note: $\displaystyle\int_{1}^{4} f(x)dx = \int_{-4}^{-1} f(-x)dx$

SECTION B

Question 1 (13 marks)

a. From the graph, $r_f = [-3,1]$. **(1 mark)**

b. Define f on your CAS including the domain, i.e. $0 \le x \le \frac{\pi}{2}$. Point A lies on the x-axis.

Solve $f(x) = 0$ for x using CAS.

$x = \frac{\pi}{6}$ So A is the point $\left(\frac{\pi}{6}, 0\right)$. **(1 mark)**

c. Using your CAS, find the derivative of $f(x)$ at the point where $x = \frac{\pi}{6}$.

i.e. $f'\left(\frac{\pi}{6}\right) = -2\sqrt{3}$ **(1 mark)**

Re-read the question!
So $a = 2$ and $b = 3$.

(1 mark)

d. Using your CAS, solve $f'(x) = -2\sqrt{3}$ for x. **(1 mark)**

$x = \frac{\pi}{6}$ or $x = \frac{\pi}{3}$

Point A occurs where $x = \frac{\pi}{6}$.

Point B must therefore occur where $x = \frac{\pi}{3}$ and $f\left(\frac{\pi}{3}\right) = -2$.

Point B is $\left(\frac{\pi}{3}, -2\right)$.

(1 mark)

e. The equation of the tangent at B is given by

$$y - {}^-2 = -2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$y = -2\sqrt{3}\left(x - \frac{\pi}{3}\right) - 2$$ **(1 mark)**

This tangent intersects the x-axis when $y = 0$.

Solve $0 = -2\sqrt{3}\left(x - \frac{\pi}{3}\right) - 2$

$$x = \frac{\pi}{3} - \frac{\sqrt{3}}{3}$$

This point of intersection with the x-axis occurs to the left of point A and at point A, $x = \frac{\pi}{6}$.

Now $\frac{\pi}{6} - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{3}\right)$

$$= -\frac{\pi}{6} + \frac{\sqrt{3}}{3}$$

So $r = \frac{\sqrt{3}}{3} - \frac{\pi}{6}$.

(1 mark)

f. Define g on your CAS. Note that it has the same rule as f but a different domain.
Solve $g(x)=\sqrt{3}-1$, for x. **(1 mark)**

$x=\frac{(12k-1)\pi}{12}$ or $x=\frac{(12k+1)\pi}{12}$ where $k\in Z$.

(1 mark)

g. Let (x',y') be an image point that lies on h.

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} -2 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} \frac{\pi}{2} \\ n \end{bmatrix}$$

$$x'=-2x+\frac{\pi}{2}, \qquad y'=my+n$$

$$2x=\frac{\pi}{2}-x' \qquad my=y'-n$$

$$x=\frac{\pi}{4}-\frac{x'}{2} \qquad y=\frac{y'-n}{m}$$

$f(x)=2\cos(2x)-1$

Let $y=2\cos(2x)-1$

So $\frac{y'-n}{m}=2\cos\left(2\left(\frac{\pi}{4}-\frac{x'}{2}\right)\right)-1$ **(1 mark)**

$$y'-n=2m\cos\left(\frac{\pi}{2}-x'\right)-m$$

$$y'-n=2m\sin(x')-m \qquad \text{since} \quad \cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$$

$$y'=2m\sin(x')-m+n$$

Now $h(x)=2\sin(x)+1$

So $2m=2$

$m=1$ **(1 mark)**

and $-m+n=1$

$n=2$

(1 mark)

Question 2 (14 marks)

a. **i.** $f(g(x)) = -\frac{1}{4}\left(-\frac{1}{2}x+2\right)\left(3-2\left(-\frac{1}{2}x+2\right)\right)\left(6-2\left(-\frac{1}{2}x+2\right)\right)^2$ **(1 mark)**

$$= \frac{1}{8}(x-4)(3+x-4)(6+x-4)^2$$

$$= \frac{1}{8}(x-4)(x-1)(x+2)^2$$

$$= \frac{1}{8}(x+2)^2(x-1)(x-4)$$

as required

(1 mark)

ii. $f(g(x))$ exists if $r_g \subseteq d_f$

Now $r_g = R$ and $d_f = R$,

so $r_g \subseteq d_f$

so $f(g(x))$ exists.

(1 mark)

b. Define $h(x) = \frac{1}{8}(x+2)^2(x-1)(x-4)$ on your CAS.

Stationary points occur when $h'(x) = 0$.

Solve $h'(x) = 0$ for x. **(1 mark)**

$x = -2$ or $x = \frac{11-3\sqrt{17}}{8}$ or $x = \frac{3\sqrt{17}+11}{8}$

Re-read the question.

So $a = -2, b = \frac{11}{8}$ and $c = 17$.

(1 mark) for a and c
(1 mark) for b

c. **i.** $h(0) = 2$

So A is the point (0, 2).

The gradient of the tangent at A is given by $h'(0) = -\frac{1}{2}$ (using CAS).

Equation of tangent is

$$y - 2 = -\frac{1}{2}(x-0)$$

$$y = -\frac{1}{2}x + 2$$

$$= g(x)$$

as required

(1 mark)

ii. Define $g(x)$ on your CAS.
To find the x-coordinates of points B and C, solve $h(x) = g(x)$ for x.**(1 mark)**

$x = -3$ or $x = 0$ or $x = 4$

$h(-3) = \frac{7}{2}$ $\quad\quad$ $h(4) = 0$

So B is the point $\left(-3, \frac{7}{2}\right)$ and C is the point $(4,0)$. **(1 mark)**

$$\text{length of } BC = \sqrt{(4 - -3)^2 + \left(0 - \frac{7}{2}\right)^2}$$

$$= \frac{7\sqrt{5}}{2} \text{ units}$$

(1 mark)

Note that because we haven't been asked to express our answer to a certain number of decimal places, we must leave it as an exact value.
Note also that the distance formula is not on the formula sheet so you must memorise it (in case you need it in Exam 1).

d. i. Start by finding the equation of the line that passes through points D and X.
This line has a gradient of 2 since it is perpendicular to the tangent with equation $y = -\frac{1}{2}x + 2.\left(\text{i.e. } 2 \times -\frac{1}{2} = -1\right)$.

It passes through the point D which is located at the point (–2, 0) (using working from part **b.**)

It's equation is $y - 0 = 2(x - {}^-2)$

$$y = 2x + 4$$

(1 mark)

To find the x-coordinate of X solve $-\frac{1}{2}x + 2 = 2x + 4$ for x.

i.e. solve $g(x) = 2x + 4$ for x

$$x = -0.8$$

$$g(-0.8) = 2.4$$

X is the point $(-0.8, 2.4)$.

(1 mark)

ii. area of triangle BCD

$$= \frac{1}{2} \times BC \times DX$$

(1 mark)

$$= \frac{1}{2} \times \frac{7\sqrt{5}}{2} \times \sqrt{(-2 - {}^-0.8)^2 + (0 - 2.4)^2}$$

$$= \frac{1}{2} \times \frac{7\sqrt{5}}{2} \times \frac{6\sqrt{5}}{5}$$

$$= 10.5 \text{ square units}$$

(1 mark)

Question 3 (16 marks)

a. **i.** Let X denote the number of parcels that require a signature at the delivery address.

$X \sim \text{Bi}(50, 0.3)$ **(1 mark)**

$\Pr(X \geq 10) = 0.959768...$ (Using CAS i.e. binom Cdf$(50, 0.3, 10, 50)$)

$= 0.9598$ (correct to 4 decimal places)

(1 mark)

ii. $\Pr(X > 15 | X \geq 10)$

$= \dfrac{\Pr(X > 15 \cap X \geq 10)}{\Pr(X \geq 10)}$ (Conditional probability formula)

$= \dfrac{\Pr(X \geq 16)}{\Pr(X \geq 10)}$ **(1 mark)**

$= \dfrac{0.430821...}{0.959768...}$ (Using CAS binom Cdf$(50, 0.3, 16, 50)$ and the result from part **i.**)

$= 0.448880...$

$= 0.4489$ (correct to 4 decimal places)

(1 mark)

b. Let Y denote the delivery time of a parcel.

$Y \sim \text{N}(260, 50^2)$

$S = \Pr(Y > 360)$

$= 0.022750...$ (Using CAS norm Cdf$(360, \infty, 260, 50)$

$= 0.0228$ (correct to 4 decimal places)

So $S = 0.0228$ (correct to 4 decimal places as required)

(1 mark)

c. <u>Method 1</u>

Let m represent more than 6 hours.

We require **(1 mark)**

$\Pr(m, m', m') + \Pr(m', m, m') + \Pr(m', m', m)$

$= 3 \times 0.022750... \times 0.977249... \times 0.977249...$

$= 0.065180...$

$= 0.0652$ (correct to four decimal places)

(1 mark)

<u>Method 2</u>

Let $V =$ the number of parcels that took longer than 6 hours to deliver.

$V \sim \text{Bi}(3, 0.022750...)$ **(1 mark)**

$\Pr(V = 1) = {}^3C_1 (0.022750...)^1 (1 - 0.022750...)^2$

$= 0.065180...$

$= 0.0652$ (correct to four decimal places)

(1 mark)

d. Let W denote the number of parcels that have a delivery time of more than six hours.

$W \sim \text{Bi}(50, 0.022750...)$ and $\hat{P} = \frac{W}{n}$ **(1 mark)**

$\Pr(\hat{P} \ge 0.06 | \hat{P} \ge 0.04)$

$= \frac{\Pr(\hat{P} \ge 0.04 \cap \hat{P} \ge 0.06)}{\Pr(\hat{P} \ge 0.04)}$ (conditional probability)

(1 mark)

$= \frac{\Pr(\hat{P} \ge 0.06)}{\Pr(\hat{P} \ge 0.04)}$

$= \frac{\Pr(W \ge 3)}{\Pr(W \ge 2)}$ since $0.06 = \frac{W}{50}$ and $0.04 = \frac{W}{50}$

$= \frac{0.105168...}{0.3152416...}$ (binom Cdf (50,0.02275...,3,50))

$= 0.333610...$ (binom Cdf (50,0.02275...,2,50))

$= 0.3336$ (correct to four decimal places)

(1 mark)

e. $n = 100, \quad \hat{p} = \frac{4}{100} = 0.04$

Using CAS, z interval_1Prop 4, 100, 0.95:

confidence interval ≈ (0.001592...,0.078407...)

=(0.002,0.078) (correct to three decimal places)

(1 mark)

f. **i.** Start by defining f on your CAS.

Solve $\int_0^m f(x)dx = 0.5$ for m using CAS. **(1 mark)**

$m = 32.794061...$

Note that we reject the other value of 83.3901… because it is outside the domain of f.

The median is 32.8 minutes correct to one decimal place.

(1 mark)

ii. This is a conditional probability question given that the delivery time must be at least greater than 20 minutes.

$\Pr(X < 30 | X > 20)$ **(1 mark)**

$= \frac{\Pr(X < 30 \cap X > 20)}{\Pr(X > 20)}$ (Conditonal probability formula)

$= \frac{\int_{20}^{30} f(x)dx}{\int_{20}^{45} f(x)dx}$ **(1 mark)**

$= \frac{368}{1205}$

$= 0.305$ (correct to 3 decimal places)

(1 mark)

Question 4 (17 marks)

a. **i.** $f(x) = \log_e(2-x)$

Let $y = \log_e(2-x)$

Since the graph of f is reflected in the y-axis, replace x with $-x$.

We have $y = \log_e(2-(-x))$

So, $g(x) = \log_e(2+x)$.

(1 mark)

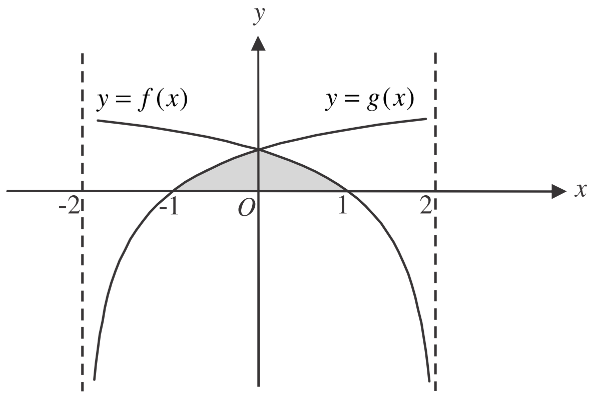
$d_g = (-2, \infty)$ (either using the graph or $2+x>0$ so $x>-2$)

(1 mark)

ii. Do a quick sketch.

The required area is shaded.

Because the graphs of f and g are symmetrical about the y-axis,



$$\text{area} = 2\int_0^1 f(x)dx$$

$$= 2(2\log_e(2)-1) \text{ square units}$$

(1 mark)

(or $\log_e(16)-2$ square units)

(Note – you must express your answer as an exact value because you have <u>not</u> been asked to approximate to a certain number of decimal places.)

b. **i.** Define h on your CAS.

<u>y-intercept</u> occurs when $x=0$

The y-intercept is $(0, \log_e(k))$ **(1 mark)**

<u>x-intercept</u> occurs when $y=0$

Solve $h(x)=0$ for x.

$$x = k-1$$

The x-intercept is $(k-1, 0)$.

(1 mark)

ii. If $h(x_1) > h(x_2)$

then $\log_e(k-x_1) > \log_e(k-x_2)$

$\log_e(k-x_1) - \log_e(k-x_2) > 0$

$$\log_e\left(\frac{k-x_1}{k-x_2}\right) > 0$$ **(1 mark)**

$\frac{k-x_1}{k-x_2} > 1$ since $k-x_2>0$ (i.e. $k>1$ and $x_2 \in (-\infty, k)$)

$k-x_1 > k-x_2$

$-x_1 > -x_2$ multiply both sides by -1

So $x_1 < x_2$

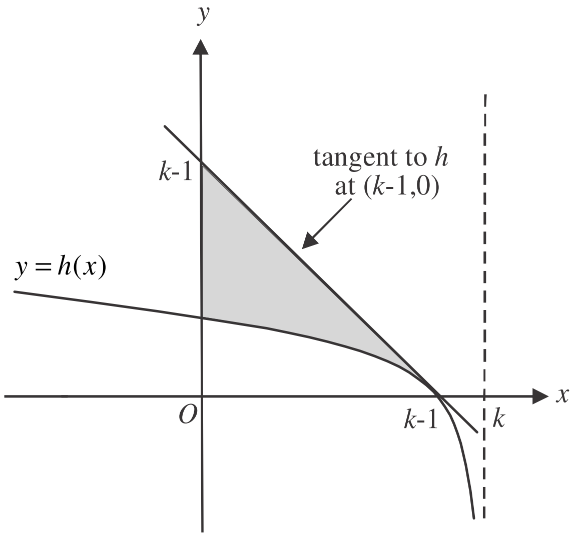
(1 mark)

iii. Having already defined h, use your CAS to find the equation of the tangent ie tangent line$(h(x), x, k-1)$ gives $y = -x + k - 1$.

(1 mark)

iv. Method 1

From part **iii**, the equation of the tangent is $y = -x + k - 1$.



The gradient of the tangent is -1 so it will cut the y-axis at $(0, k-1)$.
Alternatively, the y-intercept of this tangent occurs when $x = 0$, so $y = k - 1$.

$$\text{So } A(k) = \frac{1}{2} \times (k-1)(k-1) - \int_0^{k-1} h(x)dx$$ **(1 mark)**

$$= \frac{1}{2}(k^2 - 2k + 1) - (k\log_e(k) - k + 1)$$

$$= \frac{1}{2}k^2 - k + \frac{1}{2} - k\log_e(k) + k - 1 \quad \text{(using CAS)}$$

$$= \frac{k^2}{2} - \frac{1}{2} - k\log_e(k) \quad \text{as required.}$$ **(1 mark)**

Method 2

$$A(k) = \int_0^{k-1} ((-x + k - 1) - h(x))dx$$ **(1 mark)**

$$= \frac{(k-1)(k+1)}{2} - k\log_e(k) \quad \text{(using CAS)}$$

$$= \frac{1}{2}(k^2 - 1) - k\log_e(k)$$

$$= \frac{k^2}{2} - \frac{1}{2} - k\log_e(k) \quad \text{as required}$$

(1 mark)

v. Solve $A(k) = \int_0^{k-1} h(x)dx$ for k. **(1 mark)**

$k = 1$ or $k = 5.1156...$ but $k > 1$ so reject $k = 1$. **(1 mark)**

So $k = 5.12$ (correct to 2 decimal places).

(1 mark)

c. **i.** $h(x) = \log_e(k - x)$

Let $y = \log_e(k - x)$

Swap x and y for inverse.

$x = \log_e(k - y)$

$y = k - e^x$ (using CAS)

So $h^{-1}(x) = k - e^x$ **(1 mark)**

From the graph of h shown earlier, $r_h = R$ so $d_{h^{-1}} = R$.

(1 mark)

ii. Method 1

The tangent to h at the point $(k-1, 0)$ has equation $y = -x + k - 1$ (from part **b. iii.**).

The gradient of this tangent is -1.

Now $h^{-1}(x) = k - e^x$

so $h^{-1}{}'(x) = -e^x$

When $-e^x = -1$,

$x = 0$ **(1 mark)**

So $p = 0$.

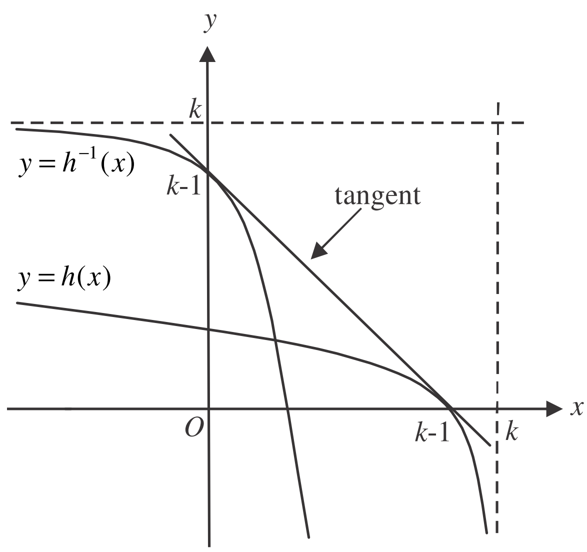
Since $h^{-1}(0) = k - e^0$

$= k - 1$

So $q = k - 1$

(1 mark)

Method 2 – graphically



The graph of h^{-1} is a reflection of the graph of h in the line $y = x$.

So the tangent to h at $(k-1, 0)$ will be a tangent to h^{-1} at $(0, k-1)$. **(1 mark)**

So $p = 0$ and $q = k - 1$ (i.e. answer the question!).

(1 mark)