YEAR 12 *Trial Exam Paper*

2017

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the exam

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Question 1a.

Worked solution

Using the quotient rule gives

$$\frac{dy}{dx}=\frac{(x^3-3x)\cos(x)-\left(3x^2-3\right)\sin(x)}{\left(x^3-3x\right)^2}$$

Mark allocation: 2 marks

- 1 method mark for attempting to use the quotient rule
- 1 answer mark for correct solution

Tips

- *You could also use the product rule to find this derivative.*
- *You do not need to simplify the answer further.*

Question 1b.

Worked solution

Using the product and chain rule gives

$$f'(x)=3x^2e^{1-4x}-4x^3e^{1-4x}$$

And then evaluate with $x=-1$, which gives $f'(-1)=3e^5+4e^5=7e^5$.

Mark allocation: 2 marks

- 1 answer mark for correct derivative
- 1 answer mark for stating $f'(-1)=7e^5$

Tips

- *Don't forget to evaluate.*
- *Simplify your answer to get full marks.*

Question 2a.

Worked solution

Using the chain rule gives

$$f'(x) = \frac{1}{2}(5-4x)^{\frac{-1}{2}} \times -4$$
$$= \frac{-2}{\sqrt{5-4x}}$$

Mark allocation: 1 mark

- 1 answer mark for correct solution in simplified form

Question 2b.

Worked solution

At $x = -1$, $\frac{dy}{dx} = \frac{-2}{3}$, and $y = 5$.

So the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$
$$y - 5 = \frac{-2}{3}(x+1)$$
$$y = \frac{-2x}{3} + \frac{13}{3}$$
$$3y + 2x = 13$$
$$2x + 3y = 13$$

Mark allocation: 3 marks

- 1 method mark for finding $m = \frac{-2}{3}$
- 1 method mark for finding $y = 5$ when $x = -1$
- 1 answer mark for correct equation, in the correct form

Tips

- *This question could also be solved using the equation* y= mx + c

Question 3a.

Worked solution

To find the x-intercept, let $f(x) = 0$

$$0 = 3 - \frac{4}{2x+1}$$
$$3 = \frac{4}{2x+1}$$
$$2x+1 = \frac{4}{3}$$
$$2x = \frac{1}{3}$$
$$x = \frac{1}{6}$$

To find the y-intercept, let $x = 0$

$$y = 3 - \frac{4}{0+1} = 3 - 4$$
$$y = -1$$

Find the asymptotes:

Method 1

To find the asymptotes, rewrite the equation as $f(x) = \dfrac{-4}{2\left(x+\frac{1}{2}\right)} + 3.$

In this form, by inspection, the equations of the asymptotes can be identified as $y = 3$ and $x = -\dfrac{1}{2}$.

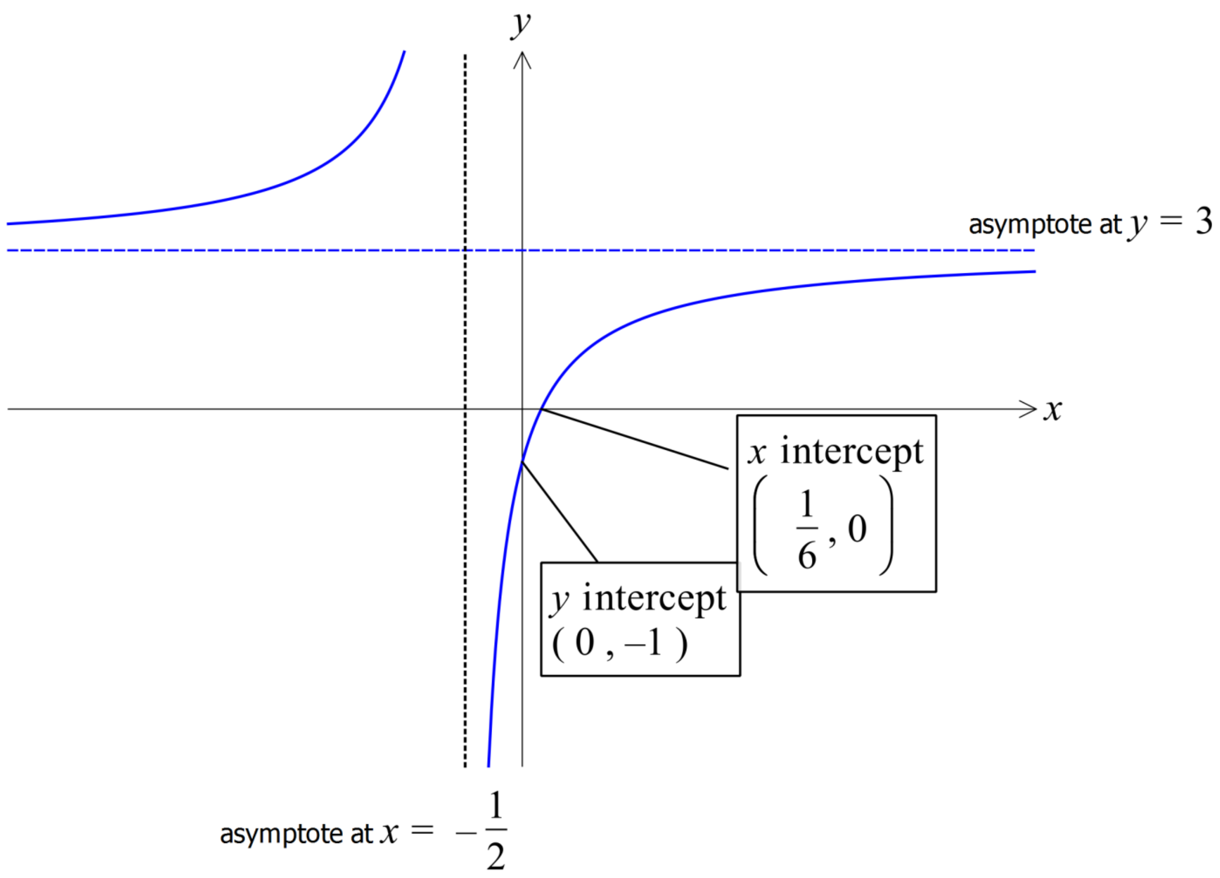
Alternatively,

Method 2

As $x \to +\infty,\ f(x) \to 3$ and as $x \to -\infty,\ f(x) \to 3$, therefore the horizontal asymptote is at $y = 3$.

Function is undefined when the denominator is equal to zero, so when $2x+1=0,\ x = -\dfrac{1}{2}$.

Therefore vertical asymptote is at $x = \dfrac{-1}{2}$.



Mark allocation: 3 marks

- 1 mark for shape; must be a negative hyperbola with asymptotic behaviour
- ½ mark for each correctly labelled intercepts and equations of asymptotes (up to 2 marks)

Tip

- *Make sure asymptotes are marked in with dotted lines.*

Question 3b.

Worked solution

$$\begin{aligned}\text{Area} &= \int_1^3 3-\frac{4}{2x+1}\,dx\\ &= \left[3x-2\log_e(2x+1)\right]_1^3\\ &= \left(9-2\log_e(7)\right)-\left(3-2\log_e(3)\right)\\ &= 6-2\log_e\left(\frac{7}{3}\right)\\ &= 6+2\log_e\left(\frac{3}{7}\right)\\ &= 6+\log_e\left(\frac{9}{49}\right)\end{aligned}$$

So $a=6$, $b=7$ and $c=49$.

Mark allocation: 3 marks

- 1 method mark for setting up the integral
- 1 method mark for correctly antidifferentiating
- 1 answer mark for the correct area in the correct form and stating the values of a, b and c

Tip

- *Make sure you write your answer for the area in the given form so that you can correctly state the values of* a, b *and* c.

Question 4a.

Worked solution

$\Pr(\text{germinating}) = \frac{1}{4}$, therefore $\Pr(\text{not germinating}) = \frac{3}{4}$

Assuming that each seed germinates or not independently of any other, then

$\Pr(\text{the 6 seeds fail to germinate}) = \left(\frac{3}{4}\right)^6$

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 4b.

Worked solution

Let X be the random variable '*Number of seeds that germinate*'.

$X \sim \text{Binomial}\left(p = \frac{1}{4}, n = 6\right)$

$$\begin{aligned}\Pr(X = 2) &= \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 \\ &= \frac{15 \times 81}{4096} \\ &= \frac{1215}{4096}\end{aligned}$$

So $k = 1215$

Mark allocation: 2 marks

- 1 method mark for knowing to use $\binom{6}{2}$
- 1 answer mark for correct value of k

Question 4c.

Worked solution

Four pots of six seeds, therefore 24 seeds in total.

We require each seed not to germinate and for this to happen on 24 occasions, so $\left(\frac{3}{4}\right)^{24}$.

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 5a.

Worked solution

$$f'(x) = \frac{-1}{(x+2)^2} + \frac{1}{(4-x)^2}$$

Mark allocation: 1 mark

- 1 answer mark for correct derivative

Tip

- *You must use the correct notation* $\mathrm{f}'(\mathrm{x})$.

Question 5b.

Worked solution

Let $f'(x) = 0$

$$\Rightarrow \frac{1}{(x+2)^2} = \frac{1}{(4-x)^2}$$

$$x^2 + 4x + 4 = 16 - 8x + x^2$$
$$12x = 12$$

Therefore, $x = 1$, $f(1) = \frac{2}{3}$.

Check the endpoints: at $x = -1$, $f(-1) = \frac{6}{5}$

and at $x = 2$, $f(2) = \frac{3}{4}$.

So range is $\left[\frac{2}{3}, \frac{6}{5}\right]$

Mark allocation: 2 marks

- 1 method mark for finding the values of $f(-1)$ and $f(2)$
- 1 answer mark for correct range

Tip

- *Always check endpoints and maximum and minimum points when finding the range of a restricted function.*

Question 5c.

Worked solution

$$f(x) = \frac{1}{x+2} + \frac{1}{4-x}$$
$$= \frac{4-x}{(x+2)(4-x)} + \frac{x+2}{(x+2)(4-x)}$$
$$= \frac{6}{(x+2)(4-x)}$$
$$= \frac{-6}{(x+2)(x-4)}$$
$$= \frac{-6}{x^2-2x-8}$$

So $k = -6$.

Mark allocation: 2 marks

- 1 method mark
- 1 answer mark for $k = -6$

Question 5d.

Worked solution

For the inverse function to exist, the graph needs to be one-to-one. For the graph to be one-to-one, restrict the domain of the graph at the turning point.
So $a = 1$.

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 5e.

Worked solution

Let $x = \dfrac{-6}{y^2 - 2y - 8}$, where $y = g^{-1}(x)$.

$$x = \frac{-6}{(y-1)^2 - 9}$$

$$(y-1)^2 - 9 = \frac{-6}{x}$$

$$(y-1)^2 = 9 - \frac{6}{x}$$

$$y - 1 = \pm\sqrt{9 - \frac{6}{x}}$$

$$y = 1 \pm \sqrt{9 - \frac{6}{x}}$$

Since dom $g(x) = [-1,1]$ and dom $g(x) = \text{ran } g^{-1}(x)$, only the negative square root is required.

$$g^{-1}(x) = 1 - \sqrt{9 - \frac{6}{x}}$$

Mark allocation: 2 marks

- 1 method mark for swapping x and y and for forming a completed square, $(y-1)^2 - 9$.
- 1 answer mark for correct inverse

Tip

- *Ensure you use the correct notation,* g^{-1}, *and give consideration to the domain.*

Question 6a.

Worked solution

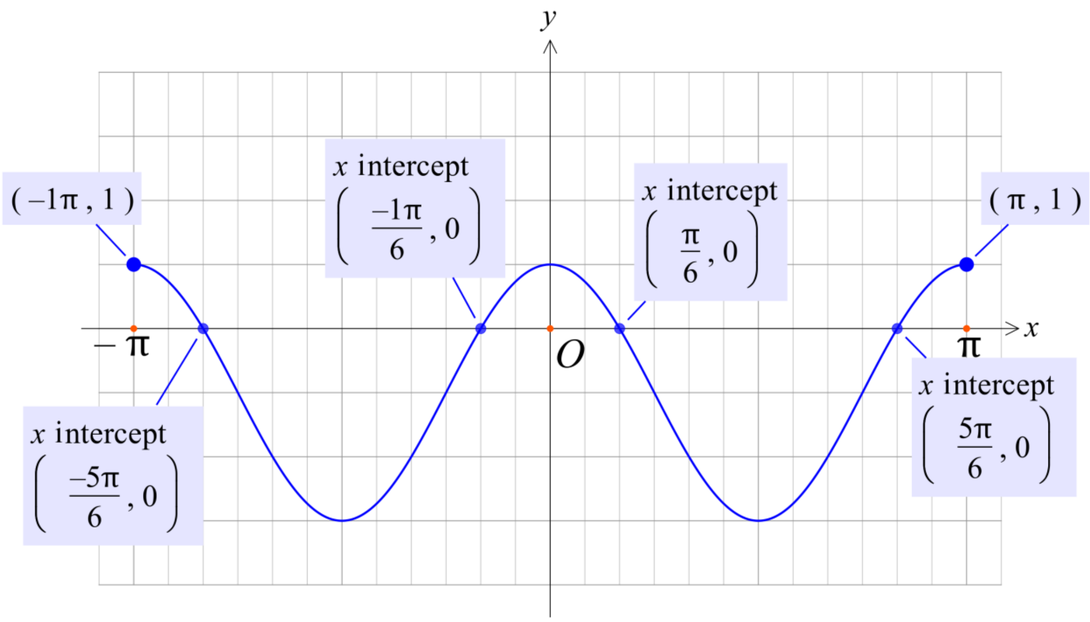
$$2\cos(2x) - 1 = 0$$
$$2\cos(2x) = 1$$
$$\cos(2x) = \frac{1}{2}, \text{ ref angle is } \frac{\pi}{3}$$
$$2x = \frac{-5\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$$
$$x = \frac{-5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Mark allocation: 2 marks

- 1 method mark for the ref angle of $\frac{\pi}{3}$
- 1 answer mark for all four angles correct

Question 6b.

Worked solution



Mark allocation: 3 marks

- 1 method mark for a correct cos graph showing two cycles and an amplitude of 2
- 1 mark for endpoints labelled correctly
- 1 mark for intercepts labelled correctly

Tip

- *Remember to label all points as coordinates.*

Question 6c.

Worked solution

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

$$= \frac{1}{\frac{\pi}{6}+\frac{\pi}{4}}\int_{\frac{-\pi}{4}}^{\frac{\pi}{6}} (2\cos(2x)-1)\ dx$$

$$= \frac{24}{10\pi}\left[\sin(2x)-x\right]_{\frac{-\pi}{4}}^{\frac{\pi}{6}}$$

$$= \frac{12}{5\pi}\left[\left(\sin\left(\frac{\pi}{3}\right)-\frac{\pi}{6}\right)-\left(\sin\left(\frac{-\pi}{2}\right)+\frac{\pi}{4}\right)\right]$$

$$= \frac{12}{5\pi}\left[\frac{\sqrt{3}}{2}-\frac{\pi}{6}+1-\frac{\pi}{4}\right]$$

$$= \frac{12}{5\pi}\left[\frac{\sqrt{3}}{2}+1-\frac{5\pi}{12}\right]$$

Mark allocation: 3 marks

- 1 method mark for setting up the integral for the average value
- 1 method mark for correctly antidifferentiating $2\cos(2x)$
- 1 answer mark for correct value

Question 7a.

Worked solution

$$
\begin{aligned}
y &= x\log_e(3x) - x \\
\frac{dy}{dx} &= x\frac{1}{x} + \log_e(3x) - 1 \\
&= 1 + \log_e(3x) - 1 \\
&= \log_e(3x)
\end{aligned}
$$

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 7b.

Worked solution

$$
\begin{aligned}
\text{Area} &= \int_{\frac{1}{3}}^{\frac{e}{3}} \log_e(3x)\ dx \\
&= \left[x\log_e(3x) - x\right]_{\frac{1}{3}}^{\frac{e}{3}} \\
&= \frac{e}{3}\log_e(e) - \frac{e}{3} - \log_e(1) + \frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

Mark allocation: 2 marks

- 1 method mark for correctly replacing $\int \log_e(3x)\ dx$ with $x\log_e(3x) - x$
- 1 answer mark for correct solution

Question 8

Worked solution

As $f(x)$ is a probability density function, $\int_0^1 (ax^2 - bx^3)\ dx = 1.$

So

$$\left[\frac{ax^3}{3} - \frac{bx^4}{4}\right]_0^1 = \frac{a}{3} - \frac{b}{4} = 1$$

$$\Rightarrow\ 4a - 3b = 12 \qquad [equation\ \ 1]$$

And with the turning point at $x = \frac{3}{4}$, then $f'(x) = 0$ at $x = \frac{3}{4}$

So $f'(x) = 2ax - 3bx^2$, $f'\left(\frac{3}{4}\right) = \frac{6a}{4} - \frac{27b}{16} = 0.$

This gives

$$\frac{3a}{2} = \frac{27b}{8}$$

$$24a = 27b \quad [equation\ 2]$$

Therefore, multiplying equation 1 by 6 gives $24a - 18b = 72.$

$$27b - 18b = 72$$
$$9b = 72$$
$$b = 8,\quad a = 9$$

Mark allocation: 3 marks

- 1 method mark for setting up the probability density function equal to 1
- 1 method mark for finding the derivative and setting it equal to zero for $x = \frac{3}{4}$
- 1 answer mark for solving simultaneous equations to get $a = 9$ and $b = 8$