

YEAR 12 *Trial Exam Paper*

2017

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the exam

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SECTION A – Multiple-choice questions

Question 1

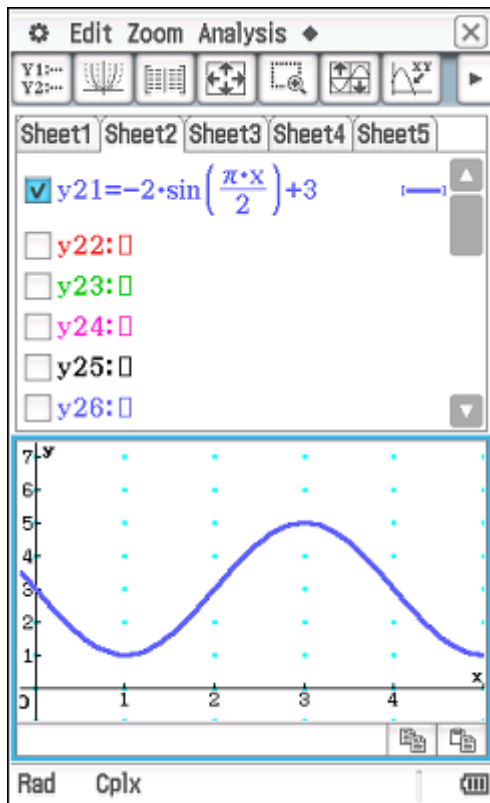
Answer: C

Explanatory notes

The period is given by $\frac{2\pi}{n} = \frac{2\pi}{\frac{\pi}{2}} = 4$.

The median is 3 and the amplitude is 2. Hence, the range is $3 - 2$ to $3 + 2$; that is, $[1, 5]$.

This can be readily checked by sketching the graph using CAS.



Question 2**Answer: B****Explanatory notes**

Using CAS to obtain the inverse function gives the inverse function as $y = \frac{1}{x^2} + 3$.

And since the range of the original becomes the domain of the inverse, the domain is R^+ .

So the answer is option B.

The screenshot shows a CAS calculator window titled "Edit Action Interactive". The main display area contains the command `solve(x = 1/sqrt(y-3), y)` and the resulting solution $\left\{ y = \frac{1}{x^2} + 3 \right\}$. Below the display is a keypad with various mathematical functions and symbols. The keypad is organized into rows: Math1 (Line, fraction, square root, pi, right arrow), Math2 (square, e, ln, log base, nth root), Math3 (absolute value, x^2, x^-1, log base 10, solve), Trig (arcsin, arccos, arctan, toDMS, degrees, radians), Var (sin, cos, tan, degrees symbol, radians symbol), abc (left arrow, copy, paste, ans, EXE). At the bottom, there are mode selection buttons for Alg, Decimal, Real, and Rad, along with a calculator icon.

**Tip**

- *You could also use the draw inverse function on the calculator to sketch the inverse.*

Question 3**Answer: C****Explanatory notes**

The roots (x -intercepts) are at $x = b$, $x = c$ and $x = d$, with a repeated root at $x = d$.

So, the factors are $(x - b)$, $(x - c)$ and $(x - d)^2$.

**Tip**

- Don't be confused by the fact that the roots at b and c are negative.

Question 4**Answer: B****Explanatory notes**

Using CAS to find the equation of the tangent line to the curve at $x = -2$ gives $y = 12x + 16$, whereby $(-1, 4)$ satisfies this equation.

The screenshot shows a CAS interface with the following elements:

- Top bar: Edit Action Interactive
- Toolbar: $\frac{1}{2}$, $\frac{1}{x}$, $\frac{d}{dx}$, $\frac{d}{dx}$, Simp, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$
- Input field: $\text{tanLine}(x^3, x, -2)$
- Output field: $12 \cdot x + 16$
- Math palette:

Math1	Line	$\frac{d}{dx}$	$\sqrt{\square}$	π	\rightarrow
Math2	\square^{\square}	e^{\square}	ln	\log_{\square}	$\sqrt[\square]{\square}$
Math3	$ \square $	x^2	x^{-1}	$\log_{10}(\square)$	solve(
Trig	$\square \square$	toDMS	{	}	()
Var	sin	cos	tan	$^{\circ}$	r
abc					
- Bottom bar: Alg, Decimal, Real, Rad, $\frac{d}{dx}$

Question 5**Answer: B****Explanatory notes**

Reflect the graph in the line $y = x$ to get the graph of the inverse and then reflect in the x -axis to get the graph of $y = -f^{-1}(x)$.

Question 6**Answer: A****Explanatory notes**

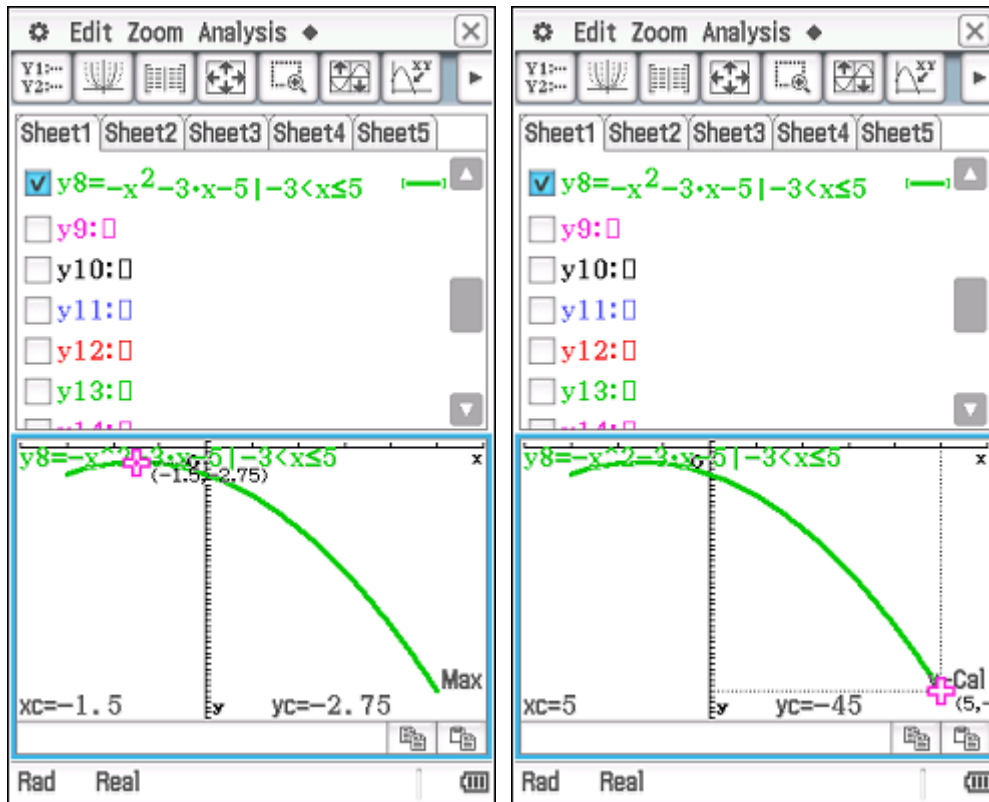
Using CAS gives

The screenshot shows a CAS interface with the following elements:

- Input field: $\text{solve}(x^3 - a \cdot x^2 + 6 \cdot x - 7 = -3 \mid x=2)$
- Result: $\{a=4\}$
- Math1 row: Line, $\frac{\square}{\square}$, $\sqrt{\square}$, π , \rightarrow
- Math2 row: Define, f , g , i , ∞
- Math3 row: solve(, dSlv, ', $\left\{ \begin{matrix} \square \\ \square \end{matrix} \right\}$, $|$
- Trig row: $<$, $>$, $()$, $\{ \}$, $[]$
- Var row: \leq , \geq , $=$, \neq , $<$
- abc row: \leftarrow , \leftarrow , \leftarrow , ans, EXE
- Bottom row: Alg, Decimal, Real, Rad, $\left(\frac{\square}{\square} \right)$

Question 7**Answer: D****Explanatory notes**

Sketching the graph using CAS shows that the range is from the right endpoint to the maximum turning point.

**Question 8****Answer: D****Explanatory notes**

For the average value of the function to be zero, the area bounded by the curve and the x -axis over the interval $[-p, 0]$ has to equal $\frac{81}{8}$.

Therefore

$$\frac{1}{2}p^2 = \frac{81}{8}$$

$$p^2 = \frac{81}{4}$$

$$p = \frac{9}{2}$$

Question 9**Answer: D****Explanatory notes**

To be a probability density function, the value of a is 14 as the area under the curve must be 1.

Because the function is symmetrical, $E(X)$ occurs in the centre, so $E(X) = 11$.

**Tip**

- Remember to look for symmetry.

Question 10**Answer: E****Explanatory notes**

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

$$\frac{1}{625} = \sqrt{\frac{\frac{1}{5} \times \frac{4}{5}}{n}}$$

The screenshot shows a TI-84 Plus calculator interface. The top menu bar includes 'Edit', 'Action', and 'Interactive'. Below the menu bar are various function keys like '0.5 1/2', '1/2', 'f(x)', 'Simp', 'f(x)', and 'v'. The main display area shows the equation $\text{solve}\left(\frac{1}{625} = \sqrt{\frac{\frac{1}{5} \cdot \frac{4}{5}}{x}}, x\right)$ and the solution $\{x=62500\}$. Below the display is a keypad with rows for 'Math1', 'Math2', 'Math3', 'Trig', 'Var', and 'abc'. The bottom row of the keypad includes '←', '→', 'ans', and 'EXE'. At the very bottom, there are mode selection buttons for 'Alg', 'Standard', 'Real', 'Rad', and a 'mode' button.

Question 11**Answer: D****Explanatory notes**The 95% confidence interval for p is

$$\left(\hat{p} - 1.96 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

**Tip**

- The confidence intervals for 90% and 95% are worth remembering:

$$90\% \quad \left(\hat{p} - 1.65 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.65 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

$$95\% \quad \left(\hat{p} - 1.96 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

Question 12**Answer: D****Explanatory notes**

$$np = 3 \text{ and } npq = \frac{9}{4}.$$

$$\Rightarrow 3q = \frac{9}{4}$$

$$\text{So } q = \frac{3}{4}$$

$$\therefore p = \frac{1}{4}$$

$$\text{Hence, } \Pr(X = 1) = \binom{12}{1} (p)^1 (q)^{11} = 12 \left(\frac{1}{4} \right)^1 \left(\frac{3}{4} \right)^{11}.$$

Question 13**Answer: A****Explanatory notes**

$$\begin{aligned}\Pr(X < 46) &= \Pr\left(Z < \frac{46 - 60}{7}\right) \\ &= \Pr(Z < -2) \\ &= \Pr(Z > 2), \text{ using symmetry}\end{aligned}$$

Question 14**Answer: D****Explanatory notes**

$y = e^{2x+4} - 3$ can be written as $y + 3 = e^{2x+4}$ and therefore $y' = y + 3$ and $x' = 2x + 4$.

So $y = y' - 3$ and $x = \frac{1}{2}(x' - 4) = \frac{1}{2}x' - 2$. The matrix equation that corresponds to these equations is option A.

Question 15**Answer: D****Explanatory notes**

Using the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{f(2x)}} \times \frac{1}{2\sqrt{f(2x)}} \times f'(2x) \times 2 \\ &= \frac{f'(2x)}{f(2x)}\end{aligned}$$

Or, using logarithmic laws, the equation for y can be written as $y = \frac{1}{2} \log_e(f(2x))$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{1}{2} \times \frac{1}{f(2x)} \times f'(2x) \times 2 \\ &= \frac{f'(2x)}{f(2x)}\end{aligned}$$

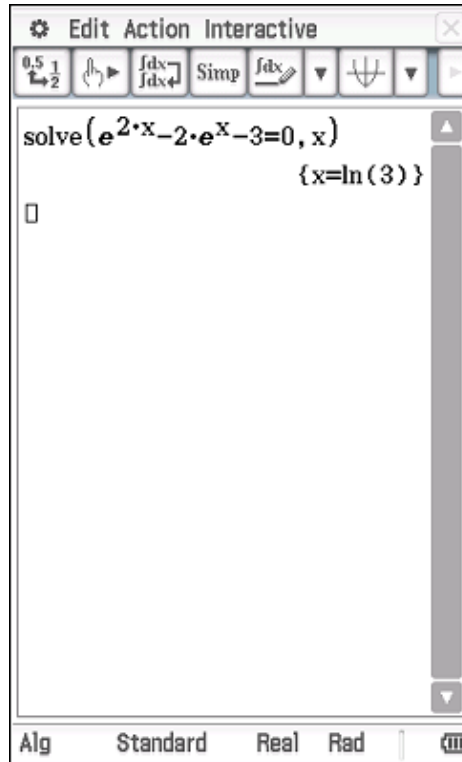
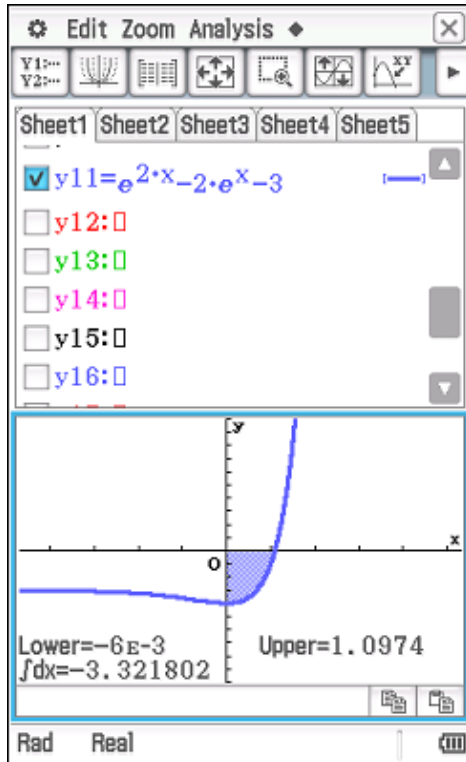
Question 18

Answer: C

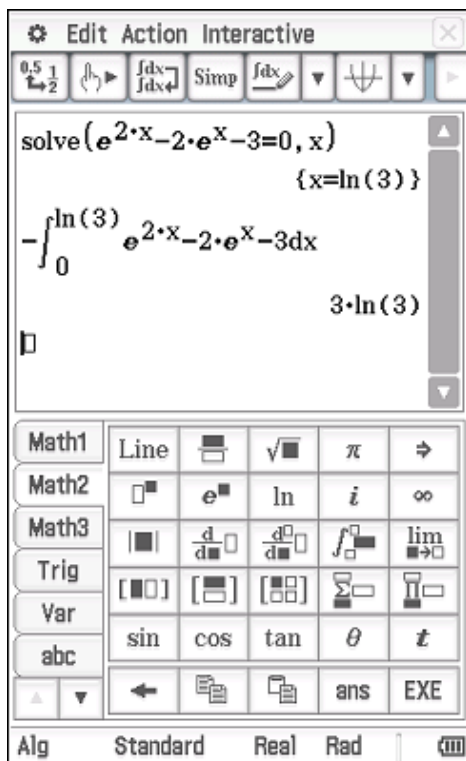
Explanatory notes

Using CAS, the area required can be seen as the shaded region (shown left).

Use the main menu to then find the exact value of the x -intercept (shown right).



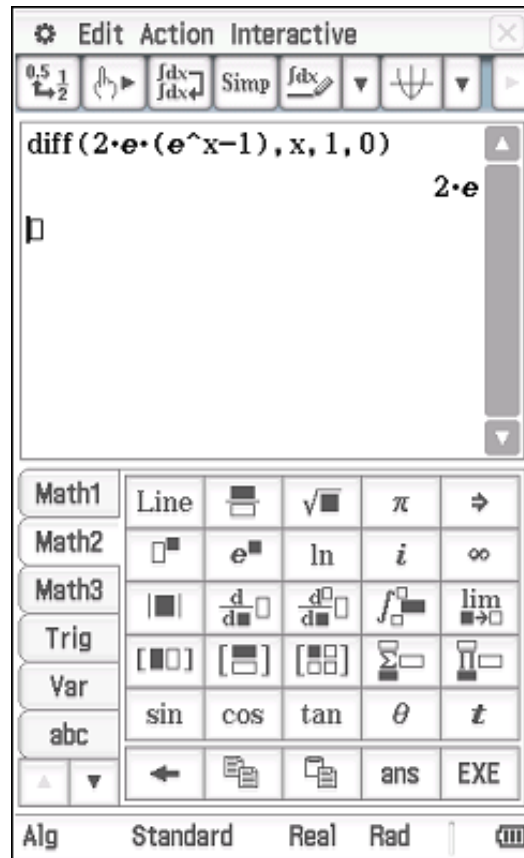
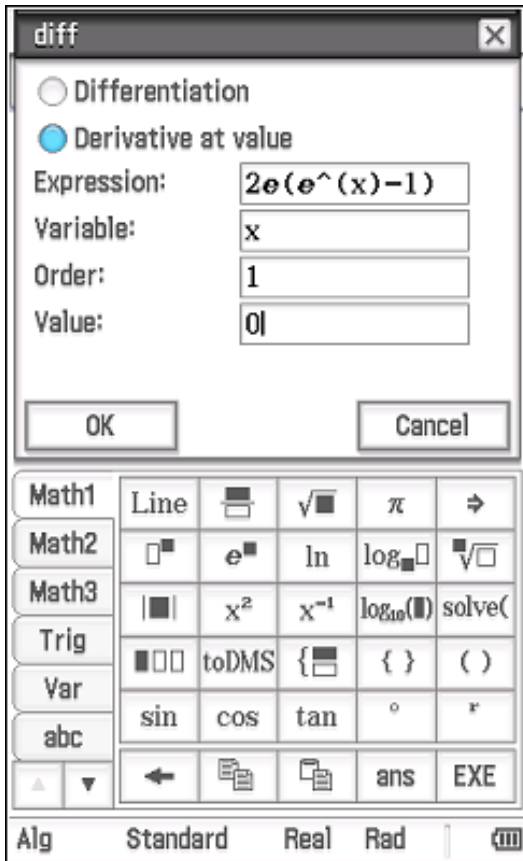
Hence, the area is



Question 19**Answer: D****Explanatory notes**

The rate of change means $\frac{dy}{dx}$.

Using CAS gives

**Question 20****Answer: C****Explanatory notes**

The graph drawn is the graph of the derivative; that is, the graph of the gradient function. The graph has a positive value over this domain and it does not have an x -intercept or change signs (therefore there are no turning points and stationary points of inflexion over this interval).

SECTION B**Question 1a.****Worked solution**

Maximum height is 4 metres.

Mark allocation: 1 mark

- 1 answer mark

Question 1b.**Worked solution**

The gradient function is given by $\frac{dy}{dx} = 2 \sin\left(\frac{\pi x}{3}\right) \times \frac{\pi}{3} = \frac{2\pi}{3} \sin\left(\frac{\pi x}{3}\right)$.

The range of this function is $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$, so the gradient is always less than or equal to $\frac{2\pi}{3}$.

Mark allocation: 1 mark

- 1 mark for finding the range of the derivative

Question 1c.**Worked solution**

Using CAS gives

The screenshot shows a CAS calculator interface. At the top, the integral $\int_0^6 2 - 2 \cdot \cos\left(\frac{\pi \cdot x}{3}\right) dx$ is entered. Below the input, the result '12' is displayed. The interface includes a menu with categories: Math1, Math2, Math3, Trig, Var, abc, and a bottom row with navigation and function keys. The bottom status bar shows 'Alg', 'Decimal', 'Real', 'Rad', and a calculator icon.

So the area is 12 square metres.

Mark allocation: 1 mark

- 1 mark for correct answer

**Tip**

- *Be guided by the number of marks allocated for the question. In this case there is 1 mark, so there is no need to show any working; just use CAS to find the answer.*

Question 1d.**Worked solution**

Two-thirds of the area is 8 and we want half of that; that is, 4 square units, to be between 3 and $3+c$.

So set up the integral and use CAS to find c .

The screenshot shows a CAS interface with the following elements:

- Toolbar:** Includes icons for fraction conversion (0.5, 1/2), undo, redo, integration (∫dx), simplification (Simp), differentiation (d/dx), and other mathematical functions.
- Input Area:** Contains the equation $\text{solve}\left(\int_3^y 2 - 2 \cdot \cos\left(\frac{\pi \cdot x}{3}\right) dx = 4, y\right)$.
- Output Area:** Displays the solution $\{y=4.119773379\}$.
- Math Palette:** A grid of mathematical symbols and functions categorized by Math1, Math2, Math3, Trig, Var, and abc.
- Mode Selector:** At the bottom, it shows 'Alg', 'Decimal', 'Real', 'Rad', and a calculator icon.

Hence, $c = 1.12$.

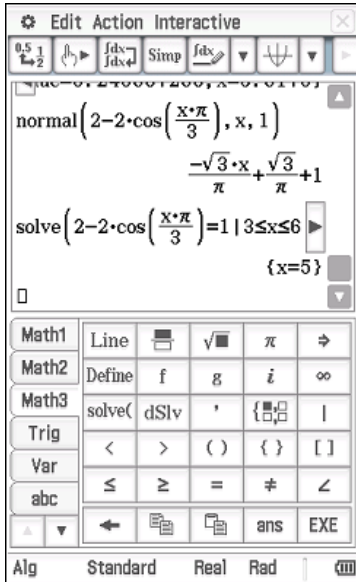
Mark allocation: 2 marks

- 1 method mark for setting an integral equal to either 8 or 4
- 1 answer mark for the correct value of c

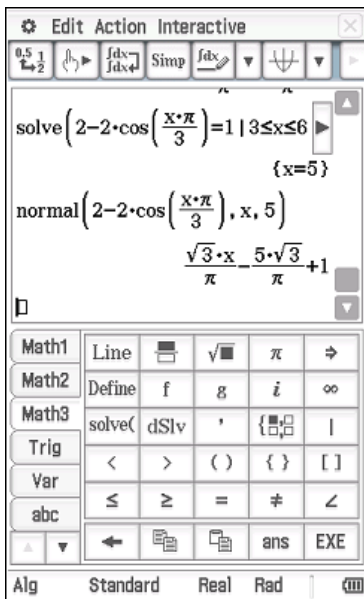
Question 1e.**Worked solution**

AB is the line drawn normal to the curve at a point when $y = 1$.

First, solve $2 - 2\cos\left(\frac{\pi x}{3}\right) = 1$ for $x \in [3, 6]$ for x , to get $x = 5$.



Then, use CAS to find the equation of the normal line drawn at $x = 5$.



The equation of the normal is $y = \frac{\sqrt{3}x}{\pi} - \frac{5\sqrt{3}}{\pi} + 1$.

Mark allocation: 2 marks

- 1 method mark for setting $2 - 2\cos\left(\frac{\pi x}{3}\right) = 1$ for $x \in [3, 6]$ to get $x = 5$
- 1 answer mark for the equation of the normal

Question 1f.**Worked solution**

$$\frac{dy}{dx} = \frac{2\pi}{3} \sin\left(\frac{\pi x}{3}\right)$$

$$\text{At } x = a, \quad \frac{dy}{dx} = \frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)$$

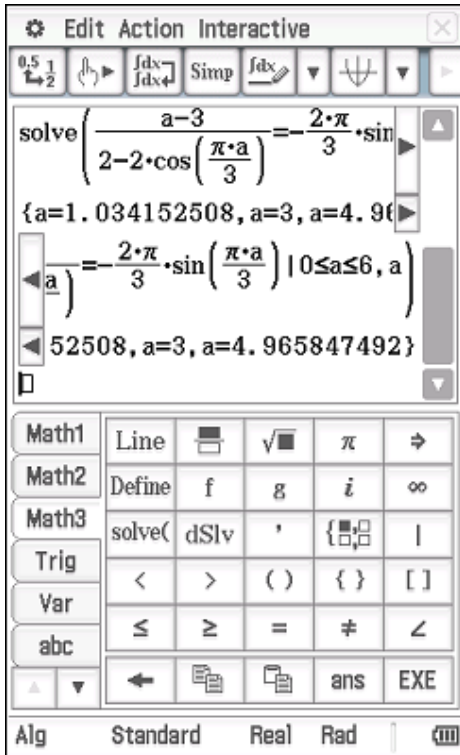
$$\Rightarrow \frac{dy}{dx}(\text{normal}) = \frac{-1}{\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)}$$

Gradient of normal line passing through $(3,0)$ and $\left(a, 2 - 2\cos\left(\frac{\pi a}{3}\right)\right)$ is $\frac{2 - 2\cos\left(\frac{\pi a}{3}\right)}{a - 3}$.

$$\text{So } \frac{2 - 2\cos\left(\frac{\pi a}{3}\right)}{a - 3} = \frac{-1}{\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)}$$

$$\frac{a - 3}{2 - 2\cos\left(\frac{\pi a}{3}\right)} = -\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)$$

Solving this using CAS gives $a = 4.966$.



Mark allocation: 3 marks

- 1 method mark for getting the gradient of the normal line $\frac{dy}{dx}(\text{normal}) = \frac{-1}{\frac{2\pi}{3}\sin\left(\frac{\pi a}{3}\right)}$
- 1 method mark for setting it equal to $\frac{-1}{f'(a)}$
- 1 answer mark for finding $a = 4.966$

Question 1g.**Worked solution**

Let the longest strut be located at $x = a$, $y = 2 - 2 \cos\left(\frac{\pi a}{3}\right)$.

$$\frac{dy}{dx}(x = a) = \frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)$$

$$\therefore m_{\text{normal}} = \frac{-1}{\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)}$$

The equation of the normal is $y - \left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right) = \frac{-1}{\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)}(x - a)$.

To find the x -intercept, let $y = 0$.

$$\Rightarrow -\left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right) = \frac{-1}{\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right)}(x - a)$$

$$\Rightarrow x_{\text{int}} = a + \frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right) \left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right)$$

Strut extends from $(x_{\text{int}}, 0)$ to $\left(a, 2 - 2 \cos\left(\frac{\pi a}{3}\right)\right)$.

Length of strut is $d = \sqrt{(x_{\text{int}} - a)^2 + \left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right)^2}$.

$$d = \sqrt{\left[\frac{2\pi}{3} \sin\left(\frac{\pi a}{3}\right) \left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right)\right]^2 + \left(2 - 2 \cos\left(\frac{\pi a}{3}\right)\right)^2}$$

Using CAS to find the maximum value of this function gives

$$\frac{d}{dx} \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)$$

$$\frac{2 \cdot \sin \left(\frac{\pi x}{3} \right) \cdot \pi}{3}$$

$$fMax \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

$$\{MaxValue=6.249607266, x=3.9179\}$$

$$\frac{d}{dx} \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)$$

$$\frac{2 \cdot \sin \left(\frac{\pi x}{3} \right) \cdot \pi}{3}$$

$$fMax \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

$$\{MaxValue=6.249607266, x=3.9179\}$$

$d_{\max} = 6.2496$ at $a = 3.9179$, so $x = 3.9179$.

So the longest strut is 6.2496 metres in length and is positioned at (3.9179, 3.1450).

$$fMin \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

$$\{MinValue=0, x=6\}$$

$$fMax \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

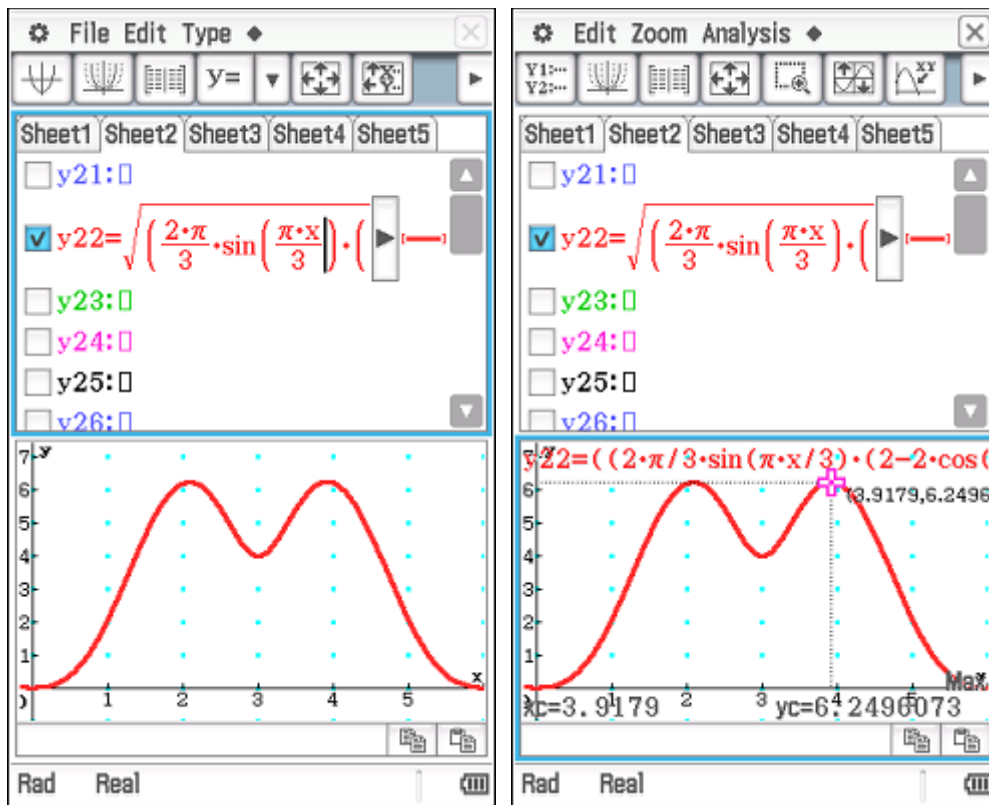
$$\{MaxValue=6.249607266, x=3.9179\}$$

$$fMin \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

$$\{MinValue=0, x=6\}$$

$$fMax \left(\sqrt{\left(\frac{2 \cdot \pi}{3} \cdot \sin \left(\frac{\pi x}{3} \right) \right) \cdot \left(2 - 2 \cdot \cos \left(\frac{\pi x}{3} \right) \right)} \right)$$

$$\{MaxValue=6.249607266, x=3.9179\}$$



Mark allocation: 3 marks

- 1 method mark for finding the equation of the normal line at $x = a$,

$$y - \left(2 - 2 \cos \left(\frac{\pi a}{3} \right) \right) = \frac{-1}{\frac{2\pi}{3} \sin \left(\frac{\pi a}{3} \right)} (x - a)$$

- 1 method mark for finding the function that gives the length of any strut drawn normal

$$\text{to the curve at } x = a, \quad d = \sqrt{\left[\frac{2\pi}{3} \sin \left(\frac{\pi a}{3} \right) \left(2 - 2 \cos \left(\frac{\pi a}{3} \right) \right) \right]^2 + \left(2 - 2 \cos \left(\frac{\pi a}{3} \right) \right)^2}$$

- 1 answer mark for maximum length of 6.2496 metres at (3.9179, 3.1450)

Question 2a.i.**Worked solution**

$$E(X) = 120$$

$$\begin{aligned} E(X) &= \int_0^a x \cdot \frac{2x}{a^2} dx \\ &= \int_0^a \frac{2x^2}{a^2} dx \\ &= \frac{1}{a^2} \left[\frac{2x^3}{3} \right]_0^a \\ &= \frac{1}{a^2} \cdot \frac{2a^3}{3} = \frac{2a}{3} \\ \Rightarrow \frac{2a}{3} &= 120, \quad a = 180 \end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for setting up the integral for $E(X)$
- 1 answer mark for the correct antiderivative and evaluation leading to $a = 180$

**Tip**

- *This is a 'show that' question, so appropriate working must be shown.*

Question 2a.ii.**Worked solution**

$$\Pr(X > 150) = \int_{150}^{180} \frac{2x}{180^2} dx$$

Using CAS gives $\Pr(X > 150) = \frac{11}{36}$.

The screenshot shows a CAS calculator interface. At the top, there is a title bar "Edit Action Interactive" and a toolbar with icons for undo, redo, simplify, and other functions. The main display area shows the integral expression $\int_{150}^{180} \frac{2 \cdot x}{180^2} dx$ and the result $\frac{11}{36}$. Below this, the same integral expression is shown again, and the decimal result 0.305555556 is displayed. At the bottom, there is a keypad with various mathematical symbols and functions, and a mode selector at the bottom.

Mark allocation: 2 marks

- 1 method mark for setting up the integral
- 1 answer mark for $\frac{11}{36}$

**Tip**

- *This question is worth 2 marks, so you must also provide a solution step.*

Question 2b.**Worked solution**

$$X_{\text{RF}} \sim N(\mu = 25, \sigma = 5)$$

Using CAS gives $\Pr(X_{\text{RF}} < 18) = 0.0808$.

The screenshot shows a CAS calculator interface with the following elements:

- Top bar: Edit Action Interactive
- Toolbar: $\frac{0.5}{2}$, $\int dx$, $\int dx$, Simp, $\int dx$, $\int dx$, $\int dx$
- Display area:
 - $\frac{11}{36}$
 - $\int_{150}^{180} \frac{2 \cdot x}{180^2} dx$
 - 0.3055555556
 - normCdf($-\infty, 18, 5, 25$)
 - 0.08075665923
 - \square
- Math palette:
 - Math1: Line, $\frac{\square}{\square}$, $\sqrt{\square}$, π , \rightarrow
 - Math2: \square^{\square} , e^{\square} , \ln , i , ∞
 - Math3: $\int \square$, $\frac{d}{d\square} \square$, $\frac{d}{d\square} \square$, $\int \square$, $\lim_{\square \rightarrow \square} \square$
 - Trig: $\square(\square)$, $\square(\square)$, $\square(\square)$, $\int \square$, $\int \square$
 - Var: \sin , \cos , \tan , θ , t
 - abc: \leftarrow , \square , \square , ans, EXE
- Bottom bar: Alg, Decimal, Real, Rad, \square

Mark allocation: 2 marks

- 1 method mark for $X_{\text{RF}} \sim N(\mu = 25, \sigma = 5)$
- 1 answer mark for 0.0808

**Tip**

- Be careful to define the distribution using a new variable. Something like X_{RF} is suitable.

Question 2c.**Worked solution**

Using symmetry, the mean is 24 mm.

$$X_{JJ} \sim N(\mu = 24, \sigma = ?), \text{ where } \Pr(X_{JJ} < 20) = 0.1.$$

Using CAS gives $\sigma = 3.12$.

The screenshot shows a CAS calculator interface with the following content:

- Top bar: Edit Action Interactive
- Toolbar: $\frac{0.5}{2}$, $\frac{1}{2}$, $\frac{f(x)}{g(x)}$, $\frac{f(x)}{g(x)}$, Simp, $\frac{f(x)}{g(x)}$, $\frac{f(x)}{g(x)}$, $\frac{f(x)}{g(x)}$
- Input area: $\int_{150}^{180} \frac{2 \cdot x}{180^2} dx$
- Output area:
 - 0.3055555556
 - normCdf($-\infty$, 18, 5, 25)
 - 0.08075665923
 - solve(normCdf($-\infty$, 20, x, 24))=
 - {x=3.121216584}
- Bottom panel: Math1, Math2, Math3, Trig, Var, abc, Alg, Decimal, Real, Rad, ANS, EXE

Mark allocation: 3 marks

- 1 answer mark for finding the mean
- 1 method mark for setting up an integral to find the standard deviation
- 1 answer mark for the correct standard deviation

**Tip**

- *Be careful to use a different notation for defining the distribution.*

Question 2d.i.**Worked solution**

$$\begin{aligned}
 \Pr(\text{JJ} \mid \text{length} < 18) &= \frac{\Pr(X_{\text{JJ}} < 18)}{\Pr(\text{JJ}) \times \Pr(X_{\text{JJ}} < 18) + \Pr(\text{RF}) \times \Pr(X_{\text{RF}} < 18)} \\
 &= \frac{0.7 \times 0.027235}{0.7 \times 0.027235 + 0.3 \times 0.0808} \\
 &= 0.440
 \end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for recognising conditional probability or for using a tree diagram and for finding $\Pr(X_{\text{JJ}} < 18) = 0.7 \times 0.027235$
- 1 method mark for having a denominator that involves 0.7 multiplied by one probability and 0.3 multiplied by another
- 1 answer mark for 0.440

Question 2d.ii.**Worked solution**

$$p = 0.7, n = 500$$

$$E(\hat{p}) = p = 0.7, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.7 \times 0.3}{500}}$$

$$\hat{P} \sim N(\mu = 0.7, \sigma = 0.0205)$$

$$\Pr(\hat{P} > 0.75) = 0.0073$$

The screenshot shows a TI-84 Plus calculator interface. The top status bar reads "Edit Action Interactive". The main display area shows the calculation of the standard deviation: $\sqrt{\frac{.7 * .3}{500}}$, resulting in 0.02049390153 . Below this, the normal cumulative distribution function is calculated: $\text{normCDF}(0.75, \infty, \sqrt{\frac{0.7 * 0.3}{500}})$, resulting in $7.348710885E-3$. The bottom of the screen shows the calculator's keypad with various mathematical functions and modes (Alg, Decimal, Real, Rad).

Mark allocation: 2 marks

- 1 mark for identifying parameters $\mu = 0.7, \sigma = 0.0205$
- 1 mark for answer $\Pr(\hat{P} > 0.75) = 0.0073$ (although if the rounded value for σ is used this will give $\Pr(\hat{P} > 0.75) = 0.0074$, so accept either 0.0073 or 0.0074)

Question 3a.i.**Worked solution**

Using CAS, the gradient of PA is $\frac{6}{a-2}$.

The screenshot shows a CAS interface with the following elements:

- Header:** Edit Action Interactive
- Toolbar:** Includes buttons for fraction conversion (0.5, 1/2), undo, redo, simplify (Simp), and other mathematical functions.
- Input Area:** The expression $\text{simplify}\left(\frac{-6 - \frac{6}{2-a}}{3-a}\right)$ is entered.
- Output Area:** The simplified result $\frac{6}{a-2}$ is displayed.
- Function Palette:** A grid of mathematical functions categorized by Math1, Math2, Math3, Trig, Var, and abc.

Math1	Line	$\frac{\square}{\square}$	$\sqrt{\square}$	π	\rightarrow
Math2	\square^{\square}	e^{\square}	\ln	\log_{\square}	$\sqrt[\square]{\square}$
Math3	$ \square $	x^2	x^{-1}	$\log_{10}(\square)$	$\text{solve}(\square)$
Trig	$\square \square \square$	toDMS	$\{\square\}$	$\{\}$	(\square)
Var	sin	cos	tan	$^{\circ}$	$^{\circ}$
abc	←	$\frac{\square}{\square}$	$\frac{\square}{\square}$	ans	EXE
- Mode Selection:** Alg, Decimal, Real, Rad, and a calculator icon.

Mark allocation: 1 mark

- 1 mark for answer in simplified form

Question 3b.i.**Worked solution**

Set up and evaluate using CAS to give an answer of 6.

The screenshot shows a CAS calculator interface with the following content:

$$\text{solve}\left(\frac{6}{a-2} = \frac{d}{dx}\left(\frac{6}{2-x}\right), x\right)$$

$$\{x = -\sqrt{a-2} + 2, x = \sqrt{a-2} + 2\}$$

$$\text{simplify}\left(-\int_3^{e+2} \frac{6}{2-x} dx\right)$$

The result of the integral is displayed as 6.

The calculator interface includes a toolbar with icons for fractions, differentiation, simplification, and integration. Below the main display is a keypad with categories: Math1 (Line, square, root, pi, arrow), Math2 (square, e, ln, i, infinity), Math3 (abs, d/dx, integral, limit), Trig (trig functions), Var (variables), abc (algebraic symbols), and a bottom row with navigation and execution keys (left arrow, copy, paste, ans, EXE). The mode is set to Alg, and the angle mode is Standard.

Mark allocation: 1 mark

- 1 mark for correct answer

Question 3c.i.**Worked solution**

Shape of the region is a trapezium with vertical sides of length 6 and $\frac{6}{a-2}$, and a width of $a-3$.

Using the formula for the area of a trapezium gives $A = \frac{1}{2}(a+b)h = \frac{3(a-1)(a-3)}{(a-2)}$.

The screenshot shows a TI-84 Plus calculator interface. The top bar includes 'Edit Action Interactive' and various function keys. The main display area shows the following steps:

- $\text{solve}\left(-\int_b^3 \frac{6}{2-x} dx = 6, b\right)$
- $\{b = -e^{-1} + 2, b = e^{-1} + 2\}$
- $\text{simplify}\left(\frac{1}{2}\left(6 + \frac{6}{a-2}\right)(a-3)\right)$
- $\frac{3 \cdot (a-1) \cdot (a-3)}{a-2}$

The bottom of the screen shows a keypad with categories: Math1, Math2, Math3, Trig, Var, abc, and a numeric keypad. At the very bottom, there are mode settings: Alg, Standard, Real, Rad, and a display mode icon.

Mark allocation: 2 marks

- 1 method mark for recognising the trapezium and finding the lengths 6 and $\frac{6}{a-2}$
- 1 answer mark for correct answer

Question 3c.ii.**Worked solution**

$$\text{Set } \frac{3(a-1)(a-3)}{(a-2)} = 6.$$

Using CAS to solve for a gives $a = \pm\sqrt{2} + 3$.

Since $a > 3$, $a = \sqrt{2} + 3$.

The screenshot shows a CAS window titled "Edit Action Interactive". The main display area contains the following text and mathematical expressions:

simplify $\left(\frac{1}{2} \left(6 + \frac{6}{a-2}\right) (a-3)\right)$

$$\frac{3 \cdot (a-1) \cdot (a-3)}{a-2}$$

solve $\left(\frac{3 \cdot (a-1) \cdot (a-3)}{a-2} = 6, a\right)$

$$\{a = -\sqrt{2} + 3, a = \sqrt{2} + 3\}$$

Below the main display is a toolbar with various mathematical symbols and functions, including Math1, Math2, Math3, Trig, Var, abc, and a bottom row with Alg, Standard, Real, Rad, and a grid icon.

Mark allocation: 2 marks

- 1 method mark for setting $\frac{3(a-1)(a-3)}{(a-2)} = 6$
- 1 answer mark for $a = \sqrt{2} + 3$

**Tip**

- *This question is worth two marks, so a working step must be shown. It is not sufficient to just give an answer.*

Question 3c.iii.**Worked solution**

The area bounded by the curve $y = \frac{6}{2-x}$, the x -axis and the lines $x = 3$ and $x = a$ is less than the area of the trapezium, so

$$-\int_3^a \frac{6}{2-x} dx < \text{area of trapezium}$$

$$\text{For } a = \sqrt{2} + 3$$

$$\Rightarrow -\int_3^{\sqrt{2}+3} \frac{6}{2-x} dx < 6$$

$$\text{Now } -\int_3^{e+2} \frac{6}{2-x} dx = 6 \text{ from part b.i., so}$$

$$\Rightarrow -\int_3^{\sqrt{2}+3} \frac{6}{2-x} dx < -\int_3^{e+2} \frac{6}{2-x} dx$$

$$\Rightarrow \sqrt{2} + 3 < e + 2$$

$$\Rightarrow \sqrt{2} + 1 < e$$

Mark allocation: 2 marks

- 1 method mark for using $-\int_3^{e+2} \frac{6}{2-x} dx = 6$ from **part b.i.**
- 1 answer mark for a clear and logical argument

**Tip**

- *This question is a 'hence' question, so it relies on using arguments and results established in earlier parts of the question. It is not sufficient to simply calculate a decimal equivalent for $\sqrt{2} + 1$ and show that $e > \sqrt{2} + 1$.*

Question 4a.**Worked solution**

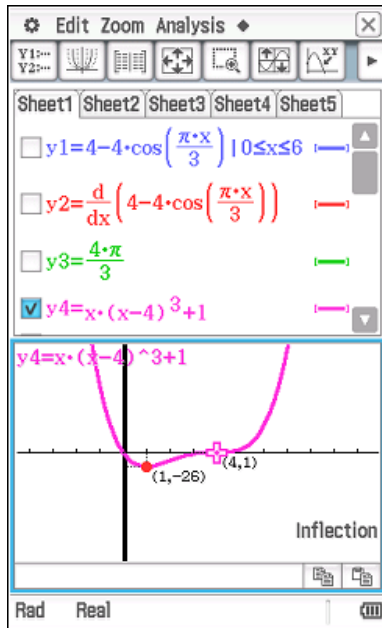
Using the product rule gives

$$\begin{aligned}f'(x) &= 3x(x-4)^2 + (x-4)^3 \\ &= (x-4)^2(3x+x-4) \\ &= (x-4)^2(4x-4) \\ &= 4(x-1)(x-4)^2\end{aligned}$$

So $a = 4$.

Mark allocation: 2 marks

- 1 mark for evidence of product rule
- 1 mark for answer

Question 4b.**Worked solution**

Stationary points: Let $f'(x) = 0 \Rightarrow 4(x-4)^2(x-1) = 0$

$$\Rightarrow x = 4, x = 1$$

At $x = 1$, $y = -26$ and at $x = 4$, $y = 1$.

So $u = 1$ and $v = 1$.

Mark allocation: 2 marks

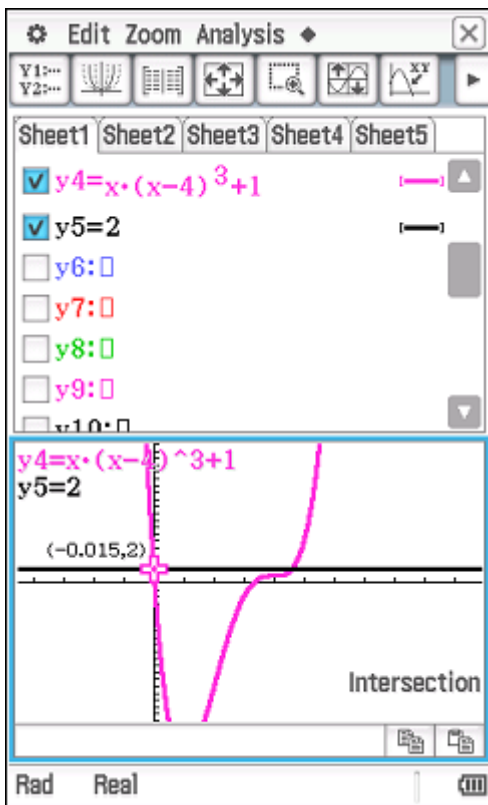
- 1 mark for correct value for u
- 1 mark for correct value for v

Question 4c.**Worked solution**

Looking at the graph of $f(x)$ gives the region of the graph from the x -intercept at $x = 0.016$ to the minimum turning point at $x = 1$. So the region is $[0.016, 1]$.

Mark allocation: 2 marks

- 1 mark for correct x -intercept
- 1 mark for correct range

Question 4d.**Worked solution**

Looking at the screen, the graph must be moved to the right at least 0.015 units, so $k > 0.015$.

Mark allocation: 1 mark

- 1 mark for answer $k > 0.015$

**Tip**

- Sometimes questions are completed more easily using a graph.

Question 4e.**Worked solution**

The graph of $y = f(x) - 1$ has x -intercepts at $x = 0$ and $x = 4$, so the area can be calculated as

$$\begin{aligned} \text{Area} &= \left| \int_0^4 x^4 - 12x^3 + 48x^2 - 64x \, dx \right| \\ &= \left| \left[\frac{x^5}{5} - \frac{12x^4}{4} + \frac{48x^3}{3} - \frac{64x^2}{2} \right]_0^4 \right| \\ &= |-51.2 - 0| \\ &= 51.2 \text{ square units} \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for setting up the integral with correct intercepts
- 1 mark for correct antiderivative
- 1 mark for answer

**Tip**

- *Be careful here as the area cannot be negative – it is best to use absolute value signs around the calculation.*

Question 4f.i.**Worked solution**

Dilation of factor of $\frac{1}{2}$ in x -direction (or from the y -axis)

Translation of 1 unit down

Mark allocation: 2 marks

- 1 mark for dilation specified correctly
- 1 mark for translation specified correctly

Question 4f.ii.**Worked solution**

$(0,0)$ and $(2,0)$ by looking at the x -intercepts of $y = f(x) - 1$ and halving

Mark allocation: 1 mark

- 1 mark for both coordinates correct

**Tip**

- *Note that the question is a 'hence' question, so it needs to be obvious that the answer has come from working with the previous answer.*

Question 4f.iii.**Worked solution**

$$\text{Area} = \frac{1}{2} \times \text{previous area}$$

$$= \frac{1}{2} \times 51.2 = 25.6 \text{ square units}$$

Mark allocation: 1 mark

- 1 mark for answer (showing appropriate working)

Note: Must have evidence of halving previous answer. If area is calculated using an integral, then zero marks allocated.

Question 5a.i.**Worked solution**

The sum of the edges is equal to E cm; therefore, $6x + 3y = E$, so $y = \frac{E}{3} - 2x$.

Using Pythagoras, or otherwise, it can be shown that the vertical height of the triangle is

$$\sqrt{x^2 - \left(\frac{x}{2}\right)^2} = \sqrt{\frac{3x^2}{4}} = \frac{\sqrt{3}x}{2}$$

The area of the triangle end is $\frac{1}{2}bh = \frac{1}{2}x \frac{\sqrt{3}x}{2} = \frac{\sqrt{3}x^2}{4}$.

The volume of the prism is

$$A_{\text{triangle}} \times y = \frac{\sqrt{3}x^2}{4} \left(\frac{E}{3} - 2x \right) = \frac{\sqrt{3}Ex^2}{12} - \frac{\sqrt{3}x^3}{2}, \text{ as required.}$$

Mark allocation: 3 marks

- 1 method mark for finding $y = \frac{E}{3} - 2x$
- 1 method mark for finding the vertical height of the triangle
- 1 answer mark for finding area of triangle, leading to volume of prism

Question 5a.ii.**Worked solution**

Maximum volume occurs when $\frac{dV}{dx} = 0$.

Using CAS gives $\frac{dV}{dx} = \frac{-(9\sqrt{3}x^2 - \sqrt{3}Ex)}{6}$.

Set $\frac{-(9\sqrt{3}x^2 - \sqrt{3}Ex)}{6} = 0$

Using CAS gives

The screenshot shows a CAS interface with the following content:

- Top bar: Edit Action Interactive
- Toolbar: $\frac{1}{2}$, $\frac{1}{2}$, $\frac{d}{dx}$, $\frac{d}{dx}$, Simp, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$
- Main display:
$$\frac{d}{dx} \left(\frac{\sqrt{3} \cdot E \cdot x^2}{12} - \frac{\sqrt{3} \cdot x^3}{2} \right)$$

$$\frac{-(9 \cdot \sqrt{3} \cdot x^2 - \sqrt{3} \cdot E \cdot x)}{6}$$
- Bottom panel: Math1 (Line, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$), Math2 ($\frac{d}{dx}$, e^x , ln, i , ∞), Math3 ($\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$), Trig (sin, cos, tan, θ , t), Var, abc, \leftarrow , \rightarrow , ans, EXE
- Bottom status bar: Alg, Standard, Real, Rad

The screenshot shows a CAS interface with the following content:

- Top bar: Edit Action Interactive
- Toolbar: $\frac{1}{2}$, $\frac{1}{2}$, $\frac{d}{dx}$, $\frac{d}{dx}$, Simp, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$
- Main display:
$$\frac{d}{dx} \left(\frac{\sqrt{3} \cdot E \cdot x^2}{12} - \frac{\sqrt{3} \cdot x^3}{2} \right)$$

$$\frac{-(9 \cdot \sqrt{3} \cdot x^2 - \sqrt{3} \cdot E \cdot x)}{6}$$

$$\text{solve}(9 \cdot \sqrt{3} \cdot x^2 - \sqrt{3} \cdot E \cdot x = 0, x)$$

$$\left\{ x=0, x=\frac{E}{9} \right\}$$
- Bottom panel: Math1 (Line, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$), Math2 ($\frac{d}{dx}$, e^x , ln, i , ∞), Math3 ($\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$, $\frac{d}{dx}$), Trig (sin, cos, tan, θ , t), Var, abc, \leftarrow , \rightarrow , ans, EXE
- Bottom status bar: Alg, Standard, Real, Rad

So the maximum volume occurs $x = \frac{E}{9}$ cm.

Mark allocation: 2 marks

- 1 mark for setting $\frac{dV}{dx} = \frac{-(9\sqrt{3}x^2 - \sqrt{3}Ex)}{6}$ equal to zero
- 1 mark for answer $x = \frac{E}{9}$

**Tip**

- For questions worth more than 1 mark, appropriate working must be shown.

Question 5b.**Worked solution**

Surface area of the prism is

$$\begin{aligned}
 & 2 \times \text{Area}_{\text{triangle}} + 3 \times \text{Area}_{\text{rectangle}} \\
 &= \frac{\sqrt{3}x^2}{2} + 3xy \\
 &= \frac{\sqrt{3}x^2}{2} + 3x\left(\frac{E}{3} - 2x\right) \\
 &= \frac{\sqrt{3}x^2}{2} + Ex - 6x^2
 \end{aligned}$$

Using CAS gives

The screenshot shows a CAS calculator window with the following content:

$$\frac{d}{dx} \left(\frac{\sqrt{3} \cdot x^2}{2} + E \cdot x - 6x^2 \right)$$

$$\sqrt{3} \cdot x - 12 \cdot x + E$$

$$\text{solve}(\sqrt{3} \cdot x - 12 \cdot x + E = 0, x)$$

$$\left\{ x = \frac{E}{-\sqrt{3} + 12} \right\}$$

The calculator interface includes a toolbar with icons for fractions, derivatives, simplification, and other mathematical operations. Below the main display is a keypad with categories like Math1, Math2, Math3, Trig, Var, and abc, along with function keys like Line, π , e^x , \ln , i , ∞ , $\frac{d}{dx}$, \int , \lim , \sin , \cos , \tan , θ , t , and ans .

So, the maximum surface area occurs when $x = \frac{E}{12 - \sqrt{3}}$ cm.

Mark allocation: 2 marks

- 1 method mark for finding prism's surface area
- 1 answer mark for $x = \frac{E}{12 - \sqrt{3}}$

END OF WORKED SOLUTIONS