The Mathematical Association of Victoria

Trial Examination 2017

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	C	**11**	B
2	B	**12**	D
3	D	**13**	C
4	E	**14**	D
5	C	**15**	E
6	B	**16**	C
7	E	**17**	A
8	D	**18**	A
9	E	**19**	E
10	C	**20**	B

Question 1 **Answer C**

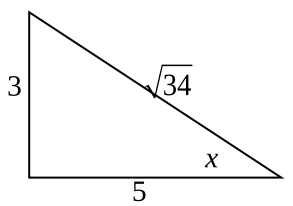
$f:\left[0,\frac{\pi}{2}\right]\to R,\ f(x)=-3\cos\left(4x+\pi\right)+1$

Amplitude = 3 translated vertically by 1 unit giving range = $\left[-2,4\right]$

period = $\frac{2\pi}{4}=\frac{\pi}{2}$

Question 2 **Answer B**

$\tan(x)=\frac{3}{5}$ and $\pi\le x\le\frac{3\pi}{2}$



In 3rd quadrant $\sin\left(x\right)=-\frac{3}{\sqrt{34}}$

Question 3 **Answer D**

$h(x) = x^4$

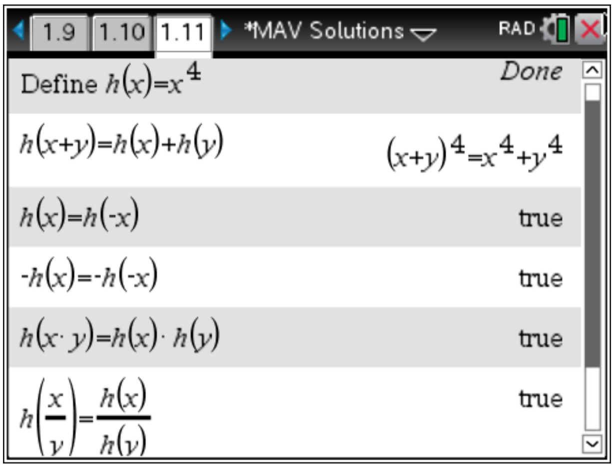
$h(x+y) = h(x) + h(y)$ is false

LHS = $(x+y)^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

RHS = $x^4 + y^4$

LHS $\neq$ RHS



Question 4 **Answer E**

$h: (-\infty, 4) \to R, h(x) = 2(4-x)^2 - 1$

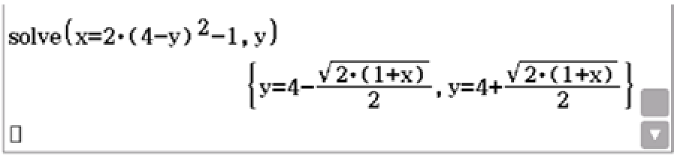
$y = 2(4-x)^2 - 1$

Swap x and y for inverse

$x = 2(4-y)^2 - 1$

$4 - y = \pm\sqrt{\frac{x+1}{2}}$

$y = 4 \pm \sqrt{\frac{x+1}{2}}$



Domain of h^{-1} = range h = $(-1, \infty)$

$h^{-1}: (-1, \infty) \to R, h^{-1}(x) = 4 - \frac{\sqrt{2x+2}}{2}$

Question 5 **Answer C**

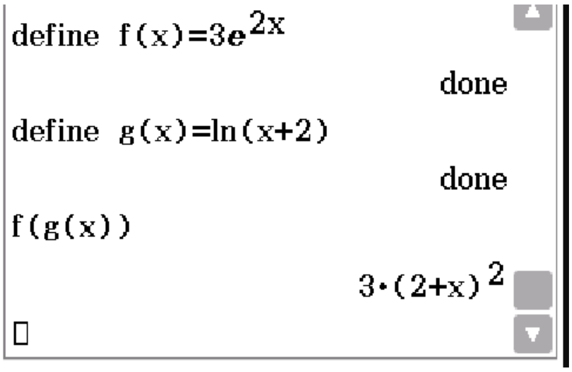
$f(x) = 3e^{2x}$ and $g(x) = \log_e(x+2)$

For $f(g(x))$, test that the range $g \subseteq$ domain f

Giving $R \subseteq R$

So $f(g(x))$ exists with

dom $f(g(x))$ = dom $g(x) = (-2, \infty)$



$h = f(g(x))$ can be defined as

$h: (-2, \infty) \to R, h(x) = 3(x+2)^2$

Question 6 **Answer B**

$mx + y = 2$

$2x - 3y = k$

Using ratios

For infinite solutions or no solution

$\frac{m}{2} = -\frac{1}{3}, m = -\frac{2}{3}$

For no solution

$-3 \neq \frac{k}{2}, k \neq -6$

$m = -\frac{2}{3}, k \in R \setminus \{-6\}$

Question 7 **Answer E**

Range = 42 – 37 = 5

Amplitude = 2.5

Vertical translation = 39.5

Period $= \frac{2\pi}{n} = 24$, giving $n = \frac{\pi}{12}$

$$y = 2.5\sin\left(\frac{\pi t}{12}\right) + 39.5$$

Question 8 **Answer D**

$f(x) = \sqrt{2x-1} + 3$ and $g(x) = \dfrac{1}{(x-2)^2} + 4$

Domain of f is $\left[\frac{1}{2}, \infty\right)$

Domain of g is $R \setminus \{2\}$

The intersection of both is

$$\left[\frac{1}{2}, 2\right) \cup (2, \infty)$$

which is the domain of $f + g$

Question 9 **Answer E**

$$f'(x) = e^{x+1} + x + 1$$

$$f(x) = e^{x+1} + \frac{x^2}{2} + x + c$$

$f(0) = 2$ giving $2 = e^{0+1} + \frac{0^2}{2} + 0 + c \therefore c = 2 - e$

then

$$f(x) = e^{x+1} + \frac{x^2}{2} + x + 2 - e$$

$$= \frac{x^2}{2} + x + 2 + e\left(e^x - 1\right)$$

Question 10 **Answer C**

Given $\int_1^3 g(x)dx = 2$

$$2\int_3^1 \left(g(x) + 1\right)dx = 2\int_3^1 g(x)dx + 2\int_3^1 1dx$$

$$= -2\int_1^3 g(x)dx - 2\int_1^3 1dx$$

$$= -2 \times 2 - 2[x]_1^3$$

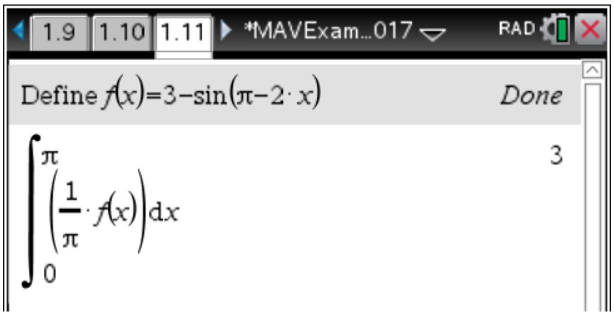
$$= -4 - 4$$

$$= -8$$

Question 11 **Answer B**

Average value $= \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{\pi - 0}\int_0^\pi f(x)dx$

where $f(x) = -\sin(\pi - 2x) + 3$



Average value = 3

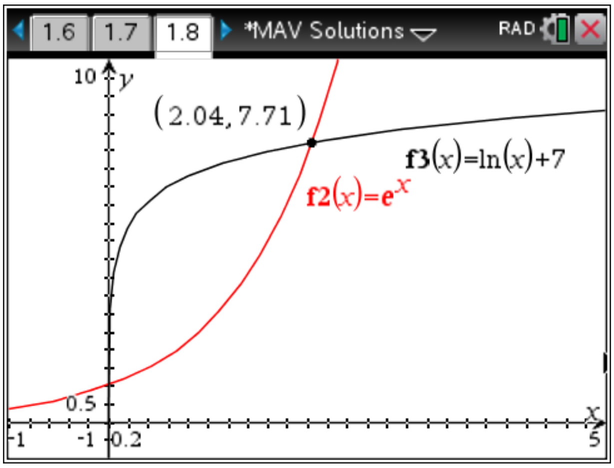
Question 12 **Answer D**

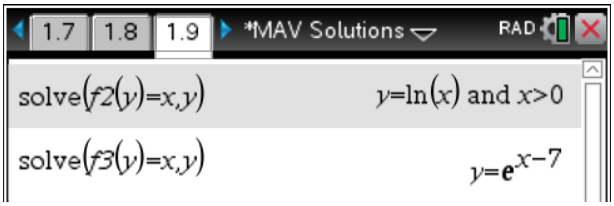
$f(x) = e^x$ and $g(x) = \log_e(x) + 7$

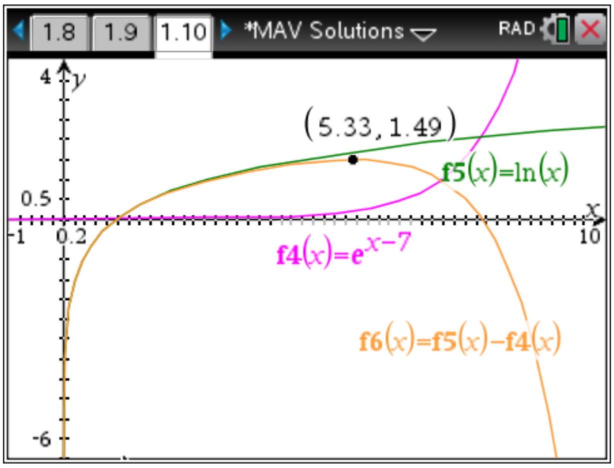
$f^{-1}(x) = \log_e(x)$ and $g^{-1}(x) = e^{x-7}$

Let $k(x) = f^{-1}(x) - g^{-1}(x)$

Maximum value is approximately 1.49.







Question 13 **Answer C**

$f(x) = \log_e(3 - x)$

Intercepts

$(0, \log_e(3)), (2, 0)$

Gradient = $-\dfrac{\log_e(3)}{2}$

Gradient of the perpendicular line = $\dfrac{2}{\log_e(3)}$

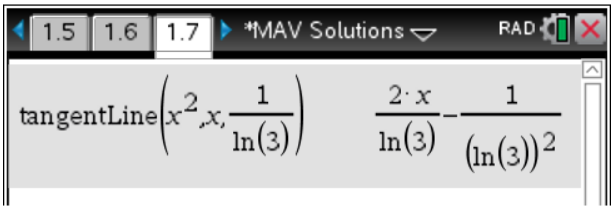
$g(x) = x^2$

$g'(x) = 2x = \dfrac{2}{\log_e(3)}, x = \dfrac{1}{\log_e(3)}$

$\left(\dfrac{1}{\log_e(3)}, \dfrac{1}{(\log_e(3))^2}\right)$

$y - \dfrac{1}{(\log_e(3))^2} = \dfrac{2}{\log_e(3)}\left(x - \dfrac{1}{\log_e(3)}\right)$

$y = \dfrac{2}{\log_e(3)}x - \dfrac{1}{(\log_e(3))^2}$



Question 14 **Answer D**

$a = 2t^2 - 1$

$v = \int (2t^2 - 1)\,dt$

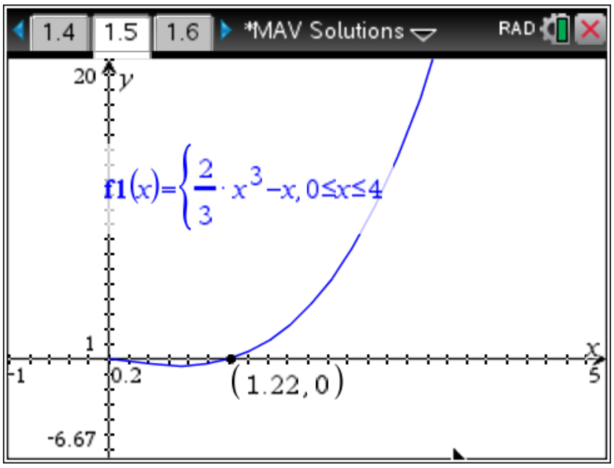
$= \frac{2}{3}t^3 - t + c$

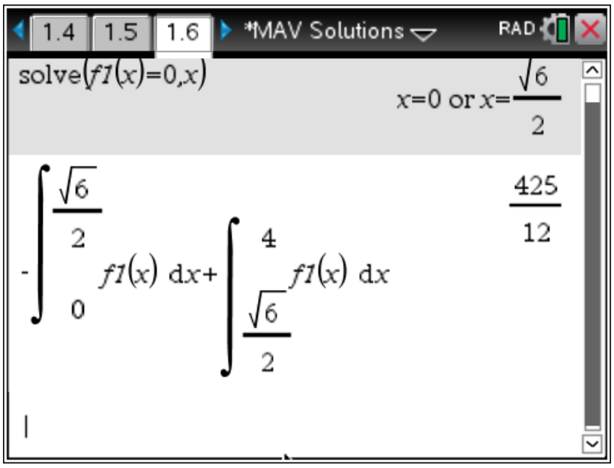
$c = 0$

Solve $v(t) = 0$ for t

$t = \frac{\sqrt{6}}{2}, \; t > 0$

Distance = $-\int_0^{\frac{\sqrt{6}}{2}} v(t)\,dt + \int_{\frac{\sqrt{6}}{2}}^{4} v(t)\,dt = \frac{425}{12}$





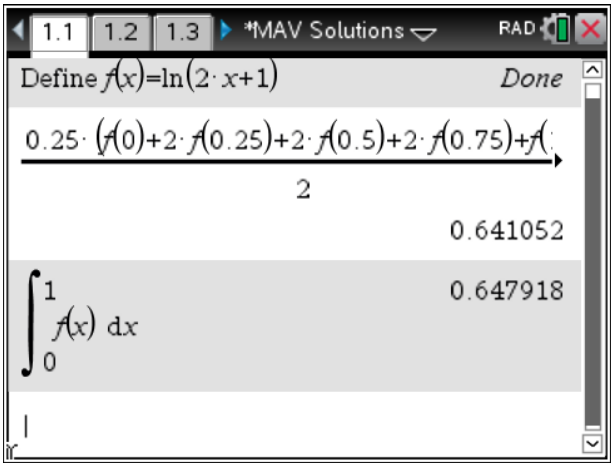
Question 15 **Answer E**

$$\frac{\left(\text{left-endpoint rectangles of width 0.25 + right-endpoint rectangles of width 0.25}\right)}{2}$$

The smaller the width of the rectangle the better the estimate. In this case the right-endpoint rectangles estimate will give an over estimate and the left-endpoint rectangles will give an underestimate. The average of the two should give the best estimate.

$$\frac{0.25(f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)}{2}$$

≈ 0.6411

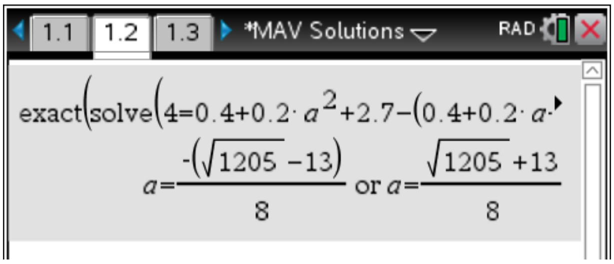


Question 16 **Answer C**

$0.1+b+0.2+0.3=1,\ b=0.4$

$\text{Var}(X)=E(X^2)-(E(X))^2 \quad 4=0.4+0.2a^2+2.7-(0.4+0.2a+0.9)^2$

$$a=\frac{13\pm\sqrt{1205}}{8}$$



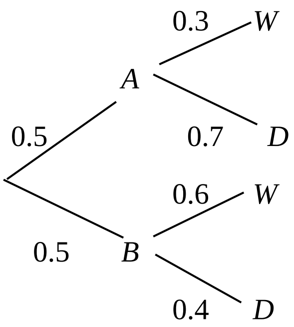
Question 17 **Answer A**

Let W represent a white chocolate and

A represent Box A.

$$\Pr(A|W)=\frac{\Pr(W\cap A)}{\Pr(W)}$$

$$=\frac{\frac{1}{2}\times\frac{3}{10}}{\frac{1}{2}\times\frac{3}{10}+\frac{1}{2}\times\frac{6}{10}}=\frac{1}{3}$$



Question 18 **Answer A**

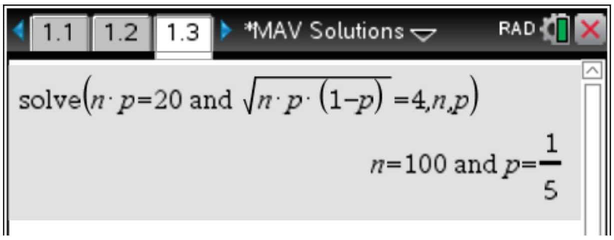
$X\sim\text{Bi}(n,p)$

$\text{E}(X)=np=20$

$\text{SD}(X)=\sqrt{npq}=4$

$20q=16, q=\frac{4}{5}, p=\frac{1}{5}$

$\frac{1}{5}n=20, n=100$



Question 19 **Answer E**

Let W represent winning.

$\Pr(W)=0.65$

$W\sim\text{Bi}(n,0.65)$

$\Pr(W\geq 2)>0.95$

$1-(\Pr(W=0)+\Pr(W=1))>0.95$

$\Pr(W=0)+\Pr(W=1)<0.05$

$0.35^n+0.65n(0.35)^{n-1}<0.05$

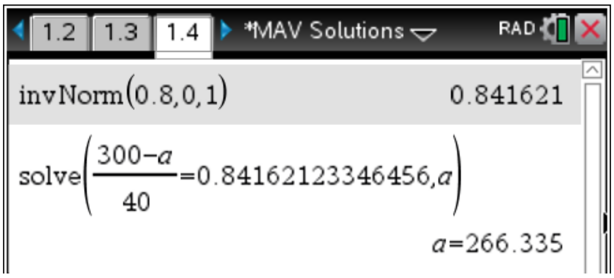
Question 20 **Answer B**

$X \sim N\left(\mu, 40^2\right)$

$\Pr\left(X > 300\right) = 0.2$

Solve $\dfrac{300-\mu}{40} = 0.8416...$ for μ

$\mu = 266$ to the nearest integer



SECTION B

Question 1

$f: \left[-\pi, \dfrac{7\pi}{2}\right] \to R,\ f(x) = -a\sin\left(\dfrac{x}{3}+\dfrac{\pi}{3}\right)-\pi$

a. period = $\dfrac{2\pi}{\frac{1}{3}} = 6\pi$ **1A**

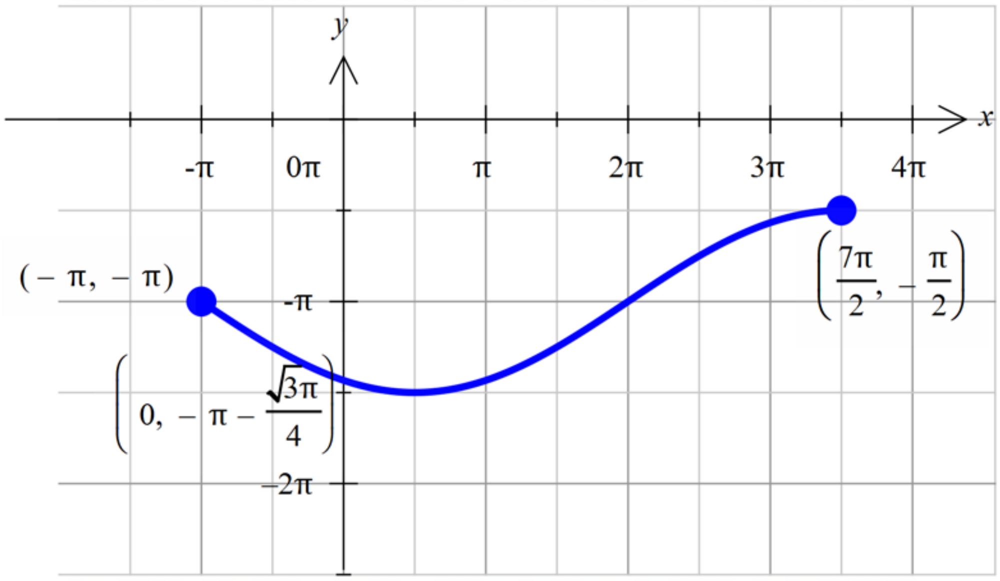
b. range of f is $\left[-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right]$

So $\dfrac{-\frac{3\pi}{2}+\frac{\pi}{2}}{2} = -\dfrac{\pi}{2}$

As there is already a negative sign in front of a, $a = \dfrac{\pi}{2}$ **1M show that**

c. shape **1A**

intercepts $\left(0, -\pi - \dfrac{\sqrt{3}\pi}{4}\right)$, endpoints $\boldsymbol{(-\pi, -\pi)}$ and $\left(\dfrac{7\pi}{2}, -\dfrac{\pi}{2}\right)$ **1A**



d. domain of *f* is restricted to $[-\pi, b]$

First stationary point after $x=-\pi$ is $\left(\frac{\pi}{2}, -\frac{3\pi}{2}\right)$

Maximum possible $b=\frac{\pi}{2}$ **1A**

e. For $b=\frac{\pi}{2}$, the domain of f_1^{-1} is $\left[-\frac{3\pi}{2}, -\pi\right]$. **1A**

f. The equation of tangent to *f* at $x=0$ is $y=-\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi$. **1A**

The equation of tangent to *f* at $x=\pi$ is $y=\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi-\frac{\pi^2}{12}$. **1A**

define $f(x)=-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi \mid -\pi \le x \le \frac{7\pi}{2}$

done

tanLine(f(x), x, 0)

$\frac{-\sqrt{3}\cdot\pi}{4}-\pi-\frac{x\cdot\pi}{12}$

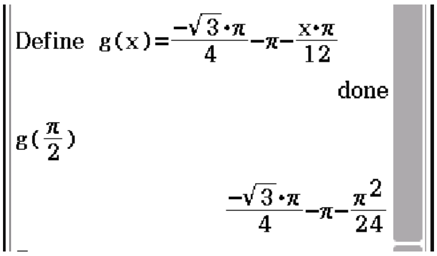
tanLine(f(x), x, π)

$\frac{-\sqrt{3}\cdot\pi}{4}-\pi+\frac{x\cdot\pi}{12}-\frac{\pi^2}{12}$

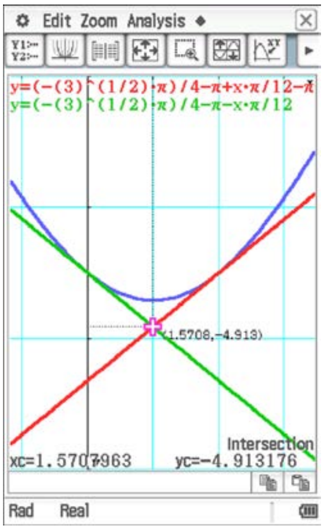
g. point of intersection of the tangents = $\left(\frac{\pi}{2}, -\frac{\sqrt{3}\pi}{4}-\pi-\frac{\pi^2}{24}\right)$ **1A**

solve$\left(\frac{-\sqrt{3}\cdot\pi}{4}-\pi-\frac{x\cdot\pi}{12}=\frac{-\sqrt{3}\cdot\pi}{4}-\pi+\frac{x\cdot\pi}{12}-\frac{\pi^2}{12}, x\right)$

$\left\{x=\frac{\pi}{2}\right\}$



h. Sketch the graph to view the required area.



Area =

$$\int_0^{\frac{\pi}{2}}\left(-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi\right)-\left(-\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi\right)dx+\int_{\frac{\pi}{2}}^{\pi}\left(-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi\right)-\left(\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi-\frac{\pi^2}{12}\right)dx$$

OR

Alternatively define the functions as $f(x)=-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi$

And $g(x)=-\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi$ and $h(x)=\frac{\pi x}{12}-\frac{\sqrt{3}\pi}{4}-\pi-\frac{\pi^2}{12}$.

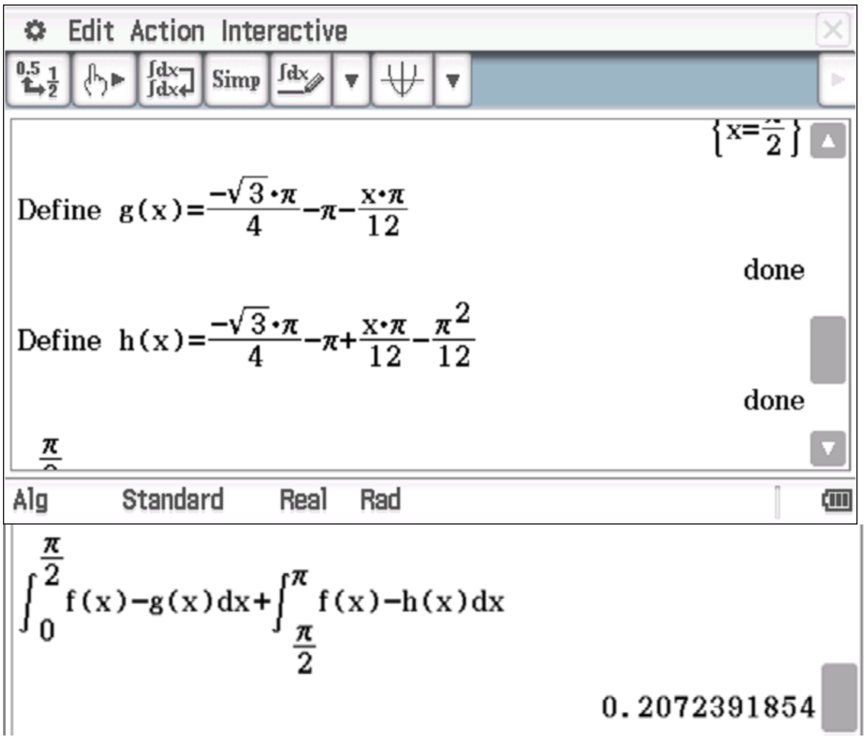
Gives Area = $\int_0^{\frac{\pi}{2}}(f(x)-g(x))dx+\int_{\frac{\pi}{2}}^{\pi}(f(x)-h(x))dx$

OR

By symmetry Area = $2\int_0^{\frac{\pi}{2}}(f(x)-g(x))dx$ **correct terminals 1A**

correct functions 1A

i. Area enclosed between f and the tangents, correct to 2 decimal = 0.21 square units **1A**



j. Transformations to get from f to f_1 where

$$f(x)=-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi=-\frac{\pi}{2}\sin\left(\frac{1}{3}(x+\pi)\right)-\pi$$

There are many solutions. Three possibilities have been given.

Translate π units up and π units to the right: $y_1=-\frac{\pi}{2}\sin\left(\frac{x}{3}\right)$ **1A**

Dilate by a factor of $\frac{2}{\pi}$ from the x-axis: $y_2=-\sin\left(\frac{x}{3}\right)$

Reflect in the x-axis: $y_2=\sin\left(\frac{x}{3}\right)$ **1A**

Dilate by a factor of $\frac{1}{3}$ from the y-axis: $f_2(x)=\sin(x)$ **1A**

OR

Dilate by a factor of $\frac{2}{\pi}$ from x-axis: $y_1=\frac{2}{\pi}\left[-\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-\pi\right]=-\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)-2$ **1A**

Dilate by a factor of $\frac{1}{3}$ from y-axis: $y_2=-\sin\left(\frac{3x}{3}+\frac{\pi}{3}\right)-2=-\sin\left(x+\frac{\pi}{3}\right)-2$

Reflect in the x-axis: $y_3=\sin\left(x+\frac{\pi}{3}\right)+2$ **1A**

Translate $\frac{\pi}{3}$ units right and 2 units down: $f_2(x)=\sin\left(\left(x-\frac{\pi}{3}\right)+\frac{\pi}{3}\right)+2-2=\sin(x)$ **1A**

OR

Translate up by π units: $y_1 = -\frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)$

Reflect over x-axis: $y_2 = \frac{\pi}{2}\sin\left(\frac{x}{3}+\frac{\pi}{3}\right)$ **1A**

Dilate by a factor of $\frac{2}{\pi}$ from the x-axis: $y_3 = \sin\left(\frac{x}{3}+\frac{\pi}{3}\right) = \sin\left(\frac{1}{3}(x+\pi)\right)$ **1A**

Translate to the right by π units: $y_4 = \sin\left(\frac{x}{3}\right)$

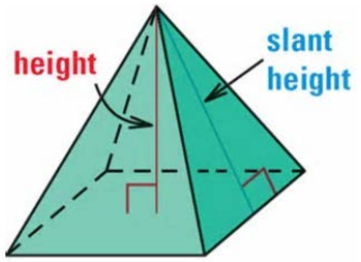
Dilate by a factor $\frac{1}{3}$ from the y-axis: $f_2(x) = \sin(x)$ **1A**

Question 2

Square based pyramid with vertical height, h metres, and length of the sides of the base, x metres.

TSA = 60 m^2.

a. Sketch the right-angled triangle required.



This gives the Pythagoras formula $y^2 = \left(\frac{x}{2}\right)^2 + h^2$

Rearrange to get

$y^2 = \frac{x^2}{4} + h^2$

$y = \pm\sqrt{\frac{x^2}{4} + h^2}$

For $y > 0$, $y = \sqrt{h^2 + \frac{x^2}{4}}$ as required. **1M show that**

b. TSA = square base + 4 slanting faces.

TSA = $x^2 + 4\times\frac{1}{2}xy$ **1A**

Using $y = \sqrt{h^2 + \frac{x^2}{4}}$

TSA = $x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}}$ **1A**

c. To find the relationship between x and h we know that TSA = 60 m^2.

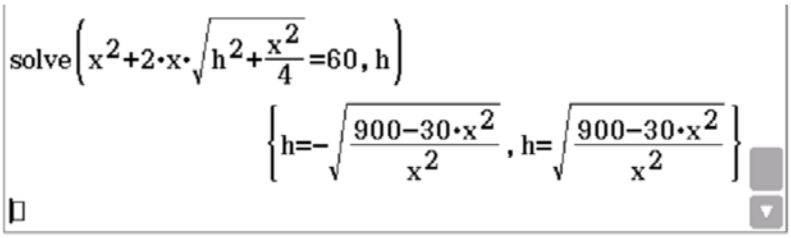
Letting $x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}}$ = 60 and solve for h

For $h > 0$, $h = \dfrac{\sqrt{900 - 30x^2}}{x}$ **1A**

Volume = $\dfrac{1}{3} \times$ area of base $\times$ height.

Volume = $\dfrac{1}{3} x^2 \times \dfrac{\sqrt{900 - 30x^2}}{x}$

Giving $V = \dfrac{1}{3} x \sqrt{900 - 30x^2}$ as required **1M show that**



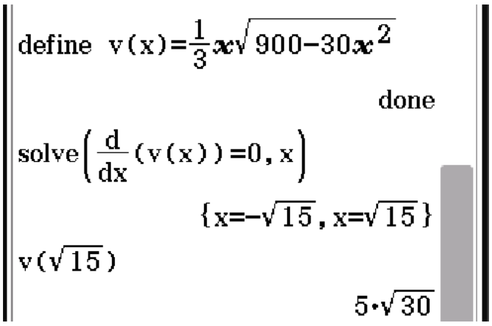
d. We need $900 - 30x^2 > 0$ giving implied domain $x \in \left(0, \sqrt{30}\right)$ **1A**

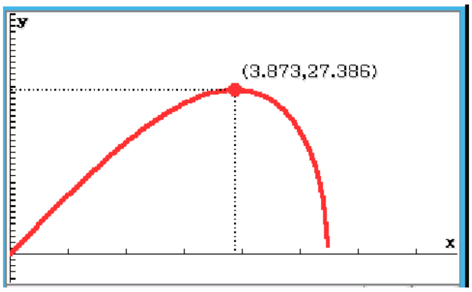
e. Solve $\dfrac{dV}{dx} = 0$

Considering the graph of V, for the given domain we have a local maximum.

The maximum volume, $V = 5\sqrt{30}$ m^3, **1A**

at $x = \sqrt{15}$ m. **1A**





f.i. TSA = 4 $\times$ area of equilateral triangle.

TSA = 4 $\times$ $\dfrac{1}{2} d^2 \sin(60^0)$ **1M**

$= 2d^2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3} d^2$ in the required form. **1A**

ii. TSA = 60 m^2, so $\sqrt{3} d^2 = 60$

giving, for $d > 0$, $d = \sqrt{\frac{60}{\sqrt{3}}} = \sqrt{20\sqrt{3}}$ **1M**

Vol = $V_T = \frac{d^3}{6\sqrt{2}}$

giving $V_T = \frac{\left(\sqrt{\frac{60}{\sqrt{3}}}\right)^3}{6\sqrt{2}}$ = 24.028 m^3 correct to 3 decimal places. **1A**

g. TSA = $\pi r(r+s)$ = 60 m^2

Giving $s = \frac{60 - \pi r^2}{\pi r}$

Volume of cone = $\frac{1}{3}\pi r^2 l$ where l is height of cone.

To find l, for $l > 0$, use

$$l^2 + r^2 = s^2$$
$$l = \sqrt{s^2 - r^2}$$

Volume of cone = $\frac{1}{3}\pi r^2\sqrt{s^2 - r^2}$ where $s = \frac{60 - \pi r^2}{\pi r}$

Therefore $V(r) = \frac{2}{3}\pi r^2\sqrt{\frac{900}{\pi^2 r^2} - \frac{30}{\pi}}$ **1M**

For maximum volume, $V'(r) = 0$

Considering the graph of V, for the given domain we have a local maximum.
Maximum volume, correct to two decimal places = 30.90 m^3 **1A**

combine(solve($\pi\cdot r\cdot(r+s)=60$, s))

$$\left\{s=\frac{60-r^2\cdot\pi}{r\cdot\pi}\right\}$$

$$\left\{s=\frac{60-r^2\cdot\pi}{r\cdot\pi}\right\}$$

define $v(r)=\frac{1}{3}\pi r^2\sqrt{s^2-r^2}$

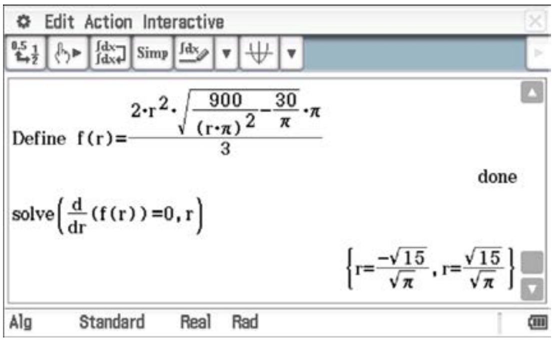
done

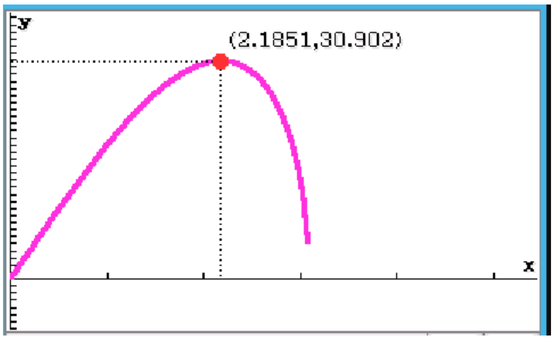
$\frac{60-r^2\cdot\pi}{r\cdot\pi} \Rightarrow s$

$$\frac{60-r^2\cdot\pi}{r\cdot\pi}$$

simplify(v(r))

$$\frac{2\cdot r^2\cdot\sqrt{\frac{900}{(r\cdot\pi)^2}-\frac{30}{\pi}}\cdot\pi}{3}$$

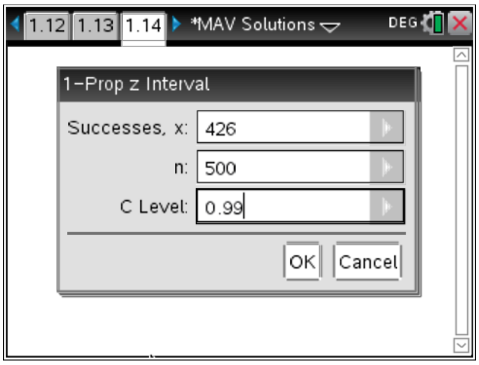


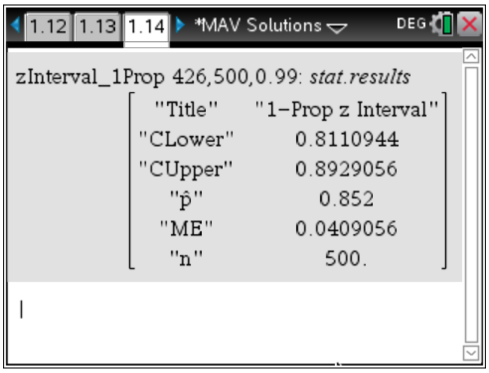


h. cone **1A**

Question 3

a. $(0.811, 0.893)$ **1A**





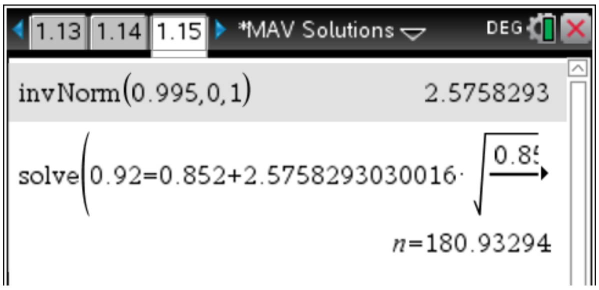
b. $0.99 \times 200 = 198$ **1A**

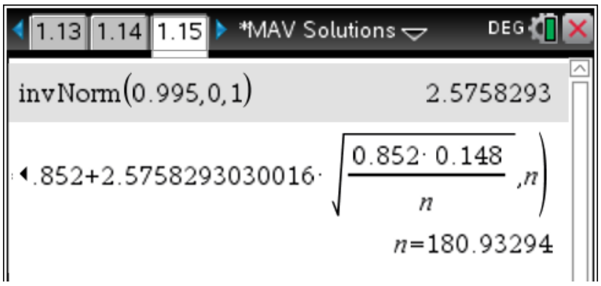
c. Yes, 0.92 is outside the 99% confidence interval.

A suitable comment. **1A**

d. Margin of error $= 0.92 - 0.852 = 2.57...\sqrt{\dfrac{0.852 \times 0.148}{n}}$ **1M**

$n = 181$ to the nearest integer **1A**

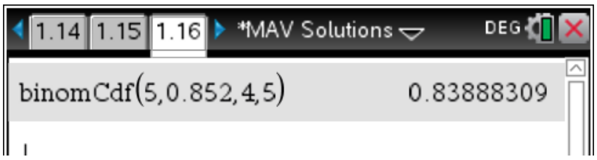




e. $X \sim \text{Bi}(5, 0.852)$ **1M**

$\Pr(X > 3) = \Pr(X \geq 4)$

= 0.839 correct to 3 decimal places **1A**



f. $\Pr(\text{first } 3 \mid \text{only } 3) = \dfrac{\Pr(\text{first } 3 \cap \text{only } 3)}{\Pr(X = 3)}$ **1M**

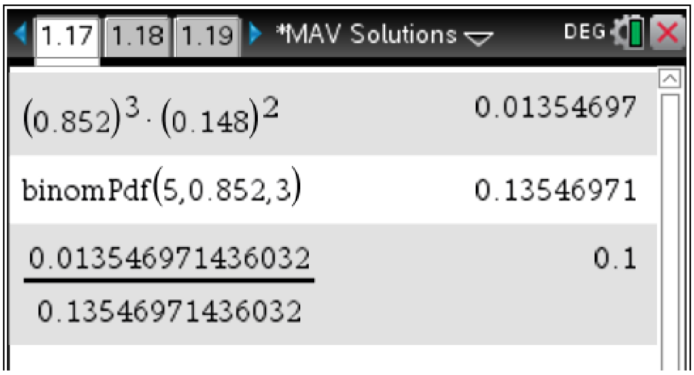
$\dfrac{(0.852)^3 \times (0.148)^2}{\Pr(X = 3)} = 0.1$ **1A**

OR

There are $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$ ways for 3 globes only to last longer than 100 hours

and each way has the same probability . **1M**

Only in one of these ways will it be the first three. Answer 0.1 **1A**



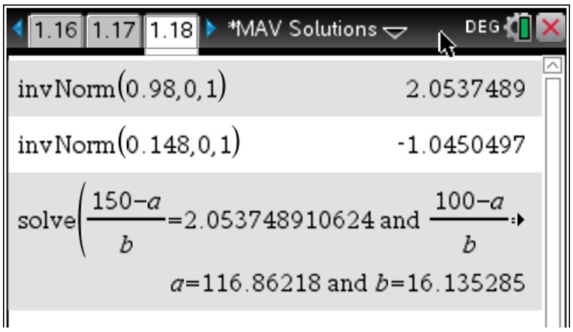
g. $Y \sim N\left(\mu, \sigma^2\right)$

$$\frac{150-\mu}{\sigma} = 2.0537...$$

$\frac{100-\mu}{\sigma} = -1.045...$ **1M**

$\mu = 116.86$ **1A**

$\sigma = 16.14$ **1A**



h. Let J represent Jessica doing her homework and

D represent David doing his homework.

$\Pr(J \cap D) = \Pr(J) \times \Pr(D) = 0.2q^2$ for independent events **1A**

$\Pr(J' \cap D') = 1 - \Pr(J) - \Pr(D) + \Pr(J \cap D)$

$\Pr(J' \cap D') = 1 - 0.2 - q^2 + 0.2q^2 = 0.8 - 0.8q^2$ **1A**

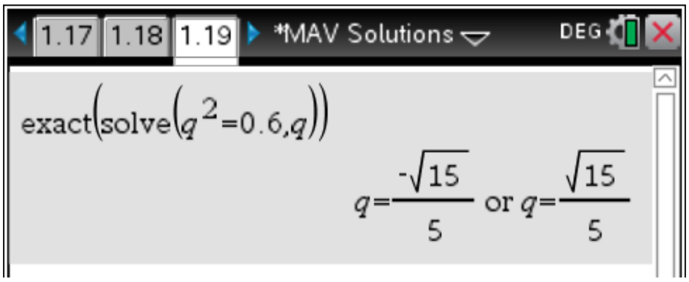
OR

For independent events $\Pr(J' \cap D') = \Pr(J') \times \Pr(D')$ **1A**

$\Pr(J' \cap D') = 0.8\left(1 - q^2\right)$ **1A**

i. Solve $q^2 = 0.6$ for q, $q > 0$

$q = \frac{\sqrt{15}}{5}$ **1A**

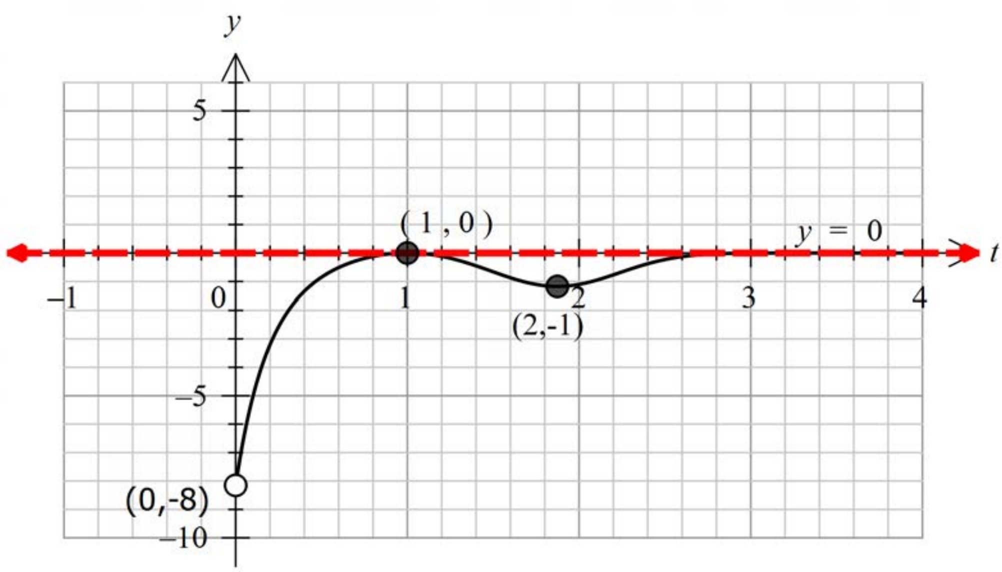


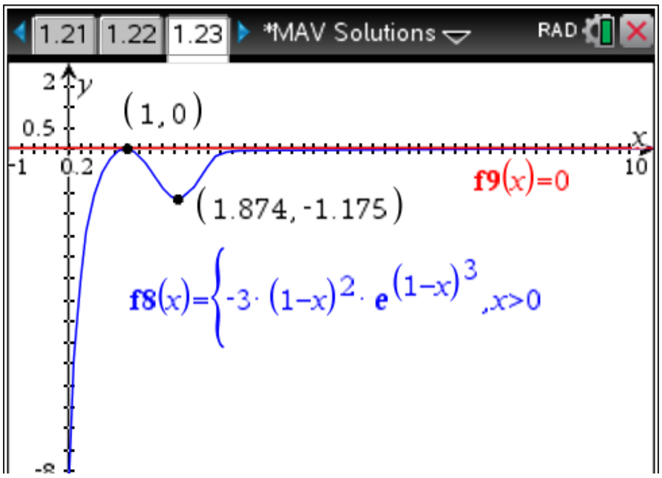
Question 4

a. Shape **1A**

Asymptote **1A**

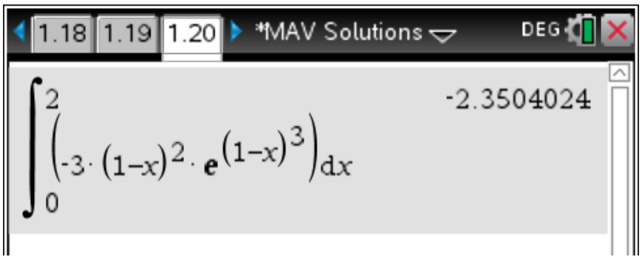
Points **1A**





b. $\int_0^2 (g'(t))\,dt = -2.35...$ **1M**

Declined by 235 rabbits **1A**



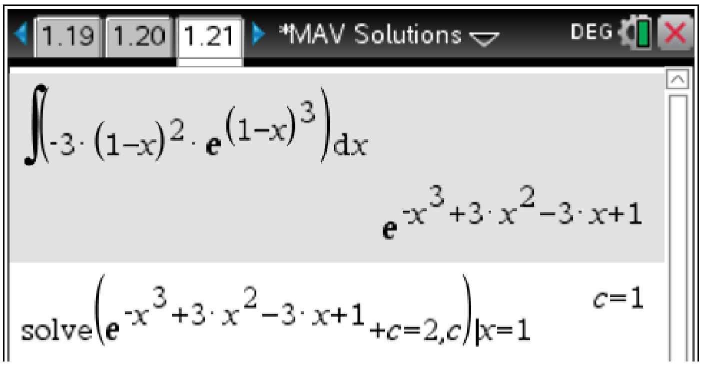
c. $g(t)=\int(g'(t))\,dt+c$

$=e^{(1-t)^3}+c$ **1A**

$g(1)=2$

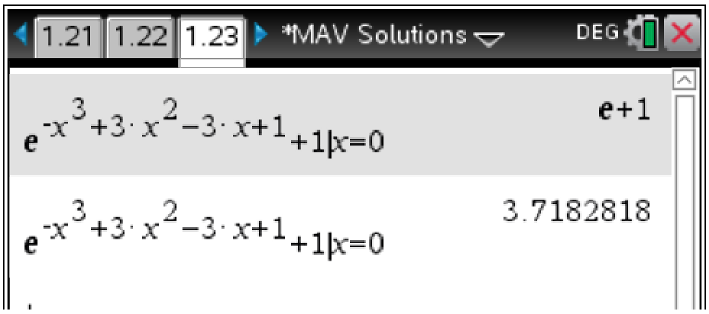
$e^{(1-1)^3}+c=2, c=1$

$g(t)=e^{(1-t)^3}+1$ **1A**



d. $g(0)=3.718...$

372 rabbits **1A**

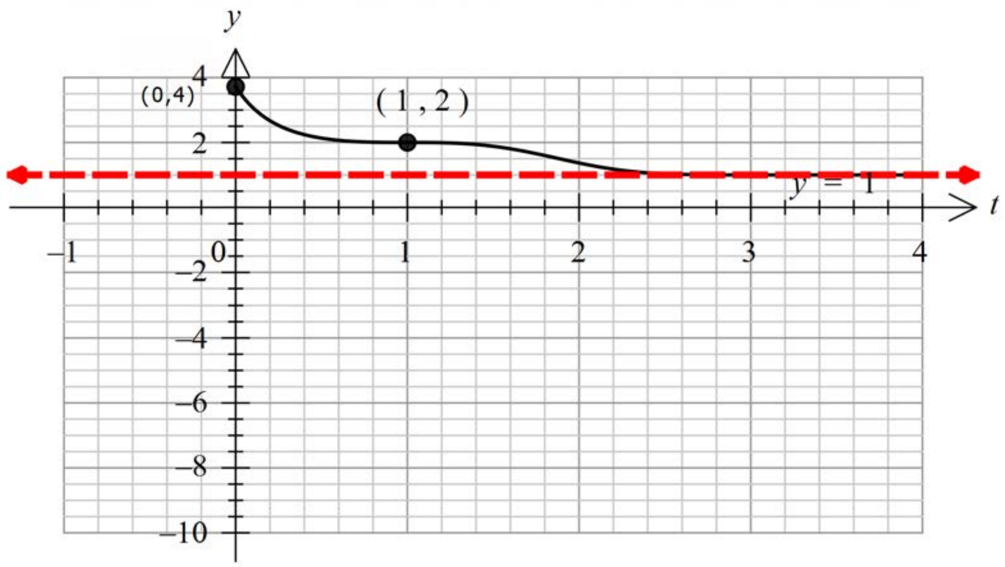


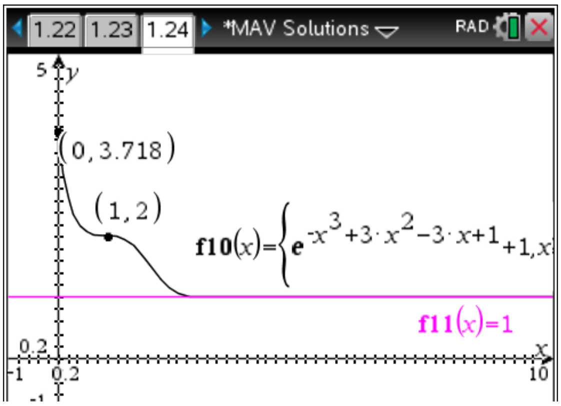
e. Asymptote $y=1$

100 rabbits **1A**

f. Shape **1A**

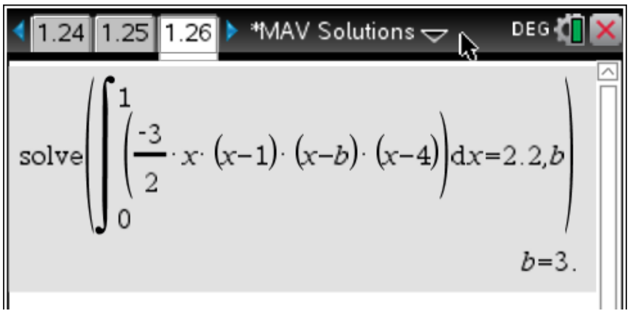
Points and asymptote **1A**





g. Solve $\int_0^1 (v(t)) = 2.2$ for b **1M**

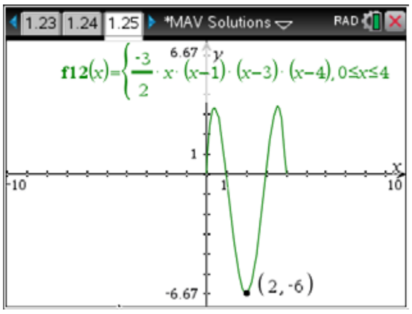
$b = 3$ **1A**

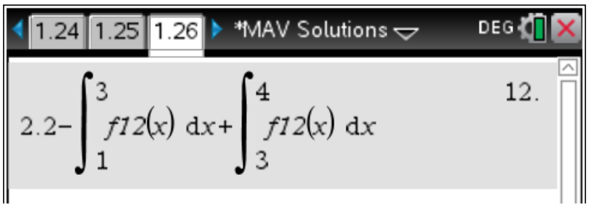


h. 6 ms^{-1} **1A**

distance = $2.2 - \int_1^3 (v(t))dt + \int_3^4 (v(t))\,dt$

= 12 m **1A**





END OF SOLUTIONS