Neap:

Trial Examination 2017

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Q						Q					
1	A	B	**C**	D	E	11	A	**B**	C	D	E
2	A	**B**	C	D	E	12	A	B	C	D	**E**
3	A	B	**C**	D	E	13	A	**B**	C	D	E
4	A	B	C	D	**E**	14	A	B	**C**	D	E
5	A	B	**C**	D	E	15	A	B	C	**D**	E
6	**A**	B	C	D	E	16	A	B	C	**D**	E
7	A	B	C	**D**	E	17	**A**	B	C	D	E
8	A	B	C	D	**E**	18	A	B	C	**D**	E
9	A	B	C	**D**	E	19	**A**	B	C	D	E
10	A	B	**C**	D	E	20	A	**B**	C	D	E

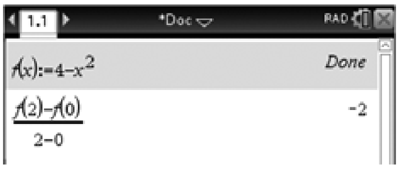
Question 1 C

$2 - 3x > 0$ for f to exist

$$x < \frac{2}{3}$$

Question 2 B

$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Question 3 C

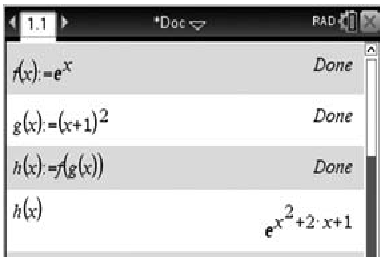
$$\text{period} = \frac{\pi}{n}$$

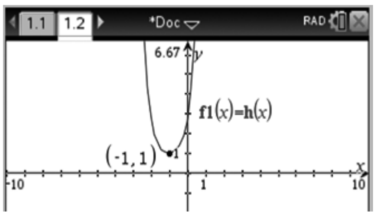
$$n = \frac{2}{3}$$

$$\therefore \text{period} = \frac{\pi}{\frac{2}{3}}$$

$$= \frac{3\pi}{2}$$

Question 4 E





Question 5 C

$f'(x) = 3x^2 - 2bx + 3$

Let $f'(x) = 0 \rightarrow 3x^2 - 2bx + 3 = 0.$

$$\Delta = (-2b)^2 - 4(3)(3)$$

$$= 4b^2 - 36$$

$\Delta > 0 \rightarrow 4b^2 - 36 > 0$

$b \in (-\infty, -3) \cup (3, \infty)$

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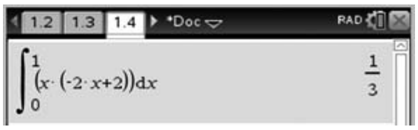
Question 6 A

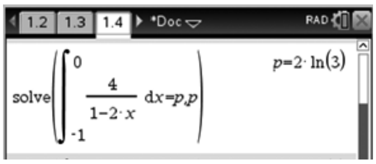
from the formula sheet: $\mathrm{E}(X) = \int x f(x)\,dx$

$$= \int_0^1 x(-2x + 2)\,dx$$



Question 7 D

$$\int_{-1}^{0} \frac{4}{1-2x}\,dx = p$$



$p = \log_e(9)$

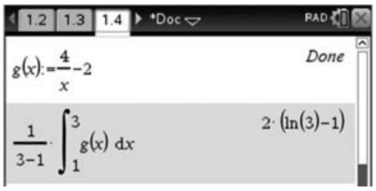
$$\frac{e^p}{2} = \frac{e^{\log_e(9)}}{2}$$

$$= \frac{9}{2}$$

Question 8 E

$$\text{average value} = \frac{1}{b-a}\int_a^b g(x)\,dx$$

$$= \frac{1}{3-1}\int_1^3 \left(\frac{4}{x}-2\right)dx$$



$$\text{average value} = 2(\log_e(3) - \log_e(e))$$

$$= 2\log_e\left(\frac{3}{e}\right)$$

$$= \log_e\left(\left(\frac{3}{e}\right)^2\right)$$

$$= \log_e\left(\frac{9}{e^2}\right)$$

Question 9 D

$-f(2x+4)-1$

$\Rightarrow -f(2(x+2))-1$

- dilation factor of $\frac{1}{2}$ from the y-axis: $(6, 2) \to (3, 2)$
- reflection in the x-axis: $(3, 2)$ maximum $\to (3, -2)$ minimum
- translation of 2 units left and 1 unit down: $(3, -2) \to (1, -3)$

Question 10 C

If A and B are mutually exclusive, then $\Pr(A \cap B) = 0$.

Construct a probability table as follows.

	Pr(*A*)	**Pr(*A*′)**	
Pr(*B*)	0	$\frac{1}{8}$	$\frac{1}{8}$
Pr(*B*′)	$\frac{1}{4}$		$\frac{7}{8}$
	$\frac{1}{4}$	$\frac{3}{4}$	1

$$\Pr(A' \cap B') = \frac{7}{8} - \frac{1}{4}$$
$$= \frac{5}{8}$$

Question 11 B

As confidence interval is symmetrical about the sample proportion, then $\hat{p} = 0.15$.

for 95% confidence interval: $\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

$$1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05$$
$$\Rightarrow 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{4n}} = \frac{0.05}{\sqrt{4}}$$
$$= 0.025$$

new 95% confidence interval $= (0.15 - 0.025, 0.15 + 0.025)$

$= (0.125, 0.175)$

Question 12 E

A function f is said to be strictly increasing when $a < b$ implies $f(a) < f(b)$ for all a and b in its domain. Remember to include the x-values at stationary points in the interval.

$\therefore (-\infty, b] \cup [d, \infty)$

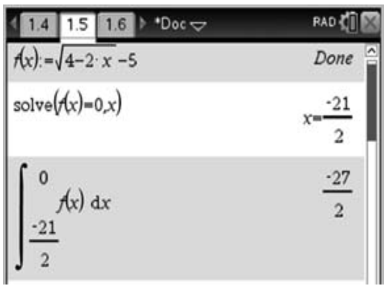
$= R \backslash (b, d)$

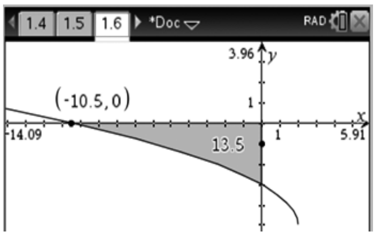
Question 13 **B**

The required area is enclosed by the graph and the coordinate axes between $x = -\frac{21}{2}$ and $x = 0$.

$$\text{area} = -\int_{-\frac{21}{2}}^{0} (\sqrt{4-2x} - 5)\,dx$$

$$= -\frac{27}{2}$$





However, the area is a positive quantity.

$\therefore \text{area} = \frac{27}{2}$

Question 14 **C**

$X \sim N(50, 60)$

$\Pr(X > b) = 0.025$

$\therefore \Pr(X < b) = 0.975$

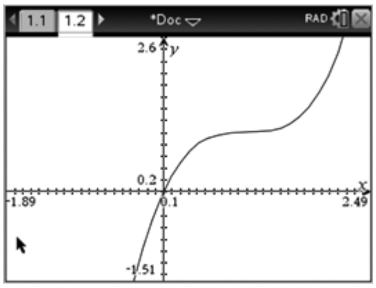
$b = 65.6797\ldots$

$\frac{b}{2} = 32.8998\ldots$

$\therefore \Pr\left(X > \frac{b}{2}\right) = 0.9840$

Question 15 D

Gradient function graph indicates the original function is always strictly increasing, with a stationary point of inflection located on the x-axis at an x-coordinate corresponding to the turning point of the gradient function. A function of the form $f(x) = (x - a)^3 + c$ would fit this profile, and when $c = 0$, $f(x)$ would pass through the origin as shown below.



Question 16 D

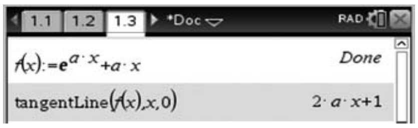
application of the chain rule:

$$\frac{dy}{dx} = f(x)g'(x)e^{g(x)} + f'(x)e^{g(x)}$$

$$\therefore \frac{dy}{dx} = e^{g(x)}(f'(x) + f(x)g'(x))$$

Question 17 A

Use a CAS calculator to find the equation of the tangent. The tangent has the equation $y = 2ax + 1$.



Question 18 D

Use a quadratic formula for an algebraic solution, or potentially set up a slider with *k* on a graph page.

$$\sin(x) = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(k)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 4k}}{2}$$

$$= \frac{2 \pm 2\sqrt{1 - k}}{2}$$

$$= 1 \pm \sqrt{1 - k}$$

Two considerations with this equation, firstly:

$1 - k < 0$ gives no solutions.

$\therefore k > 1$ is first part of solution.

Students also need to consider valid values for $\sin(x)$ as: $-1 \le \sin(x) \le 1$.

The positive solution from the quadratic formula will always result in $\sin(x) \ge 1$.

So then students need to investigate values from the negative part of the quadratic formula that produce no solutions.

$$\sin(x) < -1$$

$$1 - \sqrt{1 - k} < -1$$

$$\sqrt{1 - k} > 2$$

$$1 - k > 4$$

$$k < -3$$

The two parts of the solution combined gives $k \in (-\infty, -3) \cup (1, \infty)$.

Question 19 A

$$\Pr(X = 3 | X > 1) = \frac{1}{2p + 1}$$

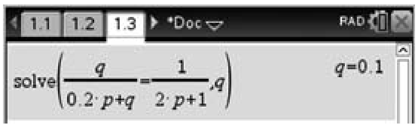
$$\Pr(X = 3 | X > 1) = \frac{\Pr(X = 3 \cap X > 1)}{\Pr(X > 1)}$$

$$\Pr(X = 3 | X > 1) = \frac{\Pr(X = 3)}{\Pr(X > 1)}$$

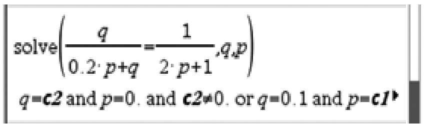
$$\Pr(X > 1) = 0.2p + q$$

$$\Pr(X = 3 | X > 1) = \frac{q}{0.2p + q}$$

$$\therefore \frac{q}{0.2p + q} = \frac{1}{2p + 1}$$



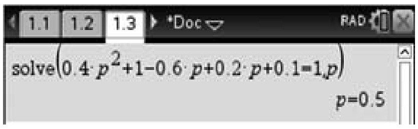
*Note: Solving for p and q will give p = 0 and a parameter for q (see below), but this solution does not keep the probability distribution valid. It is also the incorrect solution path for option **C**.*



Given $q = 0.1$, we need to find p. This can be done through trial and error with options **A** and **B** or algebraically:

$$\Sigma\Pr(X = x) = 1$$

$$\therefore 0.4p^2 + (1 - 0.6p) + 0.2p + 0.1 = 1$$



Question 20 B

$x' = \frac{1}{2}x + \frac{\pi}{2}$

$y' = -y + 3$

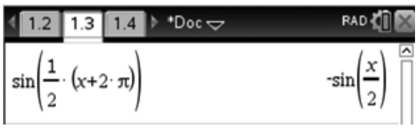
$\Rightarrow -y + 3 = 2\sin\left(\frac{1}{2}x + \frac{\pi}{2} + \frac{\pi}{2}\right) - 3$

$-y = 2\sin\left(\frac{1}{2}x + \pi\right) - 6$

$y = -2\sin\left(\frac{1}{2}(x + 2\pi)\right) + 6$

$y = 2\sin\left(\frac{1}{2}x\right) + 6$

$\therefore a = 2, b = \frac{1}{2}, c = 6$



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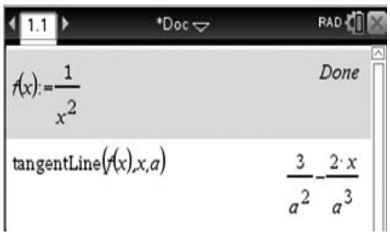
SECTION B

Question 1 (8 marks)

a. $f'(x) = -\frac{2}{x^2}$ A1

b. **i.** $y = -\frac{2}{a^3}x + \frac{3}{a^2}$ A1



ii. let $-\frac{2}{3}x + \frac{3}{a^2} = \frac{1}{x^2}$

$\therefore x = -\frac{a}{2}$ or $x = a$

but $x > 0$ and $a > 0$ M1

$\therefore x = -\frac{a}{2}$

at $x = -\frac{a}{2}$, $f\left(-\frac{a}{2}\right) = \frac{1}{\left(-\frac{a}{2}\right)^2}$ A1

$= \frac{4}{a^2}$

$\therefore Q\left(-\frac{a}{2}, \frac{4}{a^2}\right)$ as required

c. $d(a) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{3}{a^2}\right)^2}$

$= \sqrt{\frac{9a^2}{4} + \frac{9}{a^4}}$

$= 3\sqrt{\frac{a^2}{4} + \frac{1}{a^4}}$

$= 3\left(\frac{a^4}{4} + a^{-4}\right)^{\frac{1}{2}}$ A1

chain rule where $u = \left(\frac{a^4}{4} + a^{-4}\right)$

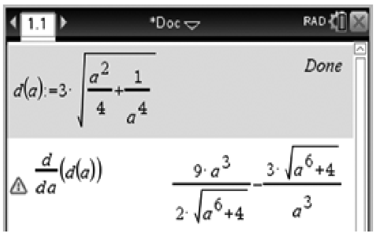
$d'(a) = \frac{du}{da} \times \frac{dd}{du}$

$= 3 \times \frac{1}{2} \times \left(\frac{a^4}{4} + a^{-4}\right)^{-\frac{1}{2}} \times \left(\frac{a}{2} - 4a^{-5}\right)$

$= \dfrac{3\left(\frac{a}{2} - \frac{4}{a^5}\right)}{2\sqrt{\frac{a^2}{4} + \frac{1}{a^4}}}$ or equivalent form *not required to simplify* M1

OR

accept CAS form as below



(solution continued on next page)

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For minimum/maximum, let $d'(a) = 0$.

$$\frac{3\left(\frac{a}{2}-\frac{4}{a^5}\right)}{2\sqrt{\frac{a^2}{4}+\frac{1}{a^4}}} = 0$$

$\therefore 3\left(\frac{a}{2}-\frac{4}{a^5}\right) = 0$ (numerator must be zero)

$\frac{a}{2}-\frac{4}{a^5} = 0$

$\frac{a^6}{2} - 4 = 0$ as $a \neq 0$

$a^6 = 8$

$a = \pm\sqrt{2}$

but $a > 0$

$\therefore a = \sqrt{2}$ *correct a value* A1

For minimum distance, $d(\sqrt{2}) = \frac{3\sqrt{3}}{2}$. *exact value of d* A1

Question 2 (11 marks)

a. period $= \frac{2\pi}{n}$

$= \frac{2\pi}{\frac{\pi}{6}}$

$= 12$ months A1

amplitude = 200 A1

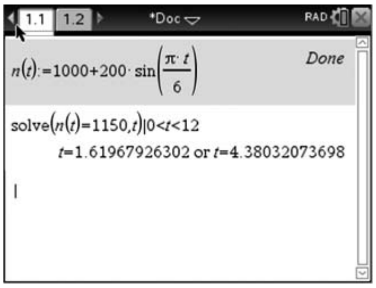
b. maximum population = 1000 + 200

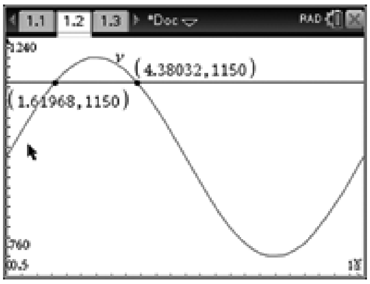
= 1200 worms A1

c. $n(t) > 1150$

total time = 4.380 – 1.619 *correct times* A1

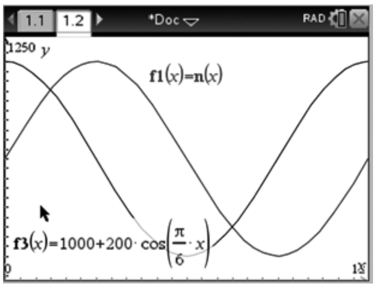
= 2.76 months (to two decimal places) A1





d. **i.** Graphically, a translation of 3 units right, or 9 units to the left to match the graphs is required.

As $d > 0$, translation must be to the left, therefore $d = 9$ is the smallest possible value. A1



ii. Starting with a translation of 9 units left and then multiples of the period (12 months) gives the following general solution for d.

$d = 9 + 12k$ where $k \in Z^+ \cup \{0\}$ *expression for d* A1

positive integers for k A1

e. $w(t) = 1000 + 200\sin\left(\frac{\pi t}{6}\right) + q(t - 12)$

$w'(t) = \frac{100\pi}{3}\cos\left(\frac{\pi t}{6}\right) + q$ A1

$w'(t) > 0$

$\frac{100\pi}{3}\cos\left(\frac{\pi t}{6}\right) + q > 0$

minimum value for $\cos\left(\frac{\pi t}{6}\right) = -1$ M1

$-\frac{100\pi}{3} + q > 0$

$q > \frac{100\pi}{3}$

A1

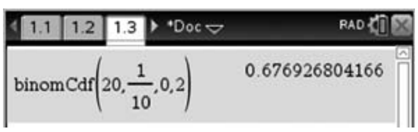
Question 3 (19 marks)

a. **i.** $X \sim Bi(n, p)$

$X \sim Bi\left(20, \frac{1}{10}\right)$ M1

$\Pr(X < 3) = \Pr(X \le 2)$

$= 0.6769$ A1

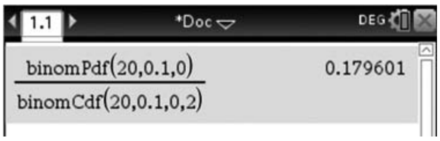


ii. $X \sim Bi\left(20, \frac{1}{10}\right)$

$$\begin{aligned}\Pr(X = 0 | X \le 2) &= \frac{\Pr(X = 0 \cap X \le 2)}{\Pr(X \le 2)} \\ &= \frac{\Pr(X = 0)}{\Pr(X \le 2)} \\ &= \frac{0.121576\ldots}{0.676926\ldots} \\ &= 0.1796\end{aligned}$$

M1

A1



b. $X \sim Bi\left(n, \frac{1}{10}\right)$ $\quad n = ?$

$$\begin{aligned}\Pr(X > 0) &\ge 0.8 \\ 1 - \Pr(X = 0) &\ge 0.8 \\ 1 - 0.9^n &\ge 0.8 \\ n &\ge 15.27 \\ n &= 16\end{aligned}$$

M1

A1

c. **i.** $\mathrm{E}(\hat{P}) = p$

$$= \frac{1}{10}$$

A1

ii. $\mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$

$$= \sqrt{\frac{\frac{1}{10}\left(1 - \frac{1}{10}\right)}{20}}$$

≈ 0.0671 (to four decimal places)

A1

iii. $\mathrm{sd}(\hat{P}) \propto \frac{1}{\sqrt{n}}$

To halve the standard deviation, increase n by a factor of 4.

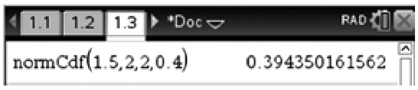
$$\begin{aligned}n &= 4 \times 20 \\ &= 80\end{aligned}$$

A1

d. **i.** $Y \sim N(\mu, \sigma^2)$

$Y \sim N(2, 0.4^2)$ M1

$\Pr(1.5 < Y < 2) \approx 0.3944$ (to four decimal places) M1



ii. $\Pr(Y \geq 2 | Y \geq 1.5) = \dfrac{\Pr(Y > 2)}{\Pr(Y > 1.5)}$ A1

$= \dfrac{\Pr(Y > 2)}{\Pr(1.5 < Y < 2) + \Pr(Y > 2)}$ M1

Note: Students must use this step as it is a 'hence' question.

$= \dfrac{0.5}{0.3944 + 0.5}$

$= 0.5591$ A1

e. $\hat{p} = \dfrac{20}{100}$

$= 0.2$

An approximate 95% confidence interval for p is given by:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Where $\hat{p} = 0.2$, $z = 2$, $n = 100$. M1

$$\left(0.2 - 2\sqrt{\frac{0.2 \times 0.8}{100}}, 0.2 + 2\sqrt{\frac{0.2 \times 0.8}{100}}\right) = \left(\frac{3}{25}, \frac{7}{25}\right)$$

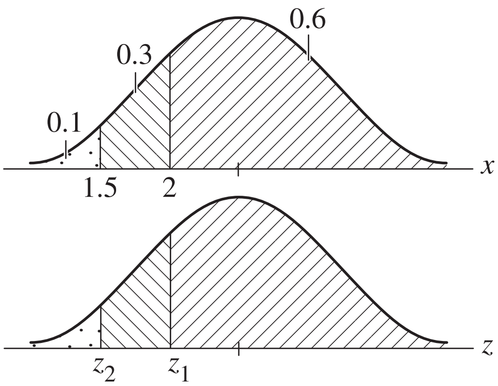
$a = 3$, $b = 7$ A1

f. $\Pr(X > 2) = \frac{12}{20}$

$= 0.6$

$\Pr(1.5 < X < 2) = \frac{6}{20}$

$= 0.3$



$z = \frac{x - \mu}{\sigma}$

$-0.25334 = \frac{2 - \mu}{\sigma}$ *correct simultaneous equations* M1

$-1.28155 = \frac{1.5 - \mu}{\sigma}$

solving simultaneously gives:

$\mu = 2.1232$ A1

$\sigma = 0.4863$ A1

Question 4 (7 marks)

a. dilation factor of $\mathbf{\frac{1}{4}}$ from the x-axis or alternatively a dilation factor of **2** from the y-axis *both correct* A1

translation of **5 in the negative *y*-direction** A1

b. $\frac{1}{4}x^2 - 5 = a^2$ *finding correct x-coordinates* M1

$\frac{1}{4}x^2 = a^2 + 5$

$x^2 = 4(a^2 + 5)$

$x = 2\sqrt{a^2 + 5}$ as $x > 0$

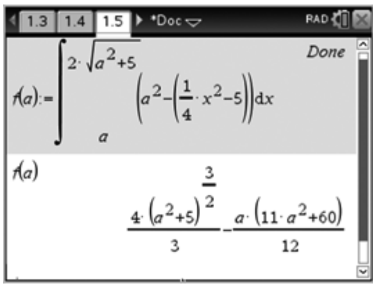
B is $(2\sqrt{a^2 + 5} - a, a^2)$.

c. **i.** Let $f(a)$ = area

$$= \int_{x_1}^{x_2} (y - g(x))\,dx$$

$$= \int_{a}^{2\sqrt{a^2+5}} \left(a^2 - \left(\frac{1}{4}x^2 - 5\right)\right)dx$$ *correct integral* M1

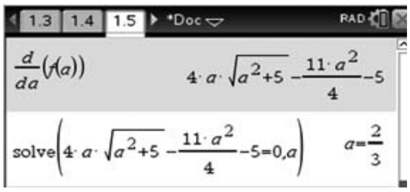
$$= 4(a^2+5)^{\frac{3}{2}} - \frac{a(11a^2+60)}{12}$$ *correct area function* A1



ii. For minimum area, let $f'(a) = 0$.

$$4a\sqrt{a^2+5} - \frac{11a^2}{4} - 5 = 0$$ M1

$$a = \frac{2}{3}$$



$a = \frac{2}{3}$, $y = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

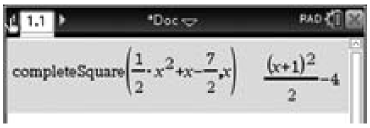
coordinates for minimum area = $\left(\frac{2}{3}, \frac{4}{9}\right)$ A1

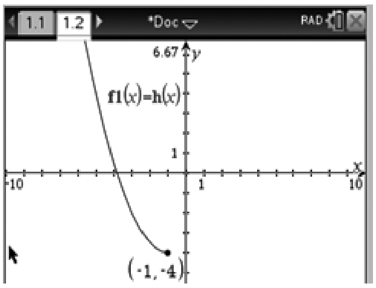
Question 5 (15 marks)

a. **i.** $h(x) = \frac{1}{2}(x^2 + 2x - 7)$

$= \frac{1}{2}(x + 1)^2 - 4$ M1

$k = -1$ A1





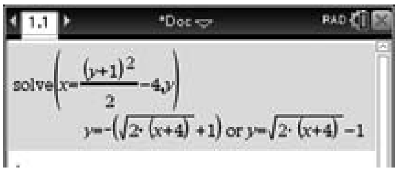
ii. Let $y = \frac{1}{2}(x + 1)^2 - 4$.

For inverse, swap *x* and *y*.

$x = \frac{1}{2}(y + 1)^2 - 4$ M1

$y = -\sqrt{2(x + 4)} - 1$ (negative $\sqrt{\ }$ as using LHS of parabola)

$h^{-1}(x) = -\sqrt{2(x + 4)} - 1$ A1

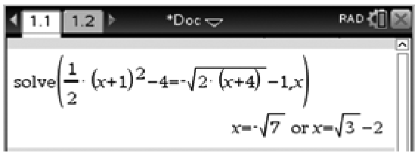


iii. domain: $[-4, \infty)$ A1

range: $(-\infty, -1]$ A1

b. Let $h(x) = h^{-1}(x)$ or $h(x) = x$ or $h^{-1}(x) = x$.

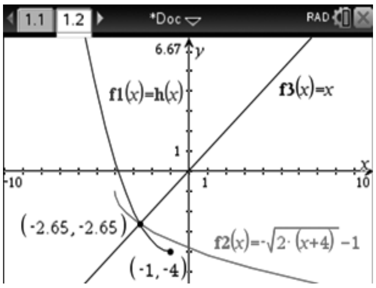
$\frac{1}{2}(x+1)^2 - 4 = -\sqrt{2(x+4)} - 1$ M1



$x = -\sqrt{7}$ as $x < -1$

point of intersection where $x = y$

point of intersection = $(-\sqrt{7}, -\sqrt{7})$ A1



c. **i.** $g(x) = h(x+b)$

$= \frac{1}{2}(x+b+1)^2 - 4$ A1

ii. As $g(x)$ is a transformation of $h(x)$ 'b' units left, then $g^{-1}(x)$ will be a transformation of $h^{-1}(x)$ 'b' units down. M1

$g^{-1}(x) = -\sqrt{2(x+4)} - 1 - b$ A1

d. For intersection at origin, $g^{-1}(x)$ must pass through (0, 0).

$g^{-1}(0) = 0$ M1

$b = -2\sqrt{2} - 1$ A1

check $g(x) \rightarrow g(-2\sqrt{2} - 1) = 0$

$g(x)$ and $g^{-1}(x)$ intersect at (0, 0) when $b = -2\sqrt{2} - 1$

 MMU34EX2_SS_2017.FM

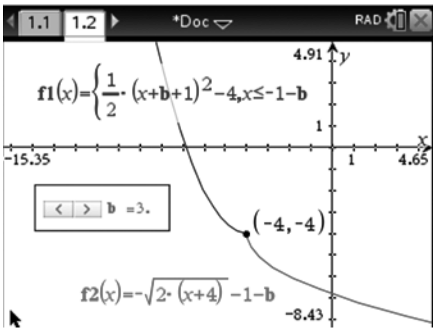
e. Let $g(x) = g^{-1}(x)$. M1

$\frac{1}{2}(x + b + 1)^2 - 4 = -\sqrt{2(x + 4)} - 1 - b$ A1

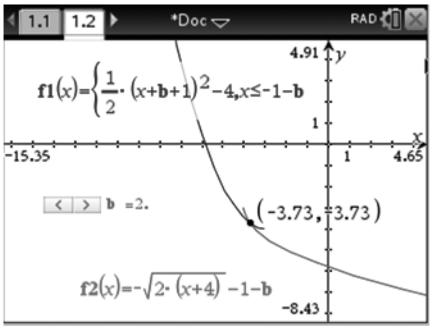
However, solving this equation either by hand or with CAS introduces questionable solutions (resulting from squaring both sides of an equation) and a graphical solution may the simplest method.

Graphically:

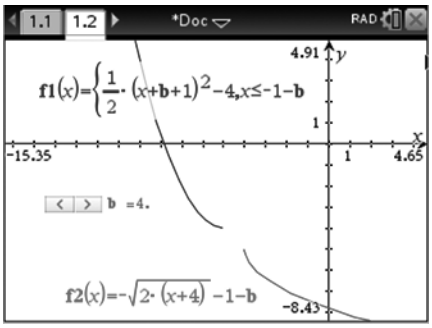
If $b = 3$, the function and its inverse intersect once at (–4, –4). *method for identifying b = 3* M1



If $b < 3$, the function and its inverse still intersect.



If $b > 3$, the function and its inverse do not intersect.



So graphs have no intersection points for $b > 3$. A1