THE SCHOOL FOR EXCELLENCE (TSFX)
UNITS 3 & 4 MATHEMATICAL METHODS 2017
WRITTEN EXAMINATION 1 – SOLUTIONS

Errors and updates relating to this examination paper will be posted at
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Note:

- $\left(\mathrm{A}\,\frac{1}{2}\times 4\downarrow\right)$ means four answer half-marks rounded **down** to the next integer.
  Rounding occurs at the end of a part of a question.
- **M**1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip).

QUESTION 1

(a) $\frac{d}{dx}\left(x^3\log_e(4x)\right)=\frac{d}{dx}\left(x^3\right)\times\log_e(4x)+x^3\times\frac{d}{dx}\left(\log_e(4x)\right)$ (Product Rule) **M1**

$$= 3x^2\log_e(4x)+x^3\times\frac{4}{4x}$$

$$=3x^2\log_e(4x)+x^3\times\frac{1}{x}$$

$$= 3x^2\log_e(4x)+x^2$$ **A1**

(b) $f(x)=\cos^2(3x)$

Let $u=\cos(3x)$ $\therefore\frac{du}{dx}=-3\sin(3x)$ (Chain Rule) **M1**

Let $y=u^2$ $\therefore\frac{dy}{du}=2u=2\cos(3x)$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-6\sin(3x)\cos(3x)$ **A1**

$f'\left(\frac{\pi}{12}\right)=-6\sin\left(\frac{3\pi}{12}\right)\cos\left(\frac{3\pi}{12}\right)=-6\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}$

$=-3$ **A1**

QUESTION 2

(a) $\int\left(4e^{-2x}+\frac{3}{\cos^2(x)}+7\right)dx=\frac{4}{-2}e^{-2x}+3\tan(x)+7x+c$

$=-2e^{-2x}+3\tan(x)+7x+c$ **A** $\frac{1}{2}$ **× 4↓**

(b) $\int_0^1(1-2x)^4dx=\left[\frac{1}{-2}\times\frac{1}{5}(1-2x)^5\right]_0^1$ **A1**

$=\left(-\frac{1}{10}\times(-1)^5\right)-\left(-\frac{1}{10}\right)$

$=\frac{1}{5}$ **A1**

QUESTION 3

$\frac{3+\cos^2(\theta)}{\sin(\theta)-2}=3\sin(\theta)$

$3+\cos^2(\theta)=3\sin(\theta)(\sin(\theta)-2)$

$3+\cos^2(\theta)=3\sin^2(\theta)-6\sin(\theta)$

$3+(1-\sin^2(\theta))=3\sin^2(\theta)-6\sin(\theta)$ **M1**

$0=4\sin^2(\theta)-6\sin(\theta)-4$

$2\sin^2(\theta)-3\sin(\theta)-2=0$ and so $(2\sin(\theta)+1)(\sin(\theta)-2)=0$

$2\sin(\theta)+1=0$ or $\sin(\theta)-2=0$ **A1**

Hence $\sin(\theta)=-\frac{1}{2}$ **<u>only</u>** as $\sin(\theta)\neq-2$

$\theta=\frac{7\pi}{6}+2n\pi$ **or** $\frac{11\pi}{6}+2n\pi$ where $n\in Z$. **A1**

QUESTION 4

(a) State the transformations needed to change $f(x) = 2x^2 - 4x - 6$ back to the graph with equation $g(x) = x^2$. The answer will be the reverse of the transformations needed to change $g(x)$ to $f(x)$.

First re-write $f(x)$ in transformation form: $f(x) = 2x^2 - 4x - 6 = 2(x-1)^2 - 8$

The transformations required to change $g(x)$ to $f(x)$ are:

- A dilation from the x axis by a factor of 2
- Translations of 1 unit to the right and 8 units down

Reversing this gives the answer required:

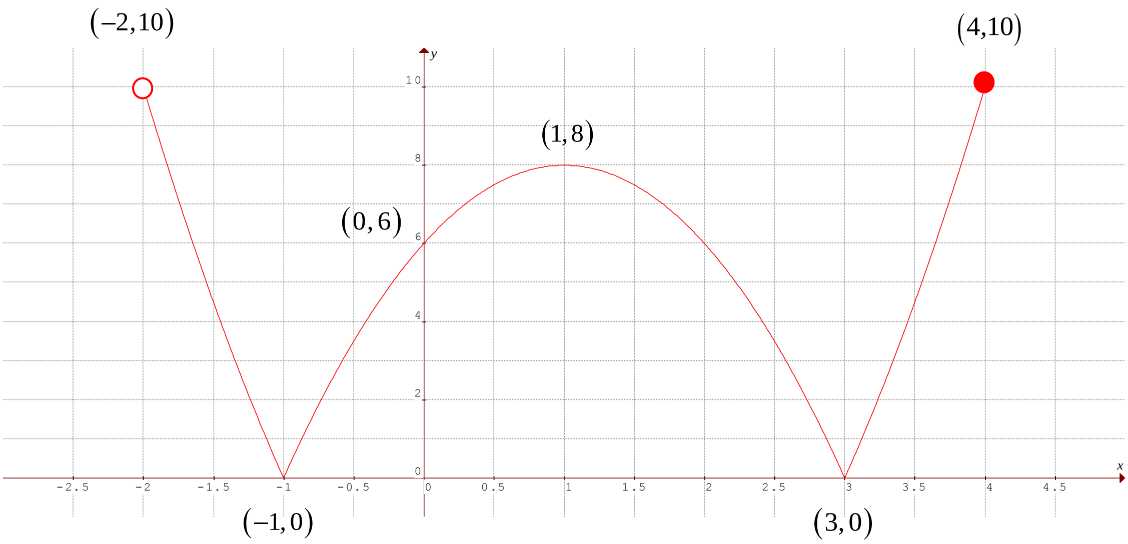
A translation of 1 unit to the right and 8 units down followed by a dilation by a factor $\frac{1}{2}$ from the x axis.

A2 all correct
A1 one correct

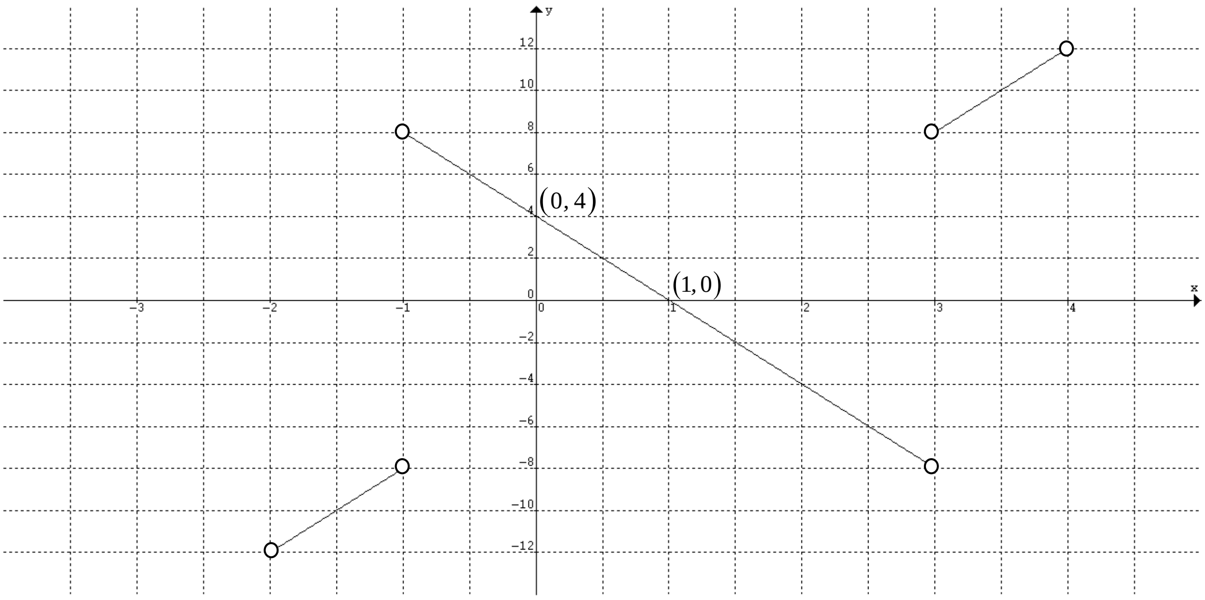
(b) The following information is required on the graph:

- $(-2,10)$ empty dot
- $(4,10)$ full dot
- Maximum turning point coordinates $(1,8)$
- Intercepts $(0,6)$, $(-1,0)$ and $(3,0)$

A $\frac{1}{2}$ **× 4**↓



(c)



Lines correct irrespective of whether endpoints are empty/full **A1**

All six end-points need to be empty **A** $\frac{1}{2}$

Intercepts at $(0, 4)$ and $(1, 0)$ **A** $\frac{1}{2}$

Round answer down

QUESTION 5

(a) $(m+1)\,x+9y=21$

$2x+(m-2)\,y=7$

Setting these equations up in matrix form gives: $\begin{bmatrix} m+1 & 9 \\ 2 & m-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 21 \\ 7 \end{bmatrix}$ **M1**

The determinant of the 2×2 matrix is zero for non-unique solutions:

$(m+1)(m-2)-18=0$

$m^2-m-2-18=0$

$m^2-m-20=(m-5)(m+4)=0$

$\therefore m=5$ or $m=-4$ for non-unique solutions. **A1**

(b) (i) When $m=5$, the given equations become:

$6x+9y=21$ (1)
$2x+3y=7$ (2)

Dividing equation (1) by 3 gives $2x+3y=7$, which is the same as equation (2).

Therefore, an infinite set of solutions exist when $m=5$. **A1**

(ii) Substitute $x=\lambda$ into equation (2):

$2\lambda+3y=7$

$\therefore y=\frac{7-2\lambda}{3}$ **A1**

QUESTION 6

$\int_{1.5}^{5}\left(2+\frac{1}{x-1}\right)dx=\left[2x+\log_e(x-1)\right]_{1.5}^{5}$ **(Anti-derivative) A1**

$=\left(10+\log_e 4\right)-\left(3+\log_3 0.5\right)$

$=7+2\log_e 2-\log_e\left(\frac{1}{2}\right)$ **A1**

$=7+2\log_e 2-\left(\log_e 1-\log_e 2\right)$ $\left[\log_e\left(\frac{1}{2}\right)=-\log_e(2)\right]$ **M1**

$=7+3\log_e 2$

Equating gives: $A+B\log_e 2=7+3\log_e 2$

Therefore: $A=7$ and $B=3$ **(Both values correct) A1**

QUESTION 7

(a) $\int_{0}^{1}a\left(x^4-x^5\right)dx=a\left[\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^1=1$ **(Let anti-deriv = 1) M1**

$a\left(\frac{1}{5}-\frac{1}{6}\right)=1$

$\therefore \frac{a(6-5)}{30}=1$

$\therefore a=30$

(b) $\int_0^1 30x\left(x^4 - x^5\right) dx = 30\left[\frac{1}{6}x^6 - \frac{1}{7}x^7\right]_0^1$ **(Correct integrand) M1**

$$= 30\left(\frac{1}{6} - \frac{1}{7}\right)$$

$$= \frac{30}{42}, \frac{5}{7}$$ **A1**

(c) The mode occurs at the maximum value of $f(x)$.

$$\frac{d}{dx}\left(30x^4 - 30x^5\right) = 0$$

$$30(4x^3 - 5x^4) = 0$$

$$30x^3(4 - 5x) = 0$$

$x = \frac{4}{5}$ or $x = 0$ **M1**

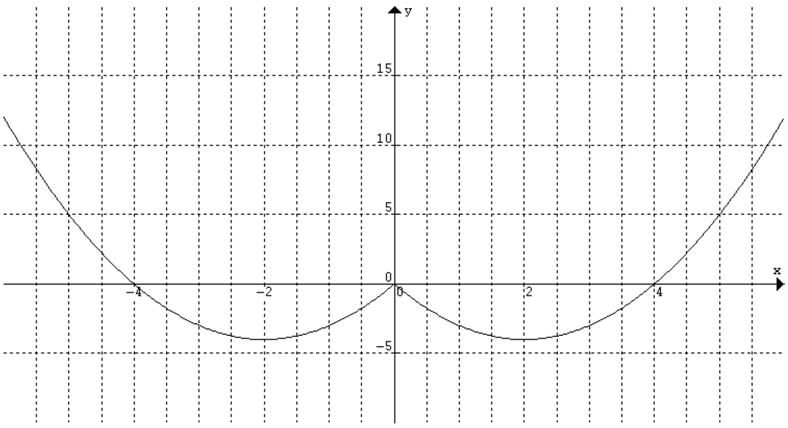
When $x = 0$: $y = 0$

When $x = \frac{4}{5}$: $y = 30 \times \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right) = \frac{4}{5}$

The mode is $\frac{4}{5}$. **A1**

QUESTION 8

(a)



(left of origin) A1
(right of origin) A1

(b) $R \setminus \{0\}$ **A1**

QUESTION 9

(a) $\hat{p} = \frac{9}{15} = \frac{3}{5}$ **A1**

Value represents the proportion of heart surgery patients that died. **A1**

(b) $$\left(p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}, \quad p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \right)$$

For a 95% confidence interval, 5% will be in the tails. i.e. 2.5% in each tail.
Therefore, the area under the normal distribution curve to the left of z is 0.975.

This means that z = 1.96

$$\left(\frac{3}{5} - 1.96\sqrt{\frac{0.6(0.4)}{15}}, \quad \frac{3}{5} + 1.96\sqrt{\frac{0.6(0.4)}{15}} \right) = (0.352,\ 0.848)$$ **A2**

(c) The Normal distribution can be used when $np \geq 10$ and $n(1-p) \geq 10$.

$np = 15 \times 0.6 = 9$ which is less than 10. **M1**

This means that the Normal model cannot be used in this case. **A1**