THE SCHOOL FOR EXCELLENCE (TSFX)

UNITS 3 & 4 MATHEMATICAL METHODS CAS 2017

WRITTEN EXAMINATION 2

Reading Time: 15 minutes
Writing Time: 2 hours

QUESTION AND ANSWER BOOKLET

Student Name:

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory **DOES NOT** need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are **NOT** permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials Supplied

- Question and answer book of 22 pages and a separate sheet of formulas.
- Working space is provided throughout the book.

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Students are **NOT** permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1 – MULTIPLE CHOICE QUESTIONS

Instructions for Section 1

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers. You should attempt every question.

No marks will be given if more than one answer is completed for any question.

QUESTION 1

The inverse f^{-1} of the function $f:(-\infty,3)\to R,\ f(x)=-x^2+6x$ can be defined as

A. $f^{-1}:R\to(-\infty,3),\ f^{-1}(x)=3-\sqrt{x-9}$

B. $f:(-\infty,9)\to R,\ f(x)=-x^2+6x$

C. $f^{-1}:(-\infty,3)\to R,\ f^{-1}(x)=3-\sqrt{9-x}$

D. $f^{-1}:(-\infty,9)\to R,\ f^{-1}(x)=3-\sqrt{9-x}$

E. $f^{-1}:(-\infty,9)\to R,\ f^{-1}(x)=3+\sqrt{9-x}$

QUESTION 2

The graph of a function has a horizontal asymptote with equation $y=a$ and a vertical asymptote with equation $x=-b$, where a and b are constants. The equation of the function could be

A. $y=\dfrac{4}{x-b}+a$

B. $y=\dfrac{4}{b-x}-a$

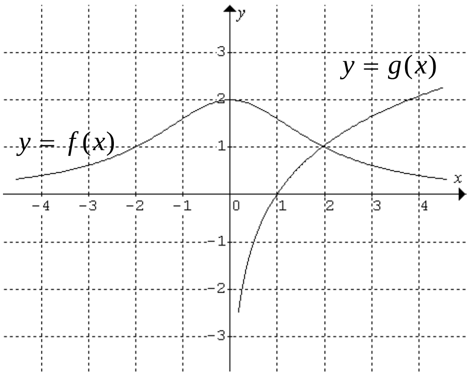
C. $y=\dfrac{1}{x+b}-a$

D. $y=\dfrac{1}{x+b}+a$

E. $y=\dfrac{1}{x+a}-b$

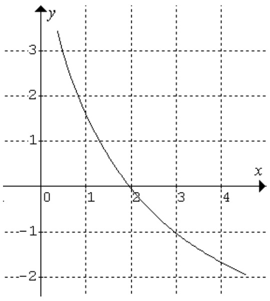
QUESTION 3

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.

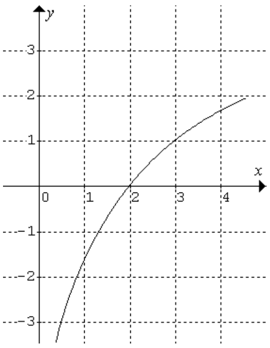


The graph of $y = f(x) - g(x)$ is

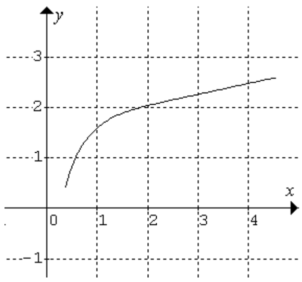
A.



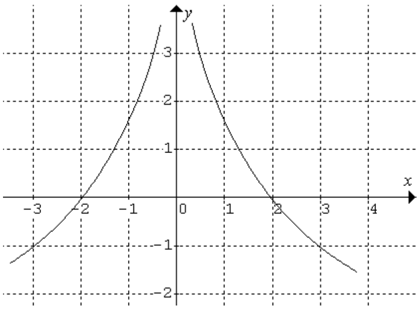
B.



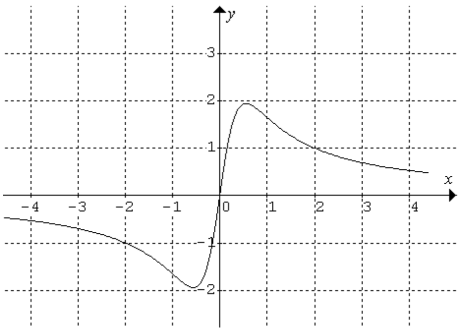
C.



D.



E.



QUESTION 4

The maximal domain for which the expression $\log_e(x^2)-\log_e(1-x)$ is defined is

A. R^+

B. $R/\{0\}$

C. $(-\infty,0)\bigcup(0,1)$

D. $(-\infty,0]\bigcup(0,1]$

E. $(0,1)$

QUESTION 5

The transformation $T:R^2\to R^2:\mathrm{T}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1&0\\0&2\end{bmatrix}\left(\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}\right)$ in the equation $y=\sqrt{x}$ is

A. $y=\sqrt{-x+2}+2$

B. $2y=\sqrt{-x+2}+1$

C. $\dfrac{y}{2}=\sqrt{2-x}+1$

D. $y=\sqrt{x-2}+1$

E. $2y=\sqrt{x+2}-1$

QUESTION 6

The equation of the tangent at the point $(0,3)$ on the curve with equation $y=f(x)$ is $y=3-x\sqrt{2}$. The equation of the tangent at the point $\left(3,\ 3+\sqrt{2}\right)$ on the curve with equation $y=f(x-3)+\sqrt{2}$ has the equation

A. $y=3+\sqrt{2}+(x-3)\sqrt{2}$

B. $y=3+\sqrt{2}-(x-3)\sqrt{2}$

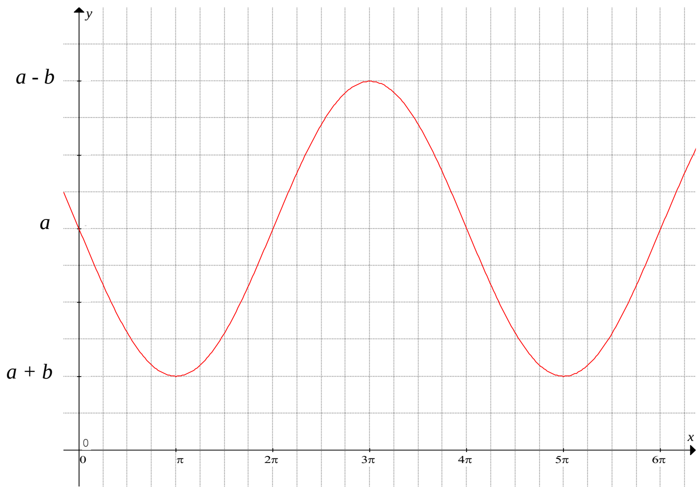
C. $y=(x-3)\sqrt{2}-3$

D. $y=(3-x)+2$

E. $y=3-x\sqrt{2}+3\sqrt{2}$

QUESTION 7

The graph shown is defined by the equation



A. $y = a - b\sin(2x)$

B. $y = a - b\sin\left(\frac{x}{2}\right)$

C. $y = a + b\sin(2x)$

D. $y = a + b\sin\left(\frac{x}{2}\right)$

E. $y = -a - b\sin\left(\frac{x}{2}\right)$

QUESTION 8

For $f(x) = 5x - 7$, which of the following statements is true?

A. $f(x+y) = f(x) + f(y) + 14$

B. $f(x+y) = f(x) - f(y) - 7$

C. $f(x+y) = f(x).f(y) + 49$

D. $f(x+y) = f(x) + f(y) + 7$

E. $f(x+y) = f(x) + f(y)$

QUESTION 9

If $f(x) = \frac{ax}{\log_e(ax)}$ where a is a positive constant, then the gradient of the normal to the curve $y = f(x)$ can be written as

A $\dfrac{a(\log_e(ax) - 1)}{(\log_e(ax))^2}$

B $\dfrac{(\log_e(ax))^2}{a(\log_e(ax) - 1)}$

C $\dfrac{(\log_e(ax))^2}{a(1 - \log_e(ax))}$

D $\dfrac{(\log_e(x))^2}{a(1 - \log_e(x))}$

E $\dfrac{-a(\log_e(ax) - 1)}{(\log_e(ax))^2}$

QUESTION 10

The position x of a particle moving along a straight line at time t is given by $x = 4 + 5\sin\left(\frac{\pi}{5}t\right)$.

When the particle's acceleration is $\frac{3}{25}\pi^2$, its position is $x =$

A. -1

B. 0

C. 1

D. 7

E. 9

QUESTION 11

The function with rule $f(x) = \begin{cases} (x-a)^3 + 2, & x \le 0 \\ bx + \cos x, & x > 0 \end{cases}$ is differentiable for all values of x if

A. $a = 0$ and $b = 1$

B. $a = 1$ and $b = -3$

C. $a = 1$ and $b = 3$

D. $a = -1$ and $b = -3$

E. $a = -1$ and $b = 3$

QUESTION 12

The function $f(x) = a\sin(x) - b\sqrt{3}\cos(x)$ will have a minimum turning point at $x = \frac{\pi}{3}$ if

A. $a = 3b$ and $a < 0$

B. $a = 3b$ and $a > 0$

C. $a = -3b$ and $a > 0$

D. $a = -3b$ and $a < 0$

E. $a = -b$ and $a < 0$

QUESTION 13

If $\int_b^a g(x)\,dx = 2$ and $\int_b^a h(x)\,dx = -3$ then $\int_b^a 3g(x) - x - h(x)\,dx$ is equal to

A. $9 + \dfrac{b^2 - a^2}{2}$

B. $9 + \dfrac{a^2 - b^2}{2}$

C. $3 + \dfrac{a^2 - b^2}{2}$

D. $3 + \dfrac{b^2 - a^2}{2}$

E. $3 + b^2 - a^2$

QUESTION 14

If $f'(x) = x^2 + \dfrac{1}{x}$ and $f(1) = \dfrac{4}{3}$, then the average rate of change between $x = 2$ and $x = 4$ is

A. 5.875

B. $26 + \log_e 8$

C. $\dfrac{56}{3} + \log_e 2$

D. $\dfrac{28}{3} + \log_e \sqrt{2}$

E. $\dfrac{28}{3} + \log_e 2$

QUESTION 15

Given that the derivative of $x\log_e(3x)$ is equal to $1 + \log_e(3x)$, then $\int 2\log_e(3x)\,dx$ is equal to

A. $x\log_e(3x) - c$

B. $2x\log_e(3x - 1) - c$

C. $2x\log_e(3x) - c$

D. $2x(\log_e(3x) - 1) - c$

E. $x\log_e(3x) - 1 - c$

QUESTION 16

The average value of the function $y = \sin\left(\frac{x}{2}\right)$ over the interval $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ is

A. $\frac{6(\sqrt{2} - \sqrt{3})}{\pi}$

B. $\frac{12(\sqrt{3} - \sqrt{2})}{\pi}$

C. $\frac{6(\sqrt{3} - \sqrt{2})}{\pi}$

D. $\frac{3(\sqrt{2} - 1)}{\pi}$

E. $\sqrt{3} - \sqrt{2}$

QUESTION 17

A and B are two events with a sample space S. $\Pr(A' \cap B) = \frac{1}{3}$ and $\Pr(A' \cap B') = \frac{2}{5}$.

If $\Pr(A \cap B) = 3 \times \Pr(A \cap B')$ then $\Pr(B) =$

A. $\frac{1}{15}$

B. $\frac{2}{15}$

C. $\frac{1}{3}$

D. $\frac{2}{5}$

E. $\frac{8}{15}$

QUESTION 19

A discrete random variable X has a mean of 4 and a standard deviation of 3.
The value of $E(X^2)$ is closest to

A. 25

B. 19

C. 16

D. 9

E. 7

The following information refers to Question 19 and Question 20:

The continuous random variable X has a probability density function that is defined by

$$f(x) = \begin{cases} \pi \sin(2\pi x), & 0 \le x \le \frac{1}{2} \\ 0, & elsewhere \end{cases}$$

QUESTION 19

The variance of the probability density function can be found by evaluating

A. $\pi \int_0^{\frac{1}{2}} x \sin(2\pi x)\, dx$

B. $\pi \int_0^{\frac{1}{2}} x^2 \sin(2\pi x)\, dx - \pi \int_0^{\frac{1}{2}} \sin(2\pi x)\, dx$

C. $\pi \int_0^{\frac{1}{2}} x^2 \sin(2\pi x)\, dx - \left(\pi \int_0^{\frac{1}{2}} x \sin(2\pi x)\, dx \right)^2$

D. $\pi \int_0^{\frac{1}{2}} x \sin(2\pi x)\, dx - \pi \int_0^{\frac{1}{2}} x^2 \sin(2\pi x)\, dx$

E. $\pi \left(\int_0^{\frac{1}{2}} x \sin(2\pi x)\, dx \right)^2$

QUESTION 20
The value of a such that $\Pr(X > a) = 0.2$ is closest to

A. 0.26
B. 0.30
C. 0.32
D. 0.35
E. 0.40

QUESTION 21
Students in a Mathematics class are concerned about their test results, which are marked out of 100. Over time it seems that the results are distributed normally with a mean of 51%. It is known that a student selected randomly will have a probability of 0.42 that they will score less than 50% for a test. The standard deviation of the distribution is closest to

A. 4.95
B. -4.95
C. 8
D. 39.89
E. 1

QUESTION 22
Which of the following statements is true?

I. When the margin of error is small, the confidence level is high.
II. When the margin of error is small, the confidence level is low.
III. A confidence interval is a type of point estimate.
IV. A population mean is an example of a point estimate.

A. I only
B. II only
C. III only
D. IV only
E. None of the above

SECTION 2 – EXTENDED ANSWER QUESTIONS

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

QUESTION 1 (13 marks)

Let $f:[0,\infty)\to R,\ f(x)=x\sqrt{x}-x$.

a. (i) Find $f'(x)$ and hence state the x coordinate of the minimum point on the graph of f.

2 marks

(ii) State the interval for which the graph of f is strictly increasing.

1 mark

b. (i) On the set of axes below sketch the graph of $f:[0,\infty)\rightarrow R,\ f(x)=x\sqrt{x}-x$, showing the coordinates of the stationary point(s) and any axial intercepts.

3 marks

Consider the function $g:[0,4]\rightarrow R,\ g(x)=x\sqrt{x}-x$.

c. (i) Find the equation of the straight line that joins the points $(1,0)$ and $(4,4)$ on the graph of g.

2 marks

(ii) Hence find the point $\left(a, \mathrm{g}(a)\right)$ where the tangent to the graph of g at $x = a$ is parallel to the line found in **c (i)**.

2 marks

A straight line with equation $y = g(a)$ from part **c (ii)** is drawn on the same set of axes and intersects the graph of $y = g(x)$.

d. (i) Write an expression that could be used to find the area enclosed between the graphs of $y = g(a)$, $y = g(x)$ and the y axis.

2 marks

(ii) Hence find the value of the area correct to 2 decimal places.

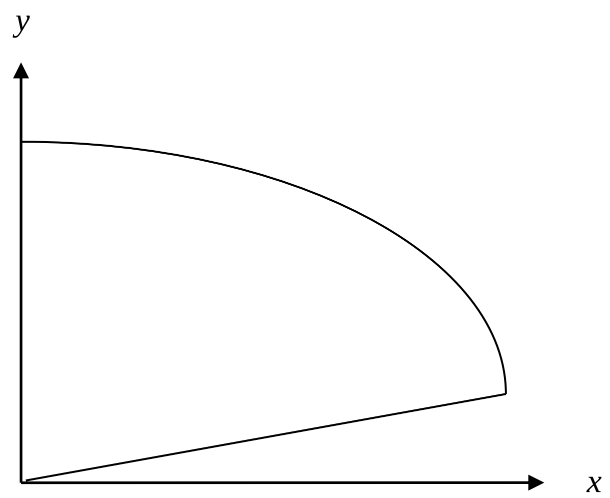
1 mark

QUESTION 2 (14 marks)
In a Melbourne art gallery a new painting is to be commissioned. A graph that describes the path of a curve on a painting in the art gallery has the origin of perpendicular axes, x and y, at the lower left-hand corner of the painting.

a. A straight line through this origin is at an angle to the x axis of $\frac{\pi}{12}^{c}$. Show that the equation of the line is $y = (2-\sqrt{3})x$.

1 mark

The painting is planned to have a mixture of curves and lines as shown in the diagram below.



The equation of the curve partly shown is of the form $y = -(x-b)(x+1)^2$. It cuts the y axis at the point $(0, 2)$. The tangent to this curve at the point where $x = p$ is perpendicular to the straight line found in part **(a).** Note that b and p are real constants.

b. Show that $b = 2$.

1 mark

c. Find the value of p giving the answer in the form of $\frac{\sqrt{\left(m\sqrt{3}+n\right)}}{3}$, where m and n are integer values.

3 marks

The gallery owner requires that the region of the painting enclosed by the two graphs is coloured in bright yellow, however, no-one working at the gallery has enough calculus knowledge to correctly find the exact area. The owner decides to find the approximate area between the straight line and the cubic curve by considering vertical strips of width 0.5 units between $x=0$ and $x=1.5$.

d. (i) Find the area of the left endpoint rectangles between the cubic curve and the x axis.

2 marks

(ii) Find the area of the left endpoint rectangles between the straight line and the x axis.

2 marks

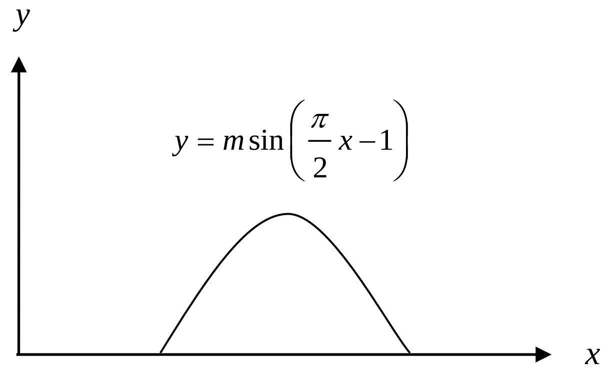
(iii) Hence find an approximate area, correct to 2 decimal places, between the two graphs.

1 mark

e. Using calculus, find the area of the region to be painted yellow, correct to 2 decimal places.

2 marks

The gallery owner's wife was not happy with the previous design. She now wants a sinusoidal curve instead of a cubic curve and definitely wants no straight lines. Her design of a section of the new painting is shown below.



f. Find the value of m, where m is a real constant, so that the area enclosed by the curve $y = m\sin\left(\frac{\pi}{2}x - 1\right)$ and the x axis has a total area of 4 square units.

2 marks

QUESTION 3 (15 marks)
An inverted right circular cone of height 10 cm contains a new type of medicine called **'Cut-the-Fat'** that will cure obesity. Special agents are guarding this medicine as when it hits the market, it is expected to be a national seller, generating great profits for the company.

Agent Lurch is hoping to steal some of the liquefied drug for himself to use before it is released to the public. He cuts a very small hole in the bottom of the cone in order to let some liquid out. He is stupid enough not to realise that once he takes what he needs, the cone will continue leaking.

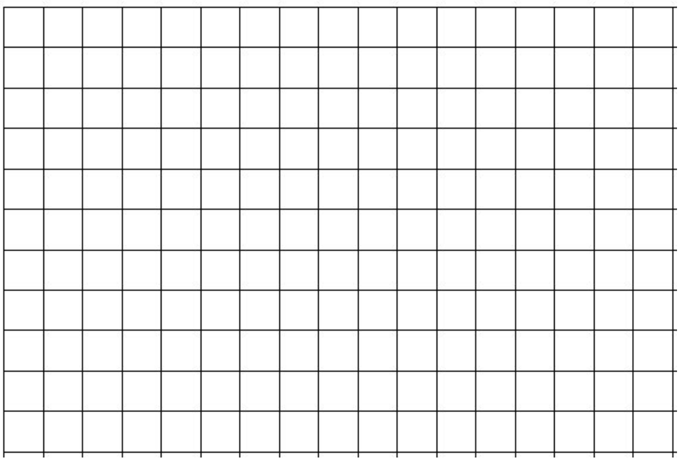
As the liquid drains out into Lurch's greedy mouth, the magnitude of the height of the liquid remaining in the cone is always twice the length of the radius. The liquid drains out of the cone at a constant rate of $\frac{1}{2} cm^3 / \sec$.

The cone was completely full with **'Cut-the-Fat'** when Lurch began his evil deed.

a. State a relationship for the radius, r, in terms of the height, h, of the liquid remaining in the cone at any time t.

1 mark

b. (i) Sketch the graph of $\frac{dV}{dt}$ on the axes below.



1 mark

(ii) Find the rate at which the volume changes wrt the height of liquid remaining in the cone when this height is $5\,cm$.

2 marks

Lurch was only planning to try a small amount of **'Cut-the-Fat'**, but was horrified to see that all of the liquid was coming out! To avoid it pouring onto his feet he immediately placed a cylindrical bowl underneath the cone.

c. Show that the maximum volume of **'Cut-the-Fat'** that could pour out of the cone is $\frac{250}{3}\pi\ cm^3$.

1 mark

d. The cylindrical bowl that Lurch placed under the cone has a diameter of x cm and a height of $12-x$ cm. Find the diameter and the height of the bowl so that it has a maximum volume.

4 marks

e. Give a mathematical reason as to whether Lurch will be able to catch all the liquid pouring out of the cone in his bowl.

1 mark

'Cut-the-Fat' will be sold to the public in small golden coloured crystal bottles. The volume of **'Cut-the-Fat'** in each bottle, X, is a normally distributed random variable with mean μ and variance σ^2.

f. (i) Find μ and σ in terms of a and b so that $\Pr(X < a) = 0.9332$ and $\Pr(X > b) = 0.841345$. a and b represent volumes of **'Cut-the-Fat'** in cm^3.

4 marks

(ii) Find the value of a and b when $\mu = 100$ and $\sigma = 20$.

1 mark

QUESTION 4 (16 marks)

The probability that Agapanthus seeds fail to germinate is $\frac{1}{300}$.

a. How many seeds would be expected to germinate in a batch of one million Agapanthus seeds?

2 marks

b. A packet contains 100 Agapanthus seeds chosen at random. Find the probability that less than 99 seeds germinate. State your answer correct to 4 decimal places.

1 mark

c. If a packet contains less than 99 viable Agapanthus seeds, the customer receives a full refund on their purchase.

(i) If the profit on a packet of Agapanthus seeds is \$1.20 and a return results in a loss of 50 cents, calculate the expected profit from a packet of seeds. State your answer correct to the nearest cent.

2 marks

(ii) If the Agapanthus seeds are distributed to sellers in boxes containing 25 packets, find the probability that 5 out of each box of 25 packets will result in a refund. State your answer correct to 4 decimal places.

2 marks

Information on a packet of 160 Yucca seeds claims that the germination rate is 92%.

d. Assuming that the sampling distribution of the sample proportion is approximately Normal:

(i) Find the mean and standard deviation of the proportion of the seeds that germinate. State your answers correct to 4 decimal places.

2 marks

(ii) What is the smallest value of n for which the sampling distribution of $\hat{p}$ is approximately Normal?

2 marks

(iii) What sample size would be required to reduce the standard deviation of the sample proportion to one-half the value you found in (i).

1 mark

(iv) What is the probability that more than 95% of the 160 Yucca seeds in the packet will germinate? State your answer correct to 4 decimal places.

2 marks

e. A 90% confidence interval for the proportion of Yucca seed that germinate is to be constructed. How many seeds must be tested if the margin of error cannot exceed 5%?

2 marks

END OF QUESTION AND ANSWER BOOKLET