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YEAR 12 *Trial Exam Paper*

2018

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips on how to approach the exam.

Question 1a.

Worked solution

$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$ and $\frac{d}{dx}\left(\cos(4x)\right) = -4\sin(4x)$.

$$\begin{aligned}\frac{dy}{dx} &= \cos(x) \times -4\sin(4x) + -\sin(x) \times \cos(4x)\\ &= -4\cos(x)\sin(4x) - \sin(x)\cos(4x)\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for application of the product rule
- 1 answer mark for the correct answer

Tip

- *Expect the first question in the exam to ask you to evaluate a derivative. This will require you to use the chain, product or quotient rules.*

Question 1b.

Worked solution

$f'(x) = 4e^{2x}$

$$\begin{aligned}f'(\log_e(2)) &= 4e^{2\log_e(2)}\\ &= 4e^{\log_e(4)}\\ &= 4 \times 4\\ &= 16\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for calculating the correct derivative $f'(x) = 4e^{2x}$
- 1 answer mark for the correct answer $f'(\log_e(2)) = 16$

Question 2a.

Worked solution

The expression is of the form $\frac{u}{v}$ and so the quotient rule can be used to find this derivative. This can be found on the formula sheet.

$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$ and $\frac{d}{dx}\left(x\right)=1$.

From the formula sheet $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{\frac{1}{x}\times x-\log_e(x)\times 1}{x^2}\\&=\frac{1-\log_e(x)}{x^2}\\&=-\frac{\log_e(x)}{x^2}+\frac{1}{x^2}\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for application of the quotient rule or the product rule
- 1 answer mark for the correct answer

Question 2b.

Worked solution

From **part 2a.**

$$\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)=-\frac{\log_e(x)}{x^2}+\frac{1}{x^2}$$

This can be rearranged to have $\frac{\log_e(x)}{x^2}$ on the left-hand side.

$$\frac{\log_e(x)}{x^2}=\frac{1}{x^2}-\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)$$

Then integrate both sides over the required interval.

$$\begin{aligned}\int_1^2\frac{\log_e(x)}{x^2}dx&=\int_1^2\frac{1}{x^2}-\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)dx\\&=\left[-\frac{1}{x}-\frac{\log_e(x)}{x}\right]_1^2\\&=\left[-\frac{1}{2}-\frac{\log_e(2)}{2}\right]-\left[-\frac{1}{1}-\frac{\log_e(1)}{1}\right]\\&=\left[-\frac{1}{2}-\frac{\log_e(2)}{2}\right]-\left[-1\right]\\&=\frac{1}{2}-\frac{\log_e(2)}{2}\end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for rearranging the result from **part a.**, either before or after integrating both sides, to construct an equation with $\frac{\log_e(x)}{x}$ on the left-hand side
- 1 answer mark for calculating the antiderivative $-\frac{1}{x}-\frac{\log_e(x)}{x}$
- 1 answer mark for the correct answer $\int_1^2\frac{\log_e(x)}{x^2}dx=\frac{1}{2}-\frac{\log_e(2)}{2}$

Tip

- *The word 'hence' tells you that you should be using the answer to the previous question to respond to this question. These sorts of 'integration by recognition' problems are very common in Exam 1.*

Question 3a.

Worked solution

$\log_e(x)$ implies $x > 0$ and $\log_e(5-2x)$ implies $x < \frac{5}{2}$; hence, $0 < x < \frac{5}{2}$.

Mark allocation: 1 mark

- 1 answer mark for $0 < x < \frac{5}{2}$ or the interval $\left(0, \frac{5}{2}\right)$

Question 3b.

Worked solution

$$\begin{aligned}
\log_e(x) + \log_e(5-2x) &= 0 \\
\log_e(x(5-2x)) &= 0 \\
x(5-2x) &= 1 \\
2x^2 - 5x + 1 &= 0 \\
x &= \frac{5 \pm \sqrt{25-8}}{4} \\
&= \frac{5 \pm \sqrt{17}}{4}
\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for the correct quadratic equation $2x^2 - 5x + 1 = 0$, or equivalent
- 1 answer mark for the correct answer $x = \frac{5+\sqrt{17}}{4}, \frac{5-\sqrt{17}}{4}$, or equivalent

Question 4a.

Worked solution

This is a conditional probability problem. The formula $\Pr(A|B)=\dfrac{\Pr(A\cap B)}{\Pr(B)}$ is given on the formula sheet.

Note that $\Pr(X=0\cap X<2)=\Pr(X=0)$.

$\Pr(X=0)=0.45$ and $\Pr(X<2)=\Pr(X=0)+\Pr(X=1)=0.75$

$$\Pr(X=0|X<2)=\frac{\Pr(X=0)}{\Pr(X<2)}=\frac{0.45}{0.75}=\frac{3}{5}$$

Mark allocation: 1 mark

- 1 answer mark for $\frac{3}{5}$. Note that the fraction must be simplified.

Tip

- *Question 4 contains some easier questions. It is important that you do not spend so much time on the more difficult parts of the paper that you miss out on easier marks. A useful strategy might be to use your reading time to identify questions that play to your strengths, and attempt those first.*

Question 4b.

Worked solution

$$\begin{aligned}\mathrm{E}(X)&=0\times 0.45+1\times 0.3+2\times 0.15+3\times 0.1\\&=0.3+0.3+0.3\\&=0.9\end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for $\mathrm{E}(X)=0.9$

Question 4c.i.

Worked solution

The expected value of $\hat{P}$ is the probability that Lily sees at least two birds on any particular day.

$\mathrm{E}(\hat{P}) = \Pr(X = 2) + \Pr(X = 3) = 0.15 + 0.1 = 0.25$

Mark allocation: 1 mark

- 1 answer mark for $\mathrm{E}(\hat{P}) = 0.25$

Question 4c.ii.

Worked solution

Let Y be the random variable representing the number of days in the week that Lily sees at least two birds. Then $\hat{P} = \dfrac{Y}{5}$ and $\hat{P} = 0.2$ corresponds to seeing at least two birds on exactly one day out of the five days.

Y is distributed as $Y \sim \mathrm{Bi}(5, 0.25)$.

$\Pr\left(\hat{P} = 0.2\right) = \Pr(Y = 1)$

$= \binom{5}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4$

$= \frac{5}{4} \times \frac{3^4}{4^4}$

$= \frac{5 \times 81}{4^5}$

$= \frac{405}{2^{10}}$

Mark allocation: 2 marks

- 1 method mark for binomial distribution expression $\binom{5}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4$
- 1 answer mark for $\Pr\left(\hat{P} = 0.2\right) = \frac{405}{2^{10}}$. Accept equivalent answers such as $\frac{405}{4^5}$.

Question 5a.

Worked solution

$$f(x) = \sqrt{x-5} = (x-5)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(x-5)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-5}}$$

$$f'(p) = \frac{1}{2\sqrt{p-5}}$$

The gradient of the tangent is $\frac{1}{2\sqrt{p-5}}$.

Mark allocation: 1 mark

- 1 answer mark for the answer $\frac{1}{2\sqrt{p-5}}$

Question 5b.

Worked solution

The tangent to f is given by the equation:

$$y - f(p) = \frac{1}{2\sqrt{p-5}}(x-p)$$

$$y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(x-p)$$

Substituting $x = 0$ and $y = 0$ gives:

$$-\sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(-p)$$

$$-\sqrt{p-5} = -\frac{p}{2\sqrt{p-5}}$$

$$2(p-5) = p$$

$$p = 10$$

Alternatively, if the tangent passes through $(0,0)$, then it has the equation $y = \frac{1}{2\sqrt{p-5}}x$, using the gradient from **part a**.

Substituting $x = p$ and $y = f(p) = \sqrt{p-5}$ gives:

$$\sqrt{p-5} = \frac{1}{2\sqrt{p-5}} \times p$$

$$p - 5 = \frac{1}{2}p$$

$$\frac{1}{2}p = 5$$

$$p = 10$$

Alternatively, a line through the origin has the equation $y = mx$, with some unknown gradient, m.

If this line is a tangent, then the equation $mx = \sqrt{x-5}$ has a single solution.

$$m^2x^2 = x - 5$$

$$m^2x^2 - x + 5 = 0$$

For a single solution $\Delta = 0$.

$$\Delta = \left(-1\right)^2 - 4 \times m^2 \times 5$$
$$= 1 - 20m^2$$

$$1 - 20m^2 = 0$$
$$m^2 = \frac{1}{20}$$
$$m = \pm\frac{1}{2\sqrt{5}}$$

Since $m > 0$, $m = \frac{1}{2\sqrt{5}}$.

This gradient can be equated to the gradient found in **part a**.

$$\frac{1}{2\sqrt{p-5}} = \frac{1}{2\sqrt{5}}$$
$$p - 5 = 5$$
$$p = 10$$

Or the gradient can be substituted back into the earlier equation to solve for *x*.

$$m^2x^2 - x + 5 = 0$$
$$\frac{1}{20}x^2 - x + 5 = 0$$
$$x^2 - 20x + 100 = 0$$
$$(x-10)^2 = 0$$
$$x = 10$$

Hence, $p = 10$.

Mark allocation: 3 marks

- 1 method mark for determining the equation of the tangent, such as $y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(x-p)$ or $y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}x + \frac{p-10}{2\sqrt{p-5}}$, or for stating the equation of the tangent as $y = \frac{1}{2\sqrt{p-5}}x$
- 1 method mark for substituting $x = 0$ and $y = 0$ or equating the *y*-intercept of the tangent to 0, or for substituting coordinate $(p, f(p))$ where appropriate.
- 1 answer mark for $p = 10$

Question 6

Worked solution

$\Pr(B)$ can be found using several methods, such as the two shown below.

$$\begin{aligned}\Pr(A' \cap B) = 0.2 &= \Pr(B) - \Pr(A \cap B)\\ &= \Pr(B) - \Pr(A) \times \Pr(B)\\ &= \Pr(B) - 0.4 \times \Pr(B)\\ &= 0.6 \times \Pr(B)\end{aligned}$$

$$\Pr(B) = \frac{0.2}{0.6} = \frac{1}{3}$$

Alternatively, if $\Pr(A) = 0.4,$ then $\Pr(A') = 0.6.$

As A and B are independent, $\Pr(A' \cap B) = 0.2 = \Pr(A') \times \Pr(B) = 0.6 \times \Pr(B),$ giving $\Pr(B) = \frac{0.2}{0.6} = \frac{1}{3}.$

Using $\Pr(B) = \frac{1}{3}$ and that A and B are independent, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ gives:

$$\Pr(A \cap B) = 0.4 \times \frac{1}{3} = \frac{2}{15}$$

Alternatively, let x denote $\Pr(A \cap B)$.

$$\Pr(B) = \Pr(A' \cap B) + \Pr(A \cap B) = 0.2 + x$$

$$\begin{aligned}x &= \Pr(A) \times \Pr(B) = 0.4 \times (0.2 + x)\\ x &= 0.08 + 0.4x\\ 0.6x &= 0.08\\ x &= \frac{0.08}{0.6} = \frac{2}{15}\end{aligned}$$

Therefore, $\Pr(A \cap B) = \frac{2}{15}.$

Alternatively, a Karnaugh map for this problem is shown below.

	A	***A′***	
B	m	0.2	n
B′			
	0.4		

This table can be used to develop the equations $n = m + 0.2$ and $m = 0.4 \times n$, giving the resulting equation $\Pr(A \cap B) = m = 0.4 \times (m + 0.2)$, which is identical to the equation solved using the previous method.

Mark allocation: 2 marks

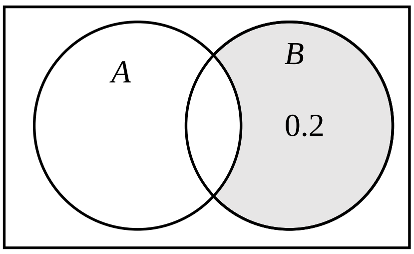
- 1 method mark for recognising and using the independence of A and B, such as $\Pr(A \cap B) = 0.4 \times (0.2 + \Pr(A \cap B))$, or for calculating $\Pr(B) = \dfrac{1}{3}$
- 1 answer mark for the solution $\Pr(A \cap B) = \dfrac{2}{15}$

Tip

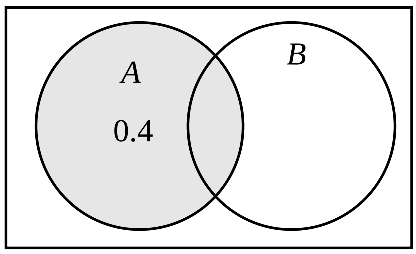
- *Venn diagrams or a Karnaugh map can be used to help visualise the probabilities given in this problem.*

  Venn diagrams would look as shown below.

$\Pr(A' \cap B) = 0.2$



$\Pr(A) = 0.4$



Question 7

Worked solution

For f to define a probability distribution $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $f(x) \geq 0$ for all x.

$$\int_0^1 x^3(1-x)^2\,dx$$

$$\int_0^1 x^3 - 2x^4 + x^5\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6\right]_0^1$$

$$= \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$

$$= \frac{15}{60} - \frac{24}{60} + \frac{10}{60} = \frac{1}{60}$$

$$k\int_0^1 x^3(1-x)^2\;dx = 1$$

$$k \times \frac{1}{60} = 1$$

$$k = 60$$

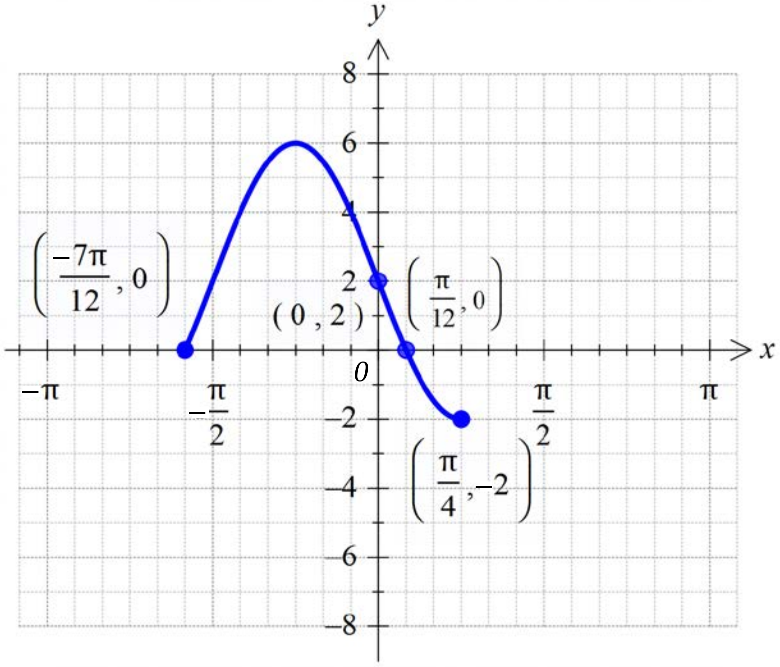
Mark allocation: 3 marks

- 1 method mark for integrating the probability density function from 0 to 1 and equating it to 1
- 1 method mark for calculating the correct antiderivative of the polynomial $\frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6$
- 1 answer mark for the single answer $k = 60$

Question 8a.

Worked solution

The final graph should include the details shown below.



From the equation, the period is $\frac{2\pi}{2}=\pi$, the amplitude is 4 and the median value is 2.

The coordinates of the end points can be found by substitution.

$$\begin{aligned} f\left(-\frac{7\pi}{12}\right) &= 2-4\sin\left(2\times-\frac{\pi}{12}\right) \\ &= 2-4\sin\left(-\frac{\pi}{6}\right) \\ &= 2-4\times-\frac{1}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= 2-4\sin\left(2\times\frac{\pi}{4}\right) \\ &= 2-4\sin\left(\frac{\pi}{2}\right) \\ &= 2-4 \\ &= -2 \end{aligned}$$

Left end point is at $\left(-\frac{7\pi}{12},0\right)$.

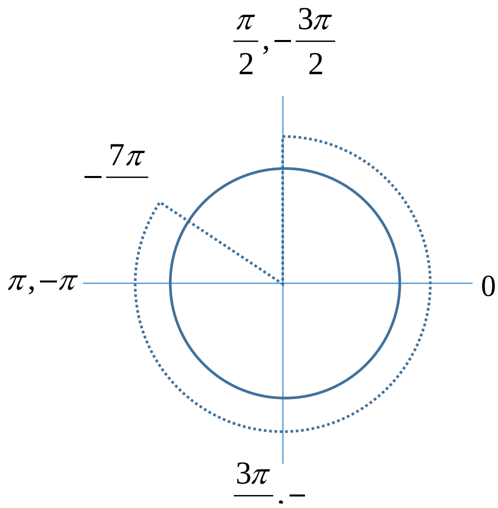
Right end point is at $\left(\frac{\pi}{4},-2\right)$.

The *x*-axis intercepts can be found by solving $2-4\sin(2x)=0$.

$$\begin{aligned} 2-4\sin(2x) &= 0 \\ \sin(2x) &= \frac{1}{2} \end{aligned}$$

The exact value of $\sin\left(\frac{\pi}{6}\right)$ should be recognised as $\frac{1}{2}$.

The domain is $-\frac{7\pi}{12}\le x\le\frac{\pi}{4}$ and this gives $-\frac{7\pi}{6}\le 2x\le\frac{\pi}{2}$, indicating that the argument of the sin function varies from the second quadrant through to the end of the first quadrant. This can be visualised using a unit circle, as shown on the following page.



Given that we have an expression where $\sin(\theta)$ is positive, solutions will be in the first and second quadrants.

$$
\begin{aligned}
2x &= \frac{\pi}{6}, -\pi - \frac{\pi}{6} \\
&= \frac{\pi}{6}, -\frac{7\pi}{6} \\
x &= \frac{\pi}{12}, -\frac{7\pi}{12}
\end{aligned}
$$

Alternatively, once it has been identified that the left end point at $x = -\frac{7\pi}{12}$ is an x-intercept, the other value $x = \frac{\pi}{12}$ can be calculated using symmetry about the local maximum at $x = -\frac{\pi}{4}$.

Mark allocation: 4 marks

- High (4 marks): A correctly drawn sketch, with all end points and axes intercepts labelled correctly. The local maximum at $\left(-\frac{\pi}{4}, 6\right)$ is correctly located.
- Medium (2–3 marks): A sinusoidal-shaped curve with the correct amplitude, median and period sketched. Some axes intercepts or end points may not be labelled. The curve may be drawn for the full horizontal length of the provided axes. The curve may be reflected across the median.
- Low (1 mark): A sinusoidal-shaped curve with a range of $[-2, 6]$ sketched. A non-sinusoidal shaped curve is drawn but the end points have been correctly located.

Tip

- *When sketching sine and cosine functions, your first step should always be to determine the period, amplitude and median value for the function.*

Question 8b.

Worked solution

$$\int_{-\frac{7\pi}{12}}^{\frac{\pi}{12}} 2-4\sin(2x)\,dx$$

$$=\left[2x+2\cos(2x)\right]_{-\frac{7\pi}{12}}^{\frac{\pi}{12}}$$

$$=\left[2\times\frac{\pi}{12}+2\cos\left(2\times\frac{\pi}{12}\right)\right]-\left[2\times-\frac{7\pi}{12}+2\cos\left(2\times-\frac{7\pi}{12}\right)\right]$$

$$=\left[\frac{\pi}{6}+\sqrt{3}\right]-\left[-\frac{7\pi}{6}-\sqrt{3}\right]$$

$$=\frac{4\pi}{3}+2\sqrt{3}$$

Mark allocation: 3 marks

- 1 method mark for a correctly composed integral expression $\int_{-\frac{7\pi}{12}}^{\frac{\pi}{12}} 2-4\sin(2x)\,dx$
- 1 method mark for evaluating the integral and substituting the end points of the interval: for example, $\left[2\times\frac{\pi}{12}+2\cos\left(2\times\frac{\pi}{12}\right)\right]-\left[2\times-\frac{7\pi}{12}+2\cos\left(2\times-\frac{7\pi}{12}\right)\right]$
- 1 answer mark for the correct answer $\frac{4\pi}{3}+2\sqrt{3}$ or $\frac{4\pi+6\sqrt{3}}{3}$

Question 9a.

Worked solution

Let $y = \dfrac{a}{x-b}$.

To find the rule for the inverse, create a new relation by swapping *x* and *y*.

$$x' = \frac{a}{y'-b}$$

$$y' = \frac{a}{x'} + b$$

The range of *f* is $R \setminus \{0\}$; hence, the domain of f^{-1} is $R \setminus \{0\}$.

Therefore, the inverse is $f^{-1}: R \setminus \{0\} \to R,\ f^{-1}(x) = \dfrac{a}{x} + b.$

Mark allocation: 2 marks

- 1 answer mark for the rule $f^{-1}(x) = \dfrac{a}{x} + b$
- 1 answer mark for the domain $R \setminus \{0\}$

Tip

- *When a question asks for you to find the inverse of a function, it is important to always give both the domain and the rule.*

Question 9b.

Worked solution

The rule for f is $f(x)=\dfrac{a}{x-b}$ and the rule for f^{-1} is $f^{-1}(x)=\dfrac{a}{x}+b.$ The required transformations are:

- Translate b units in the negative x direction.
- Translate b units in the positive y direction.

Hence, $g=1$, $h=-b$ and $k=b$.

Alternatively, from equations for f and f^{-1}:

$y=\dfrac{a}{x-b}$ gives $\dfrac{1}{a}y=\dfrac{1}{x-b}.$

$y_T=\dfrac{a}{x_T}+b$ gives $\dfrac{1}{a}(y_T-b)=\dfrac{1}{x_T}.$

From these equations:

$x_T=x-b$ and $\dfrac{1}{a}(y_T-b)=\dfrac{1}{a}y,$ giving $y_T=y+b.$

Comparing these two equations to the expanded equations from the matrix $x_T=x+h$ and $y_T=g\times y+k$ gives the solutions $g=1,$ $h=-b$ and $k=b$.

Mark allocation: 2 marks

- 1 answer mark for $g=1$
- 1 answer mark for $h=-b$ and $k=b$

Question 9c.

Worked solution

$f(x) = f^{-1}(x)$

$$\frac{a}{x-b} = \frac{a}{x} + b$$

$$a \cdot x = a(x-b) + b \cdot x(x-b)$$

$$a \cdot x = ax - ab + bx^2 - b^2 x$$

$$0 = bx^2 - b^2 x - ab$$

Quadratic equation has no solutions when $\Delta < 0$.

$b^4 + 4ab^2 < 0$

$$b^2 + 4a < 0, \quad b \neq 0$$

$$a < -\frac{b^2}{4}$$

When $b = 0$ the two functions are the same for all values of *a* and, hence, solutions exist for all values of *a*.

Mark allocation: 3 marks

- 1 method mark for equating $f(x)$ with $f^{-1}(x)$ to give $\frac{a}{x-b} = \frac{a}{x} + b$
- 1 method mark for rearranging to give $0 = bx^2 - b^2 x - ab$
- 1 answer mark for $a < -\frac{b^2}{4}$ and $b \neq 0$

END OF WORKED SOLUTIONS