Year 2018

VCE

Mathematical Methods

Trial Examination 2

Solutions

KILBAHA MULTIMEDIA PUBLISHING	**TEL: (03) 9018 5376**
PO BOX 2227	**FAX: (03) 9817 4334**
KEW VIC 3101	**kilbaha@gmail.com**
AUSTRALIA	**http://kilbaha.com.au**

SECTION A

ANSWERS

1	A	B	**C**	D	E
2	A	B	C	D	**E**
3	A	B	C	**D**	E
4	A	**B**	C	D	E
5	A	B	C	D	**E**
6	A	B	C	**D**	E
7	**A**	B	C	D	E
8	**A**	B	C	D	E
9	A	**B**	C	D	E
10	A	B	C	**D**	E
11	A	B	**C**	D	E
12	A	B	C	D	**E**
13	A	**B**	C	D	E
14	A	**B**	C	D	E
15	A	B	C	D	**E**
16	**A**	B	C	D	E
17	A	B	**C**	D	E
18	A	B	C	**D**	E
19	A	B	**C**	D	E
20	**A**	B	C	D	E

SECTION A

Question 1 **Answer C**

$f(x)=2x-a=a$

$2x=2a \quad \Rightarrow x=a$ included

$f(x)=2x-a=-a$

$\Rightarrow x=0$ not included, so the domain is $(0,a]$

Question 2 **Answer E**

$P(x)=(x+2)(x-4)Q(x)+a(x+2)+b$

$P(-2)=6 \quad \Rightarrow b=6$

$P(4)=12 \quad \Rightarrow 6a+b=12 \quad \Rightarrow a=1$

$P(x)=(x+2)(x-4)Q(x)+x+8$

$\frac{P(x)}{(x+2)(x-4)}=Q(x)+\frac{x+8}{(x+2)(x-4)}$ the remainder is $x+8$.

Question 3 **Answer D**

All of the functions **A. B. C.** and **E.** join up when $x=2$,

D. $\log_2(3x-5)$ when $x=2$, gives $\log_2(1)=0$

Question 4 **Answer B**

$3x-(k+2)y=2 \quad \Rightarrow \quad y=\frac{3x}{k+2}-\frac{2}{k+2}$ and $-2kx+10y=k-7 \quad \Rightarrow y=\frac{kx}{5}+\frac{k-7}{10}$

equal gradients when $\frac{3}{k+2}=\frac{k}{5} \quad \Rightarrow k(k+2)=15 \quad \Rightarrow k^2+2k-15=(k+5)(k-3)=0$

There is a unique solution when $k \in R\backslash\{-5,3\}$, Colin is correct.

When $k=3$ the equations become $3x-5y=2$ and $-6x+10y=-4$,

these are multiples, so there is an infinite number of solution when $k=3$.

When $k=-5$ the equations become $3x+3y=2$ and $10x+10y=-12$,

these are inconsistent, the lines are parallel with different y intercepts,

so there is no solution when $k=-5$, Ben is correct.

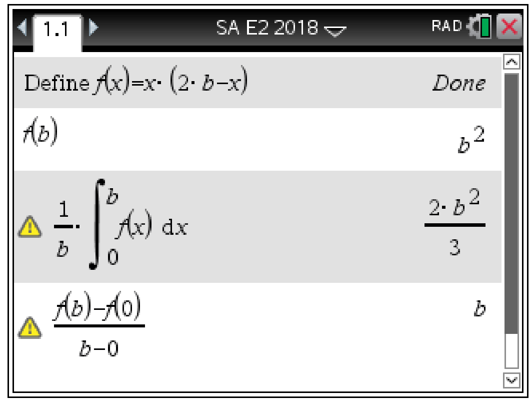
Question 5 **Answer E**

$h(x)=f(g(x)) \quad \Rightarrow \quad h(1)=f(g(1))=f(2)=3$

$h'(x)=g'(x)f'(g(x))$

$h'(1)=g'(1)f'(g(1))=g'(1)f'(2)=-2\times-4=8$

Question 6 **Answer D**



$f:[0,b]\to R$, $f(x)=x(2b-x)=2bx-x^2$ where $b>0$.

$f(b)=b^2$ so the range is $\left[0,b^2\right]$ Alan is incorrect.

The average value of the function is

$$\overline{f}=\frac{1}{b-0}\int_0^b x(2b-x)\,dx=\frac{2b^2}{3}$$

The average rate of change of the function is

$$\frac{f(b)-f(0)}{b-0}=\frac{b^2}{b}=b\text{, so Ben is correct.}$$

The gradient $f':(0,b)\to R$, $f'(x)=2b-2x=2(b-x)$, since $0<x<b$, $f'(x)>0$ the function is always increasing and that the gradient at an end-point $x=b$ does not exist, David is correct.

Question 7 **Answer A**

6 B, 5R, total 11, at least one of each colour

$$\Pr(RBB+BRB+BBR+BRR+RBR+RRB)$$

$$=3\Pr(RBB)+3\Pr(BRR)=3\times\frac{5}{11}\times\frac{6}{10}\times\frac{5}{9}+3\times\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}=\frac{9}{11}$$

Question 8 **Answer A**

The graph has a vertical asymptote at $x=-a<0$ so $a>0$.

The graph crosses the *x*-axis when $y=0$, at a negative value

$$\Rightarrow \log_e(x+a)+b=0 \quad \Rightarrow \quad \log_e(x+a)=-b \quad x+a=e^{-b},\quad x=e^{-b}-a<0 \quad \Rightarrow \quad a>e^{-b}$$

The graph crosses the *y*-axis when $x=0$, at a positive value

$$\Rightarrow \log_e(a)+b>0 \quad \Rightarrow \quad \log_e(a)>-b \quad a>e^{-b}$$

Question 9 **Answer B**

$$\int_{-1}^{2}\left(3x^2-2f(x)+g(x)\right)dx=18$$

$$\int_{-1}^{2}3x^2dx-2\int_{-1}^{2}f(x)\,dx+\int_{-1}^{2}g(x)\,dx=18$$

$$\left[x^3\right]_{-1}^{2}+2\int_{2}^{-1}f(x)\,dx-\int_{2}^{-1}g(x)\,dx=18$$

$$(8-(-1))+2\int_{2}^{-1}f(x)\,dx-3=18$$

$$2\int_{2}^{-1}f(x)\,dx=12$$

$$\int_{2}^{-1}f(x)\,dx=6$$

Question 10 **Answer D**

$f(x) = kx^n \quad , \quad g(x) = \log_e(x)$

$f(a) = g(a) \Rightarrow ka^n = \log_e(a) \Rightarrow k = \dfrac{\log_e(a)}{a^n}$ **A.** is true

$f'(x) = nkx^{n-1} \quad , \quad g'(x) = \dfrac{1}{x}$

$f'(a) = g'(a) \Rightarrow nka^{n-1} = \dfrac{1}{a} \Rightarrow nka^n = 1 \Rightarrow k = \dfrac{1}{na^n}$ **B.** is true

$a^n = \dfrac{1}{nk} \;,\; \log_e(a) = k\,a^n = \dfrac{1}{n} \Rightarrow a = e^{\frac{1}{n}} \Rightarrow a^n = e \Rightarrow \dfrac{1}{nk} = e \Rightarrow k = \dfrac{1}{en}$ **C.** is true

$a^n = \dfrac{1}{k}\log_e(a) = \dfrac{1}{nk} \Rightarrow n = \dfrac{1}{\log_e(a)} = \log_a(e)$ **E.** is true **D.** is false

Question 11 **Answer C**

$y_1 = kx + 1$ and $y_2 = x^2 - kx + 2$

$y_1 = y_2 \Rightarrow kx + 1 = x^2 - kx + 2$

$x^2 - 2kx + 1 = 0$

$\Delta = 4k^2 - 4 = 4(k^2 - 1)$

do not intersect when $\Delta < 0 \Rightarrow k^2 - 1 < 0 \Rightarrow -1 < k < 1$

Question 12 **Answer E**

$f : (-\infty, a) \to R \;,\; f(x) = 2ax - x^2 = x(2a - x)$

$f : \quad y = 2ax - x^2$

$f^{-1} \quad x = 2ay - y^2$

$y^2 - 2ay + a^2 = a^2 - x$

$(y - a)^2 = a^2 - x$

$y - a = \pm\sqrt{a^2 - x}$ take negative since dom $f^{-1} = \text{ran } f = (-\infty, a^2)$

$y = f^{-1}(x) = a - \sqrt{a^2 - x}$

$f^{-1} : (-\infty, a^2) \to R \;,\; f^{-1}(x) = a - \sqrt{a^2 - x}$

Question 13 **Answer B**

If $f(x)$ is a non-zero even function, then $f(x) = f(-x)$, the graph is symmetrical about the *y*-axis so $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. If $g(x)$ is a non-zero odd function, then $g(x) = -g(-x)$, then $\int_{-a}^{a} g(x)\,dx = 0$, so $\int_{-a}^{a} \left(f(x) + g(x)\right)dx = 2\int_0^a f(x)\,dx$

Question 14 **Answer B**

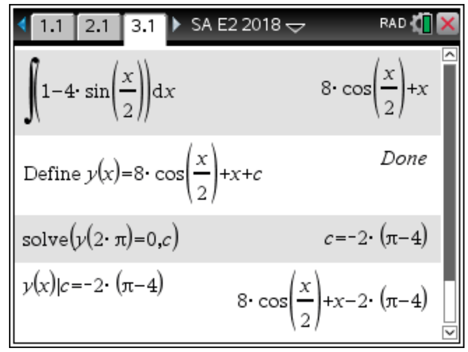
$$\frac{dy}{dx}=1-4\sin\left(\frac{x}{2}\right)$$

$$y=\int\left(1-4\sin\left(\frac{x}{2}\right)\right)dx=x+8\cos\left(\frac{x}{2}\right)+c$$

to find c, $x=2\pi$, $y=0$

$$\Rightarrow 0=2\pi+8\cos(\pi)+c \Rightarrow c=8-2\pi$$

$$y=x+8\cos\left(\frac{x}{2}\right)+8-2\pi$$

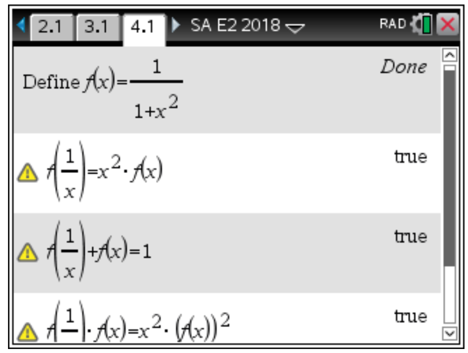


Question 15 **Answer E**

$$f(x)=\frac{1}{1+x^2}$$ **A. B. C. D.** are all true

E. is false

$$\left(f\left(\sqrt{x}\right)\right)^2=\left(\frac{1}{1+x}\right)^2=\frac{1}{(1+x)^2}\neq f(x)=\frac{1}{1+x^2}$$



Question 16 **Answer A**

The graph of $y=\cos(x)$ is transformed into the graph of

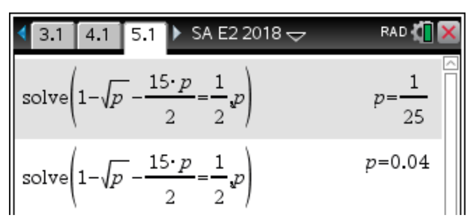
$$y=-3\sin(x)=-3\cos\left(\frac{\pi}{2}-x\right)=-3\cos\left(x-\frac{\pi}{2}\right)$$ since $\cos(-x)=\cos(x)$ by

A dilation by a scale factor of 3 parallel to the *y*-axis, a reflection in the *x*-axis, and a translation of $\frac{\pi}{2}$ to the right parallel to the *x*-axis.

Question 17 **Answer C**

$$\Pr(A)=\sqrt{p}\ ,\ \Pr(B)=10p\ ,\ \Pr(A|B)=\frac{1}{4}=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{\Pr(A\cap B)}{10p}\ \Rightarrow\ \Pr(A\cap B)=\frac{5p}{2}$$

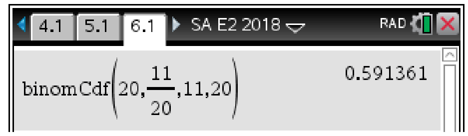
	A	A'	
B	$\frac{5p}{2}$	$10p-\frac{5p}{2}$	$10p$
B'	$\sqrt{p}-\frac{5p}{2}$	$\Pr(A'\cap B')$	$1-10p$
	$\sqrt{p}$	$1-\sqrt{p}$	



$$\Pr(A'\cap B')=\frac{1}{2}=1-\sqrt{p}-\left(10p-\frac{5p}{2}\right)=1-10p-\left(\sqrt{p}-\frac{5p}{2}\right)=1-\sqrt{p}-\frac{15p}{2}$$

solving $1-\sqrt{p}-\frac{15p}{2}=\frac{1}{2}$ gives $p=0.04$

Question 18 **Answer D**



55% have blue eyes. $p = \frac{11}{20} = 0.55$

$\hat{P} = \frac{X}{20} \quad X \stackrel{d}{=} \text{Bi}\left(n = 20, p = \frac{11}{20}\right)$

$$\Pr\left(\hat{P} > 0.5\right) = \Pr(X > 10)$$
$$= \Pr(X \ge 11) = 0.5914$$

Question 19 **Answer C**

Given $\Pr(a < Z < b) = P$ and $\Pr(Z > a) = A \;\Rightarrow\; \Pr(Z < a) = 1 - A$

since $b > a$, $\Pr(Z < a) + \Pr(a < Z < b) + \Pr(Z > b) = 1$, then

$$\Pr(Z > b) = 1 - \left(\Pr(a < Z < b) + \Pr(Z < a)\right) = 1 - P - (1 - A) = A - P$$

$$\Pr(Z > b \mid Z > a) = \frac{\Pr(Z > b)}{\Pr(Z > a)} = \frac{A - P}{A} = 1 - \frac{P}{A}$$

Question 20 **Answer A**

$\sum \Pr(X = x) = 1 \quad \Rightarrow a + b + c = 1 \quad (1)$

$E(X) = \sum x \Pr(X = x) = \frac{1}{2} \quad \Rightarrow b + 2c = \frac{1}{2} \quad (2)$

$E\left(X^2\right) = \sum x^2 \Pr(X = x) \quad \Rightarrow E\left(X^2\right) = b + 4c$

$\text{var}(X) = E\left(X^2\right) - \left(E(X)\right)^2 = b + 4c - \left(\frac{1}{2}\right)^2 = b + 4c - \frac{1}{4} = \frac{1}{2} \quad \Rightarrow b + 4c = \frac{3}{4} \quad (3)$

$(3) - (2) \;\; 2c = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \quad \Rightarrow \quad c = \frac{1}{8}$ into

$(2) \;\; b = \frac{1}{2} - 2c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \qquad (1) \;\; a = 1 - b - c = 1 - \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$

$a = \frac{5}{8}, \; b = \frac{1}{4}, \; c = \frac{1}{8}$

END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

a. the maximum on $f(x)=5+\sin\left(\frac{\pi x}{6}\right)$ is when $\sin\left(\frac{\pi x}{6}\right)=1 \Rightarrow x=3$

$f(3)=6 \quad N(3,6)$, centre of the circle $C(4,3)$ A1

$d(CN)=\sqrt{(4-3)^2+(3-6)^2}$

$=\sqrt{10}$ A1

b.i. $g(x)=2\sin^2\left(\frac{\pi x}{4}\right) \quad g(5)=2\sin^2\left(\frac{5\pi}{4}\right)=2\times\left(-\frac{1}{\sqrt{2}}\right)^2=1 \quad S(5,1)$

$g'(x)=\pi\sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)$

$g'(5)=\pi\sin\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{4}\right)=\pi\times\left(-\frac{1}{\sqrt{2}}\right)^2=\frac{\pi}{2}$ A1

$T: \quad y-1=\frac{\pi}{2}(x-5)$

$T: \quad y=\frac{\pi x}{2}+1-\frac{5\pi}{2}=\frac{\pi x}{2}+\frac{2-5\pi}{2}$ A1

Define $f1(x)=5+\sin\left(\frac{\pi\cdot x}{6}\right)$	*Done*
Define $f2(x)=2\cdot\left(\sin\left(\frac{\pi\cdot x}{4}\right)\right)^2$	*Done*
$f2(5)$	1
$\frac{d}{dx}(f2(x))$	$\pi\cdot\sin\left(\frac{\pi\cdot x}{4}\right)\cdot\cos\left(\frac{\pi\cdot x}{4}\right)$
$\frac{d}{dx}(f2(x))\|x=5$	$\frac{\pi}{2}$
tangentLine$(f2(x),x,5)$	$\frac{\pi\cdot x}{2}-\frac{5\cdot\pi-2}{2}$

ii. solving $\frac{\pi x}{2}+\frac{2-5\pi}{2}=5+\sin\left(\frac{\pi x}{6}\right)$ with $0<x<12$ M1

gives $x=7.178$ $f(7.178)=4.421$

$B(7.178, 4.421)$ A1

iii. $d(BS)=\sqrt{(7.178-5)^2+(4.421-1)^2}$

$=4.056$ A1

Define $t(x)$=tangentLine$(f2(x),x,5)$	*Done*
⚠ solve$(t(x)=f1(x),x)\|0<x<12$	x=7.17819
$t(7.178189371571)$	4.42149
$f1(7.178189371571)$	4.42149
$\sqrt{(7.17819-5)^2+(1-4.4215)^2}$	4.056

c.i. The circular island has a radius of 1 and an area of π.

The area of water in the pool is

$$\int_0^{12}\left(5+\sin\left(\frac{\pi x}{6}\right)-2\sin^2\left(\frac{\pi x}{4}\right)\right)dx-\pi$$ A1

ii. $$\int_0^{12}\left(5+\sin\left(\frac{\pi x}{6}\right)-2\sin^2\left(\frac{\pi x}{4}\right)\right)dx-\pi=48-\pi$$

$V=\frac{3}{2}(48-\pi)=72-\frac{3\pi}{2}$ m^3 A1

$\int_0^{12}(f1(x)-f2(x))dx$	48

d. $s(x) = f(x) - g(x) = 5 + \sin\left(\frac{\pi x}{6}\right) - 2\sin^2\left(\frac{\pi x}{4}\right)$

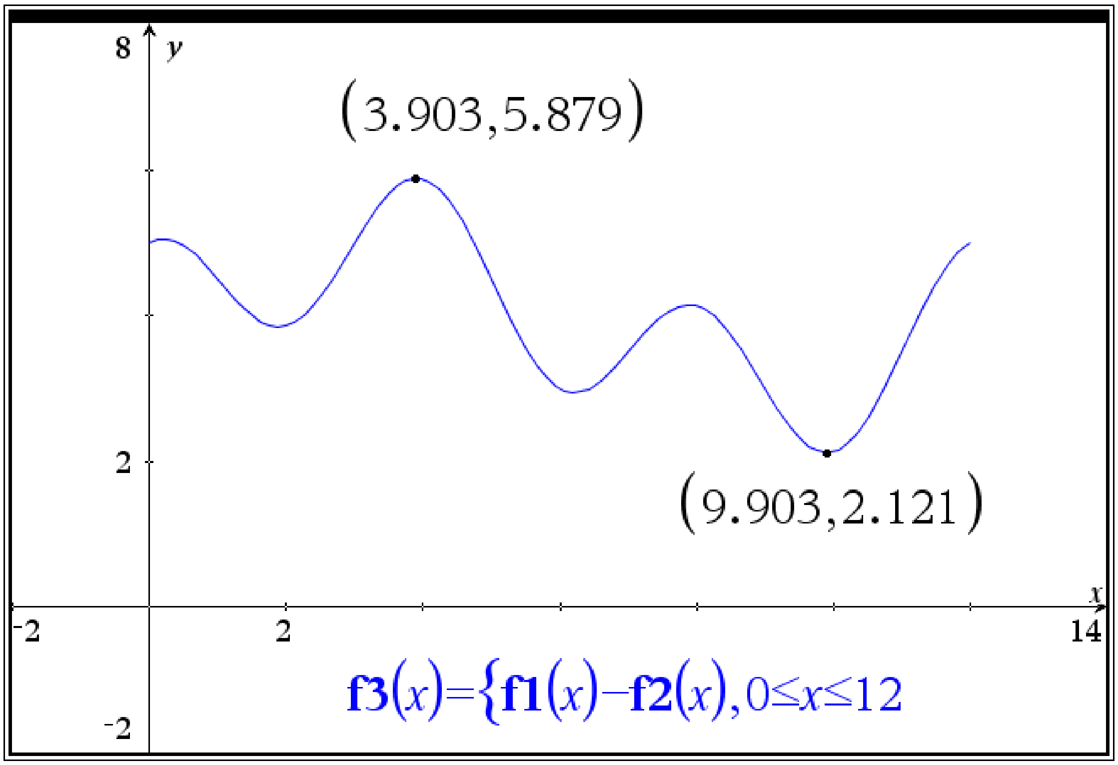
graphically find the maximum and minimum of *s*, or using $\frac{ds}{dx} = 0$ M1

the maximum value of *s* occurs when $x = 3.903$ and the maximum width is

$s(3.903) = 5.879$ m A1

the minimum value of *s* occurs when $x = 9.903$ and the minimum width is

$s(9.903) = 2.121$ m A1



Question 2

a.i. $f(x) = x^3 - 6x^2 + cx + d$

$f'(x) = 3x^2 - 12x + c$

$\Delta = 144 - 12c > 0$ for two stationary points

$c < 12$ A1

ii. Given that $c = 9$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1) = 0$

for two stationary points $x = 1$, $x = 3$ A1

$f(1) = 1 - 6 + 9 + d = 4 + d$

$f(3) = 27 - 54 + 27 + d = d$ M1

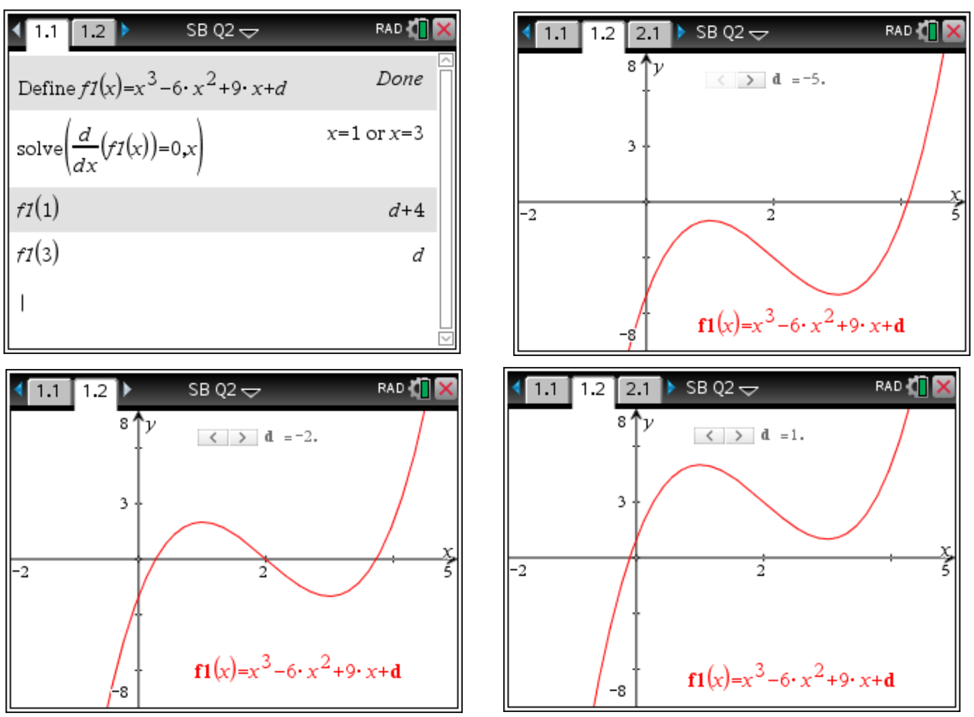
so stationary points at $(1, 4+d)$ is the maximum turning

and $(3, d)$ is the minimum turning point.

The graph crosses the *x*-axis three times if

$4 + d > 0$ and $d < 0$ that is

$-4 < d < 0$ A1



iii. If the stationary point is at $x = 1$ then from **a.ii.** $c = 9$

one stationary point is $(1, 4+d) = (1, 9)$ so

$d = 5$ and the other stationary point is $(3, 5)$ A1

b.i. $f'(x) = 3x^2 - 12x + c$

$\Delta = 144 - 12c = 0$ for one stationary point

$c = 12$ A1

ii. $f(x) = x^3 - 6x^2 + cx + d$

$f(x+p) = x^3$

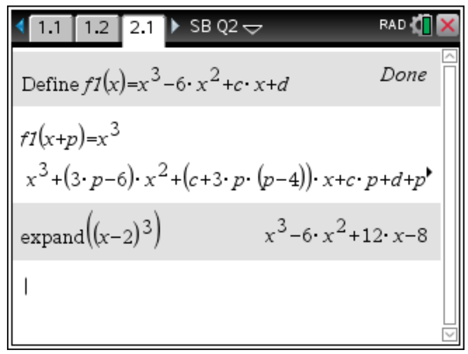
$f(x+p) = (x+p)^3 - 6(x+p)^2 + c(x+p) + d$

$= x^3 + 3x^2 p + 3xp^2 + p^3 - 6x^2 - 12xp - 6p^2 + cx + cp + d$ M1

$= x^3 + (3p-6)x^2 + (c + 3p^2 - 12p)x + cp + d + p^3 - 6p^2$

$= x^3$

equation coefficients



(1) $3p - 6 = 0 \Rightarrow p = 2$

(2) $c + 3p^2 - 12p = 0$

$c = 12p - 3p^2 = 24 - 12$

$\Rightarrow c = 12$

(3) $cp + d + p^3 - 6p^2 = 0$

$\Rightarrow d = 6p^2 - p^3 - cp = 24 - 8 - 24 = -8$

$p = 2$, $c = 12$, $d = -8$ A1

alternatively $f(x) = (x-2)^3 = x^3 - 6x^2 + 12x - 8$

c. crosses the y-axis at $y = -4 \Rightarrow d = -4$, $f(x) = x^3 - 6x^2 + cx - 4$

$f(a) = a^3 - 6a^2 + ac - 4$ the tangent is $t(x) = -3x + 4$

so $t(a) = 4 - 3a$ and $t(a) = f(a)$

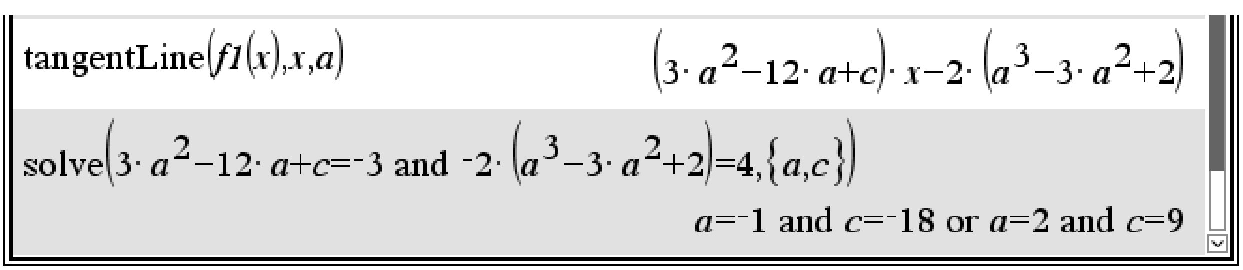
(1) $a^3 - 6a^2 + ac - 4 = 4 - 3a$

$f'(x) = 3x^2 - 12x + c$ the tangent has a gradient of -3, therefore M1

(2) $f'(a) = 3a^2 - 12a + c = -3$

solving (1) and (2) gives $a = 2$ and $c = 9$ or $a = -1$ and $c = -18$ A1

both pairs of answers are acceptable.



Define $f1(x)=x^3-6\cdot x^2+c\cdot x-4$	*Done*
$f1(a)$	$a^3-6\cdot a^2+a\cdot c-4$
Define $t(x)=4-3\cdot x$	*Done*
$t(a)$	$4-3\cdot a$
$f1(a)=t(a)$	$a^3-6\cdot a^2+a\cdot c-4=4-3\cdot a$
$eq1:=a^3-6\cdot a^2+a\cdot c-4=13-3\cdot a$	$a^3-6\cdot a^2+a\cdot c-4=13-3\cdot a$
$\frac{d}{dx}(f1(x))\|x=a$	$3\cdot a^2-12\cdot a+c$
$eq2:=3\cdot a^2-12\cdot a+c=^-3$	$3\cdot a^2-12\cdot a+c=^-3$
solve$(eq1$ and $eq2,\{a,c\})$	$a=^-1$ and $c=^-18$ or $a=2$ and $c=9$

d.i. $f(x)=x^3-6x^2+cx+d$, $A=\int_0^2\left(x^3-6x^2+cx+d\right)dx$

using four equally spaced intervals, $h=\frac{1}{2}$

(1) $L=\frac{1}{2}\left[f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)\right]=15\tfrac{1}{4}=\frac{61}{4}$ A1

$\Rightarrow\quad 6c+8d-33=61$

(2) $R=\frac{1}{2}\left[f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)\right]=16\tfrac{1}{4}=\frac{65}{4}$ A1

$\Rightarrow\quad 10c+8d-65=65$

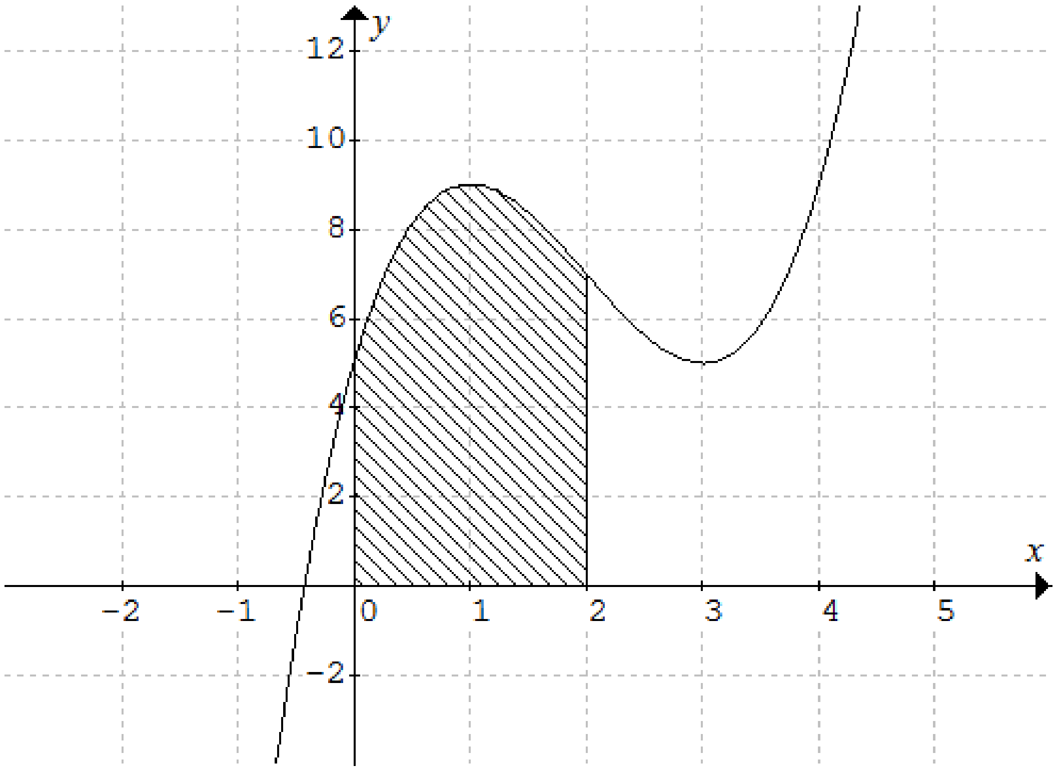
alternatively $(2)-(1)\Rightarrow\frac{1}{2}\left(f(2)-f(0)\right)=1$

$f(2)-f(0)=8-24+2c+d-d=2c-16=2$

or solving $(1),(2)$

$c=9$, $d=5$, $A=\int_0^2\left(x^3-6x^2+9x+5\right)dx=16$ A1

ii. stationary points $(1,9)$, $(3,5)$ and the y-intercept at $(0,5)$ G1



Define $f(x)=x^3-6\cdot x^2+c\cdot x+d$	*Done*
$\frac{1}{2}\cdot\left(f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)\right)=\frac{61}{4}$	$\frac{6\cdot c+8\cdot d-33}{4}=\frac{61}{4}$
$eq1:=\left(\frac{6\cdot c+8\cdot d-33}{4}=\frac{61}{4}\right)\cdot 4$	$6\cdot c+8\cdot d-33=61$
$\frac{1}{2}\cdot\left(f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)\right)=\frac{65}{4}$	$\frac{10\cdot c+8\cdot d-65}{4}=\frac{65}{4}$
$eq2:=\left(\frac{10\cdot c+8\cdot d-65}{4}=\frac{65}{4}\right)\cdot 4$	$10\cdot c+8\cdot d-65=65$
solve$(eq1$ and $eq2,\{c,d\})$	$c=9$ and $d=5$
$\int_0^2 f(x)\,dx \mid c=9$ and $d=5$	16

Question 3

a. Let the burning time in minutes be $B \overset{d}{=} N\left(60, 10^2\right)$

i. $\Pr\left(45 \le B \le 55\right) = 0.2417$ A1

ii. $\Pr\left(B \le 45\right) = 0.0668$ A1

iii. For the cake $C \overset{d}{=} \text{Bi}\left(n = 10, p = 0.0668\right)$

$\Pr\left(C \ge 1\right) = 1 - \Pr\left(C = 0\right) = 1 - \left(1 - 0.0668\right)^{10}$

$= 0.4991$ A1

normCdf(45,55,60,10)	0.24173
normCdf(-∞,45,60,10)	0.066807
p:=0.06680722872027	0.066807
binomCdf(10,p,1,10)	0.499143
$1-(1-0.0668)^{10}$	0.499104

b.i. $\hat{P} = \dfrac{S}{80}$ $S \overset{d}{=} \text{Bi}\left(n = 80, p = 0.0668\right)$

$\hat{P} = 0.1 \Rightarrow S = 8$, $\hat{P} = 0.05 \Rightarrow S = 4$ M1

$\Pr\left(\hat{P} \ge 0.1 \mid \hat{P} \ge 0.05\right)$

$= \Pr\left(S \ge 8 \mid S \ge 4\right) = \dfrac{\Pr\left(S \ge 8\right)}{\Pr\left(S \ge 4\right)} = \dfrac{0.16486}{0.789845}$

$= 0.2087$ A1

binomCdf(80,p,8,80)	0.164864
binomCdf(80,p,4,80)	0.789845
$\dfrac{0.16486440850829}{0.7898451707682}$	0.20873

ii. $\hat{p}=\dfrac{16}{80}=\dfrac{1}{5}=0.2 \qquad z_{0,9}=1.645$

$$\hat{p}\pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$=0.2\pm 1.645\sqrt{\frac{0.2\times 0.8}{80}}$$

$$=(\ 0.126\ ,0.274)$$ A1

$\text{invNorm}(0.95,0,1)$ 1.64485

$0.2+1.6448\cdot\sqrt{\dfrac{0.2\cdot 0.8}{80}}$ 0.273558

$0.2-1.6448\cdot\sqrt{\dfrac{0.2\cdot 0.8}{80}}$ 0.126442

zInterval_1Prop 16,80,0.9: *stat.results*

"Title"	"1−Prop z Interval"
"CLower"	0.12644
"CUpper"	0.27356
"p̂"	0.2
"ME"	0.07356
"n"	80.

c. long life candles $L\overset{d}{=}N(\mu=?,\sigma^2=?)$

$\Pr(L>252)=0.25 \Rightarrow \Pr(L<252)=0.75$ M1

$(1)\quad \dfrac{252-\mu}{\sigma}=0.675$

$\Pr(L<246)=0.09$

$(2)\quad \dfrac{246-\mu}{\sigma}=-1.3408$ A1

solving (1), (2) gives $\mu=250$, $\sigma=3$ minutes A1

invNorm(0.75,0,1)	0.67449
invNorm(0.09,0,1)	⁻1.34076
$\frac{246-m}{s}=^{-}1.3408$	$\frac{246-m}{s}=^{-}1.3408$
$\frac{252-m}{s}=0.675$	$\frac{252-m}{s}=0.675$
solve$\left(\frac{246-m}{s}=^{-}1.3408 \text{ and } \frac{252-m}{s}=0.675,\{m,s\}\right)$	
	$s=2.97649$ and $m=249.991$

d. I ice-cream cake, C chocolate cake

$I \to I = 0.65 \quad \Rightarrow \quad I \to C = 0.35$

$C \to C = 0.55 \quad \Rightarrow \quad C \to I = 0.45$

$\Pr(I\,I\,I\,C)+\Pr(I\,I\,C\,I)+\Pr(I\,C\,I\,I)$ M1

$=0.65^2\times0.35+0.65\times0.35\times0.45+0.35\times0.45\times0.65$

$=0.3526$ A1

e.i $\Pr(T>2\,|\,T>1)=\dfrac{\Pr(T>2)}{\Pr(T>1)}=\dfrac{\frac{9}{16}\int_2^{\infty} t\,e^{-\frac{3t}{4}}\,dt}{\frac{9}{16}\int_1^{\infty} t\,e^{-\frac{3t}{4}}\,dt}$ M1

$=\dfrac{10}{7}e^{-\frac{3}{4}}$ A1

ii. $E(T)=\frac{9}{16}\int_0^{\infty} t^2\,e^{-\frac{3t}{4}}\,dt=\dfrac{8}{3}$ A1

$E(T^2)=\frac{9}{16}\int_0^{\infty} t^3\,e^{-\frac{3t}{4}}\,dt=\dfrac{32}{3}$

$\text{var}(T)=E(T^2)-(E(T))^2=\dfrac{32}{3}-\left(\dfrac{8}{3}\right)^2$

$\text{var}(T)=\dfrac{32}{9}=3\dfrac{5}{9}$ A1

iii. Solving $\frac{9}{16}\int_0^{m} t\,e^{-\frac{3t}{4}}\,dt=0.5$ gives $m=2.238$ A1

Define $f(t)=\frac{9}{16}\cdot t\cdot e^{\frac{-3\cdot t}{4}}$ — *Done*

$\int_{0}^{\infty} f(t)\, dt$ — 1

$\int_{2}^{\infty} f(t)\, dt$ — $\frac{5\cdot e^{\frac{-3}{2}}}{2}$

$\int_{1}^{\infty} f(t)\, dt$ — $\frac{7\cdot e^{\frac{-3}{4}}}{4}$

$\frac{\frac{5\cdot e^{\frac{-3}{2}}}{2}}{\frac{7\cdot e^{\frac{-3}{4}}}{4}}$ — $\frac{10\cdot e^{\frac{-3}{4}}}{7}$

$\int_{0}^{\infty} (t\cdot f(t))\, dt$ — $\frac{8}{3}$

$\int_{0}^{\infty} (t^{2}\cdot f(t))\, dt$ — $\frac{32}{3}$

$\frac{32}{3}-\left(\frac{8}{3}\right)^{2}$ — $\frac{32}{9}$

⚠ solve$\left(\int_{0}^{m} f(t)\, dt=0.5, m\right)|m>0$ — $m=2.2378$

Question 4

a. maps onto $y' = 2\sqrt{x'+2}$ $\quad \frac{y'}{2} = \sqrt{x'+2}$

$x'+2 = x \Rightarrow x' = x-2$ M1

$\frac{y'}{2} = y \Rightarrow y' = 2y$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $\quad a = 1$, $b = 2$, $h = -2$, $k = 0$ A1

b. $x+2 \geq 0 \Rightarrow x \geq -2$

$B = [-2, \infty)$ A1

c. $f: \; y = 2\sqrt{x+2}$

$f^{-1}: \; x = 2\sqrt{y+2}$

$\sqrt{y+2} = \frac{x}{2}$

$y+2 = \frac{x^2}{4}$ M1

$y = \frac{x^2}{4} - 2$

but dom $f = [-2, \infty)$ ran $f = [0, \infty) =$ dom f^{-1}

$f^{-1}: [0, \infty) \to R$, $f^{-1}(x) = \frac{x^2}{4} - 2$ A1

d. solving $f^{-1}(u) = f(u)$ or $f(u) = u$ or $f^{-1}(u) = u$, with $u > 0$

it is easiest to solve $f(u) = u$

$2\sqrt{u+2} = u$

$4(u+2) = u^2$

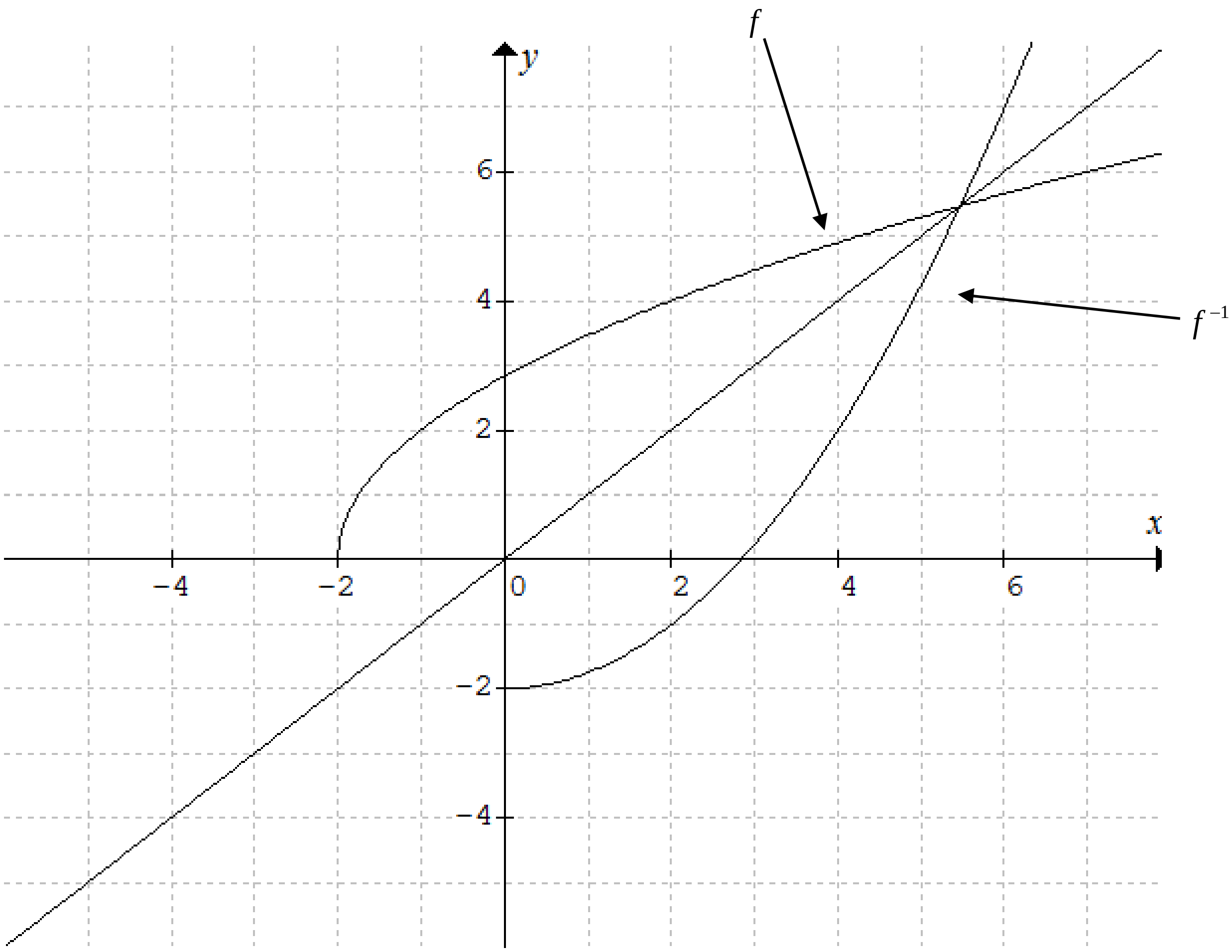
$u^2 - 4u + 4 = 4 + 8 = 12$

$(u-2)^2 = 12$ so $u = 2 \pm 2\sqrt{3}$ but $u > 0$

$u = 2 + 2\sqrt{3}$ A1

Define $f1(x)=2\cdot\sqrt{x+2}$	*Done*
$\text{solve}(f1(y)=x,y)$	$y=\frac{x^2}{4}-2 \text{ and } x\geq 0$
Define $f2(x)=\frac{x^2}{4}-2$	*Done*
$\text{solve}(f1(x)=x,x)$	$x=2\cdot(\sqrt{3}+1)$
$\text{solve}(f2(x)=x,x)\mid x>0$	$x=2\cdot(\sqrt{3}+1)$
$\text{solve}(f2(x)=f1(x),x)\mid x>0$	$x=2\cdot(\sqrt{3}+1)$
$u:=2\cdot(\sqrt{3}+1)$	$2\cdot(\sqrt{3}+1)$

e. G2



f.i. $A=\int_{-2}^{u}\left(f\left(x\right)-x\right)dx=\int_{-2}^{2+2\sqrt{3}}\left(2\sqrt{x+2}-x\right)dx$

$=4\sqrt{3}+\dfrac{22}{3}$ A1

ii. $A=\int_{0}^{u}\left(x-f^{-1}\left(x\right)\right)dx=\int_{0}^{2+2\sqrt{3}}\left(x-\dfrac{x^{2}}{4}+2\right)dx$

$=4\sqrt{3}+\dfrac{16}{3}$ A1

$$\int_{-2}^{u}(f1(x)-x)\,dx \qquad 4\cdot\sqrt{3}+\frac{22}{3}$$

$$\int_{0}^{u}(x-f2(x))\,dx \qquad \frac{4\cdot(3\cdot\sqrt{3}+4)}{3}$$

g. $f'\left(x\right)=\dfrac{1}{\sqrt{x+2}}$

$f'\left(u\right)=f'\left(2+2\sqrt{3}\right)=m_1=\dfrac{1}{\sqrt{4+2\sqrt{3}}}$ A1

$\left(f^{-1}\right)'\left(x\right)=\dfrac{x}{2}$

$\left(f^{-1}\right)'\left(u\right)=\left(f^{-1}\right)'\left(2+2\sqrt{3}\right)=m_2=1+\sqrt{3}$

$\tan\left(\theta\right)=\dfrac{m_2-m_1}{1+m_1m_2}=\dfrac{3+\sqrt{3}}{4}$

or alternatively $\theta_1=\tan^{-1}\left(m_1\right)$, $\theta_2=\tan^{-1}\left(m_2\right)$

$\theta=\theta_2-\theta_1=\tan^{-1}\left(\dfrac{3+\sqrt{3}}{4}\right)=49.8^{0}$ A1

$m1:=\frac{d}{dx}(f1(x))\|x=u$	$\frac{\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)\cdot\sqrt{2}}{2}$
$m2:=\frac{d}{dx}(f2(x))\|x=u$	$\sqrt{3}+1$
$\tan^{-1}\left(\frac{m2-m1}{1+m1\cdot m2}\right)$	49.7922
$\theta1:=\tan^{-1}(m1)$	$\tan^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$
$\theta2:=\tan^{-1}(m2)$	$\tan^{-1}(\sqrt{3}+1)$
$\theta2-\theta1$	49.7922

h. a translation of k units to the left parallel to the x-axis, or away from the y-axis and a dilation by a factor of k, parallel to the y-axis or away from the x-axis. A1

i.

$$g:\quad y=k\sqrt{x+k}$$

$$g^{-1}:\quad x=k\sqrt{y+k}$$

$$\sqrt{y+k}=\frac{x}{k}$$

$$y+k=\frac{x^2}{k^2}$$

$$y=\frac{x^2}{k^2}-k$$

but $D=\text{dom}\,g=[-k,\infty)$, ran $g=[0,\infty)=\text{dom}\,g^{-1}$

$$g^{-1}:[0,\infty)\to R\ ,\ g^{-1}(x)=\frac{x^2}{k^2}-k$$ A1

j. solving $g^{-1}(c)=g(c)$ or $g(c)=c$ or $g^{-1}(c)=c$, with $c>0$ gives

$$c=\frac{k^2+\sqrt{k^3(k+4)}}{2}$$ A1

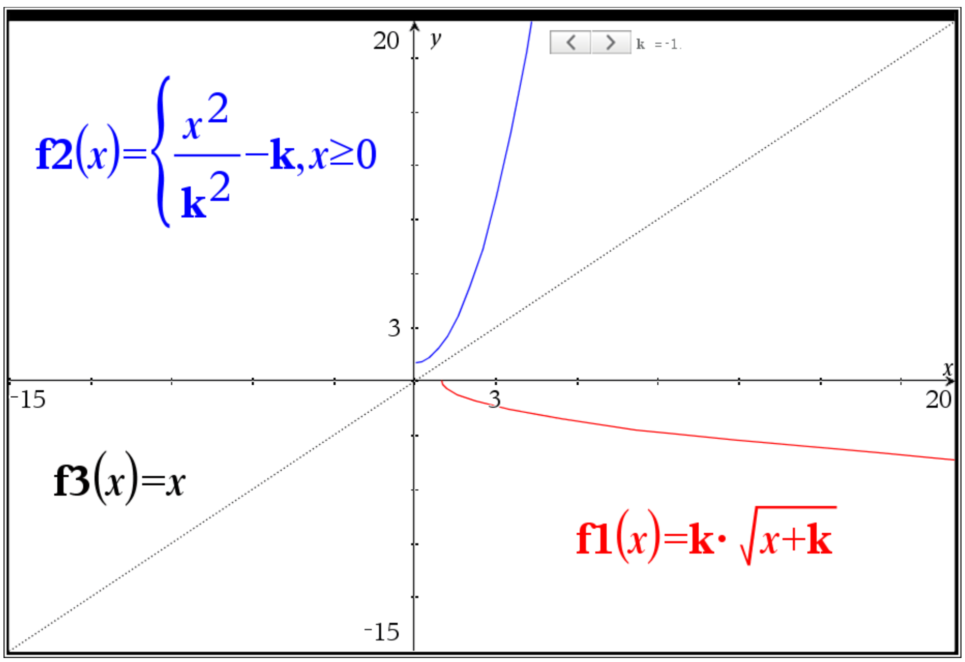
Define $g(x)=k\cdot\sqrt{x+k}$ *Done*

Define $gi(x)=\frac{x^2}{k^2}-k$ *Done*

solve$(g(x)=x,x)$ $x=\frac{-\left(\sqrt{k^3\cdot(k+4)}-k^2\right)}{2}$ or $x=\frac{\sqrt{k^3\cdot(k+4)}+k^2}{2}$

$c:=\frac{\sqrt{k^3\cdot(k+4)}+k^2}{2}$ $\frac{\sqrt{k^3\cdot(k+4)}+k^2}{2}$

k. $k<0$ A1



l. Let $m_1 = \frac{d}{dx}\left(g\left(x\right)\right)$ when $x = c$ and $m_2 = \frac{d}{dx}\left(g^{-1}\left(x\right)\right)$ when $x = c$ M1

given $\theta = 45^0$ $\tan\left(45^0\right) = 1$ solving $\frac{m_2 - m_1}{1 + m_1 m_2} = 1$ for k, with $k > 0$

as the graphs do not intersect when $k < 0$ gives

$k = 4$ A1

or alternatively $\theta_1 = \tan^{-1}\left(m_1\right)$, $\theta_2 = \tan^{-1}\left(m_2\right)$, solving $\theta_2 - \theta_1 = 45^0$ for k

$m1:=\frac{d}{dx}(g(x))\|x=c$	$\frac{k\cdot\sqrt{2}}{2\cdot\sqrt{\sqrt{k^3\cdot(k+4)}+k\cdot(k+2)}}$
⚠ $m2:=\frac{d}{dx}(gi(x))\|x=c$	$\frac{\sqrt{k^3\cdot(k+4)}+k^2}{k^2}$
⚠ $\text{solve}\left(\frac{m2-m1}{1+m1\cdot m2}=1,k\right)$	$k=4$
$\theta1:=\tan^{-1}(m1)$	$\tan^{-1}\left(\frac{k\cdot\sqrt{2}}{2\cdot\sqrt{\sqrt{k^3\cdot(k+4)}+k\cdot(k+2)}}\right)$
$\theta2:=\tan^{-1}(m2)$	$\tan^{-1}\left(\frac{\sqrt{k^3\cdot(k+4)}+k^2}{k^2}\right)$
⚠ $\text{solve}(\theta2-\theta1=45,k)$	$k=4.$
$\theta2-\theta1\|k=4$	45

END OF SECTION B SUGGESTED ANSWERS