VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2018

SOLUTIONS MS

STUDENT NUMBER | | | | | | | | | Letter

# MATHEMATICAL METHODS

## Written examination 1

**Friday 1 June 2018**

**Reading time: 2.00 pm to 2.15 pm (15 minutes)**
**Writing time: 2.15 pm to 3.15 pm (1 hour)**

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|
| 9 | 9 | 40 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

**Materials supplied**
- Question and answer book of 13 pages
- Formula sheet
- Working space is provided throughout the book.

**Instructions**
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

**At the end of the examination**
- You may keep the formula sheet.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

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## Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

**Question 1** (4 marks)

**a.** Let $f(x) = \dfrac{e^x}{\left(x^2 - 3\right)}$.

Find $f'(x)$. 2 marks

$$f'(x) = \frac{e^x(x^2-3) - 2xe^x}{(x^2-3)^2} \quad A1$$

$$= \frac{e^x\left(x^2 - 2x - 3\right)}{\left(x^2-3\right)^2} \quad A1$$

**b.** Let $y = (x + 5)\log_e(x)$.

Find $\dfrac{dy}{dx}$ when $x = 5$. 2 marks

$$\frac{dy}{dx} = \log_e(x) + \frac{x+5}{x} \quad A1$$

when $x = 5$, $\dfrac{dy}{dx} = \log_e(5) + 2$ A1

**Question 2** (4 marks)

Let $f(x) = -x^2 + x + 4$ and $g(x) = x^2 - 2$.

**a.** Find $g(f(3))$. 2 marks

$$g(f(x)) = \left(-x^2 + x + 4\right)^2 - 2$$

$$g(f(3)) = 2$$

A1A1

**b.** Express the rule for $f(g(x))$ in the form $ax^4 + bx^2 + c$, where $a$, $b$ and $c$ are non-zero integers. 2 marks

$$f(g(x)) = -\left(x^4 - 4x^2 + 4\right) + x^2 + 2 \quad M1$$

$$= -x^4 + 4x^2 - 4 + x^2 + 2$$

$$= -x^4 + 5x^2 - 2 \quad A1$$

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**Question 3** (2 marks)

Evaluate $\int_0^1 e^x - e^{-x} dx$.

$= \left[e^x + e^{-x}\right]_0^1$ *A*1

$= \left(e + \frac{1}{e}\right) - (1+1)$

$= e + \frac{1}{e} - 2$ *A*1

**Question 4** (3 marks)

Solve $\log_3(t) - \log_3(t^2 - 4) = -1$ for $t$.

$\frac{t}{t^2-4} = 3^{-1}$ and $t > 0$ and $t^2 - 4 > 0$ A1

$\frac{t}{t^2-4} = \frac{1}{3}$

$3t = t^2 - 4$

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0$ *A*1

$t = 4$ as $t > 2$ *A*1

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**Question 5** (3 marks)

Let $h: R^+ \cup \{0\} \to R,\ h(x) = \dfrac{7}{x+2} - 3$.

**a.** State the range of $h$. 1 mark

$\left(-3, \dfrac{1}{2}\right]$ A1

**b.** Find the rule for $h^{-1}$. 2 marks

$$x = \frac{7}{y+2} - 3$$

$$(x+3)(y+2) = 7 \quad A1$$

$$y + 2 = \frac{7}{x+3}$$

$$h^{-1}(x) = \frac{7}{x+3} - 2 \quad A1$$

**Question 6** (4 marks)

The discrete random variable $X$ has the probability mass function

$$\Pr(X = x) = \begin{cases} kx & x \in \{1, 4, 6\} \\ k & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

**a.** Show that $k = \frac{1}{12}$. 2 marks

| $x$ | 1 | 3 | 4 | 6 |
|---|---|---|---|---|
| $Pr(X=x)$ | $k$ | $k$ | $4k$ | $6k$ |

A1 table

$12k = 1 \Rightarrow k = \frac{1}{12}$ A1

**b.** Find $E(X)$. 1 mark

| $x$ | 1 | 3 | 4 | 6 |
|---|---|---|---|---|
| $Pr(X=x)$ | 1/12 | 1/12 | 4/12 | 6/12 |

$$E(X) = \frac{1}{12} + \frac{3}{12} + \frac{16}{12} + \frac{36}{12} = \frac{56}{12} = \frac{14}{3}$$ A1

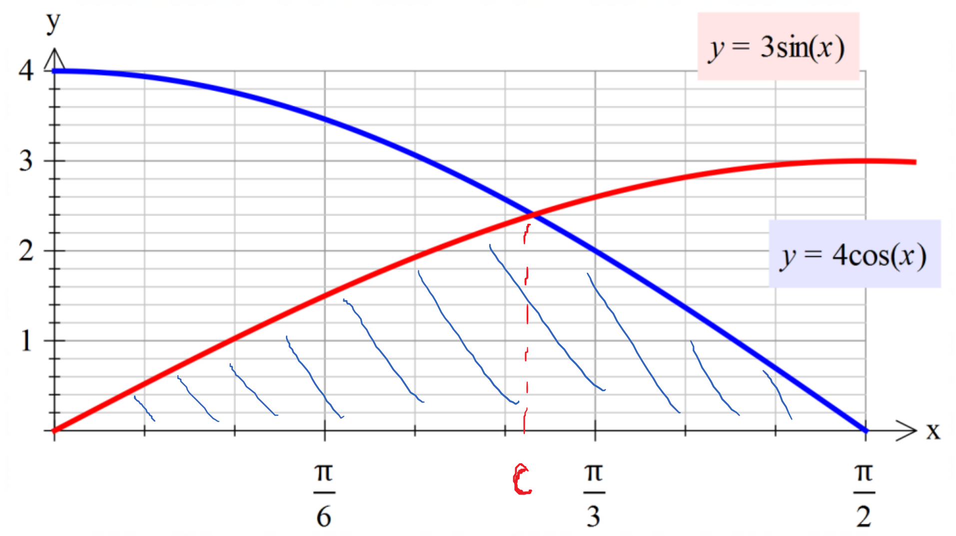
**c.** Evaluate $\Pr(X \geq 3 \mid X \geq 2)$. 1 mark

$$= \frac{\Pr(X \geq 3)}{\Pr(X \geq 2)} = 1$$ A1

**Question 7** (9 marks)

Let $f:\left[0, \frac{\pi}{2}\right] \to R,\ f(x) = 4\cos(x)$ and $g:\left[0, \frac{\pi}{2}\right] \to R,\ g(x) = 3\sin(x)$.

**a.** Sketch the graph of $f$ and the graph of $g$ on the axes provided below. 2 marks




A1A1 for each correct graph

**b.** Let $c$ be such that $f(c) = g(c)$, where $c \in \left[0, \frac{\pi}{2}\right]$.

Find the value of sin($c$) and the value of cos($c$). 3 marks

$$4\cos x = 3\sin x$$
$$16\cos^2 x = 9\sin^2 x$$
$$16\cos^2 x = 9\left(1-\cos^2 x\right)$$
$$\cos^2 x = \frac{9}{25} \qquad A1$$
$$\cos(c) = \frac{3}{5},\ \sin(c) = \frac{4}{5} \qquad A1A1$$

**c.** Let $A$ be the region enclosed by the horizontal axis, the graph of $f$ and the graph of $g$.

**i.** Shade the region $A$ on the axes provided in **part a.** and also label the position of $c$ on the horizontal axis. A1 1 mark

**ii.** Calculate the area of the region $A$. 3 marks

$$A = \int_0^c 3\sin x \; dx + \int_c^{\frac{\pi}{2}} 4\cos x \; dx$$ A1

$$= [-3\cos x]_0^c + [4\sin x]_c^{\frac{\pi}{2}}$$ A1

$$= -3\cos c + 3\cos 0 + 4\sin\left(\frac{\pi}{2}\right) - 4\sin c$$

$$= -3\times\frac{3}{5} + 3 + 4 - 4\times\frac{4}{5}$$

$$= 2$$ A1

**Question 8** (3 marks)

Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.

From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the proportion $p$ of customers who bring their own shopping bags to this large shopping centre was determined to be $\left(\frac{4853}{50000}, \frac{5147}{50000}\right)$.

**a** Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

$$5147 - 4853 = 294/2 = 147$$

$$4853 + 147 = 5000$$

$$\hat{p} = \frac{5000}{50000} = 0.1 \qquad A1$$

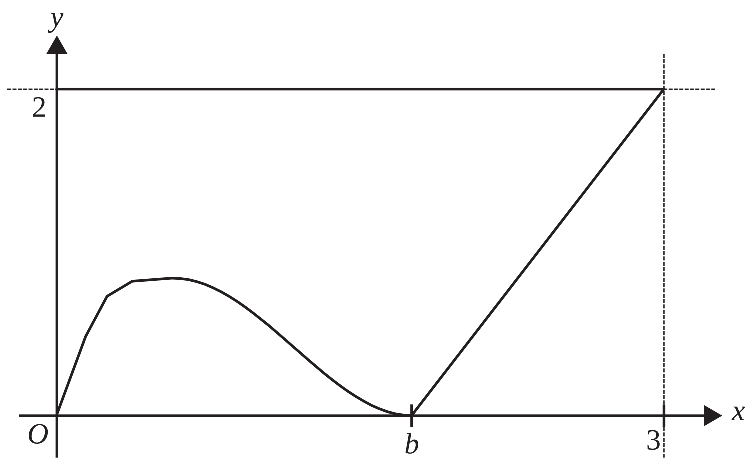
**b** Use the fact that $1.96 = \frac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained. 2 marks

$$\frac{49}{25}\sqrt{\frac{0.1 \times 0.9}{n}} = \frac{147}{50000} \qquad A1$$

$$n = 40000 \qquad A1$$

**Question 9** (8 marks)

The diagram below shows a trapezium with vertices at (0, 0), (0, 2), (3, 2) and ($b$, 0), where $b$ is a real number and $0 < b < 2$.




On the same axes as the trapezium, part of the graph of a cubic polynomial function is drawn. It has the rule $y = ax(x - b)^2$, where $a$ is a non-zero real number and $0 \le x \le b$.

**a.** At the local maximum of the graph, $y = b$.

Find $a$ in terms of $b$. 3 marks

$$y = ax^3 - 2abx^2 + ab^2x$$

$$\frac{dy}{dx} = 3ax^2 - 4abx + ab^2 \quad \text{A1}$$

$$3ax^2 - 4abx + ab^2 = 0$$

$$x = \frac{4ab \pm \sqrt{4a^2b^2}}{6a} \quad \text{A1}$$

$$b = a \times \frac{b}{3}\left(\frac{b}{3} - b\right)^2$$

$$3b = ab \times \frac{4b^2}{9}$$

$$a = \frac{27}{4b^2} \quad A1$$



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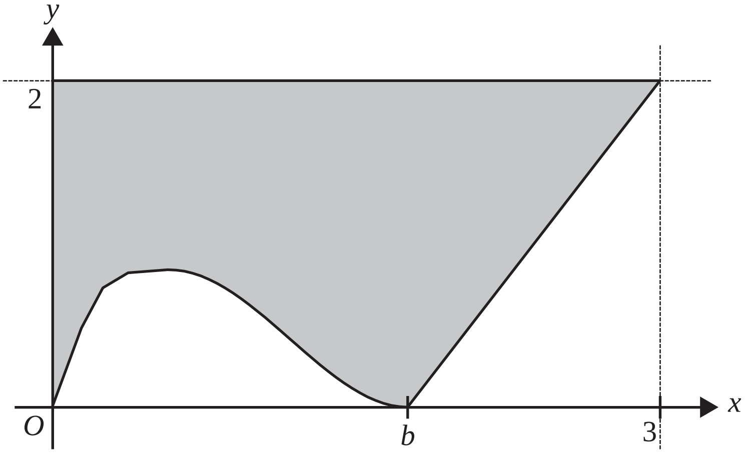
The area between the graph of the function and the $x$-axis is removed from the trapezium, as shown in the diagram below.




**b.** Show that the expression for the area of the shaded region is $b+3-\frac{9b^2}{16}$ square units. 3 marks

$$A=\int_0^b 2-ax(x-b)^2dx+\int_b^3 \frac{2}{3-b}(x-3)+2\ dx$$ A1

$$=\left[2x-\frac{ax^4}{4}+\frac{2abx^3}{3}-\frac{ab^2x^2}{2}\right]_0^b+\left[\frac{2}{3-b}\left(\frac{x^2}{2}-3x\right)+2x\right]_b^3$$ A1

$$=2b-\frac{ab^4}{4}+\frac{2ab}{3}b^3-\frac{ab^2}{2}b^2+\frac{2}{3-b}\left(\frac{9}{2}-9\right)+6-\frac{2}{3-b}\left(\frac{b^2}{2}-3b\right)-2b$$ A1

$$=b+3-\frac{9b^2}{16}$$



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**c.** Find the value of $b$ for which the area of the shaded region is a maximum and find this maximum area. 2 marks

$$A = b + 3 - \frac{9b^2}{16}$$

$$\frac{dA}{db} = 1 - \frac{9}{16} \times 2b$$

$$1 - \frac{9}{16} \times 2b = 0$$

$$b = \frac{8}{9} \qquad A1$$

$$A\left(\frac{8}{9}\right) = \frac{31}{9} \qquad A1$$

**END OF QUESTION AND ANSWER BOOK**

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# MATHEMATICAL METHODS

## Written examination 1

### FORMULA SHEET

**Instructions**

| |
|---|
| This formula sheet is provided for your reference.<br>A question and answer book is provided with this formula sheet. |

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| **Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.** |

# Mathematical Methods formulas

## Mensuration

| | | | |
|---|---|---|---|
| area of a trapezium | $\frac{1}{2}(a+b)h$ | volume of a pyramid | $\frac{1}{3}Ah$ |
| curved surface area of a cylinder | $2\pi rh$ | volume of a sphere | $\frac{4}{3}\pi r^3$ |
| volume of a cylinder | $\pi r^2 h$ | area of a triangle | $\frac{1}{2}bc\sin(A)$ |
| volume of a cone | $\frac{1}{3}\pi r^2 h$ | | |

## Calculus

| | |
|---|---|
| $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ | $\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$ |
| $\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$ | $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$ |
| $\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ | $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$ |
| $\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$ | $\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$ |
| $\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ | $\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$ |
| $\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ | $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$ |
| $\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$ | |

| | | | |
|---|---|---|---|
| product rule | $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ | quotient rule | $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ |
| chain rule | $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ | | |

## Probability

| | | | |
|---|---|---|---|
| $\Pr(A) = 1 - \Pr(A')$ | | $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ | |
| $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ | | | |
| mean | $\mu = \mathrm{E}(X)$ | variance | $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X - \mu)^2) = \mathrm{E}(X^2) - \mu^2$ |

| **Probability distribution** | | **Mean** | **Variance** |
|---|---|---|---|
| discrete | $\Pr(X = x) = p(x)$ | $\mu = \sum x\, p(x)$ | $\sigma^2 = \sum (x - \mu)^2\, p(x)$ |
| continuous | $\Pr(a < X < b) = \int_a^b f(x)dx$ | $\mu = \int_{-\infty}^{\infty} x\, f(x)dx$ | $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$ |

## Sample proportions

| | | | |
|---|---|---|---|
| $\hat{P} = \dfrac{X}{n}$ | | mean | $\mathrm{E}(\hat{P}) = p$ |
| standard deviation | $\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$ | approximate confidence interval | $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$ |

**END OF FORMULA SHEET**