The Mathematical Association of Victoria

Trial Examination 2018

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	D	**11**	B
2	A	**12**	E
3	E	**13**	B
4	C	**14**	D
5	C	**15**	B
6	E	**16**	D
7	C	**17**	C
8	E	**18**	A
9	D	**19**	A
10	A	**20**	B

Question 1 **Answer D**

$f: R \to R,\ f(x) = -2\sin(3x) + 1$

Amplitude = 2

period = $\dfrac{2\pi}{3}$

Question 2 **Answer A**

$f(x) = \dfrac{1}{(1-x)^2} + 2$

- a reflection in the *x*-axis

$y_1 = -\dfrac{1}{(1-x)^2} - 2$

- a dilation of a factor of 2 from the *y*-axis.

$g(x) = -\dfrac{1}{\left(1-\dfrac{x}{2}\right)^2} - 2 = -\dfrac{4}{(2-x)^2} - 2$

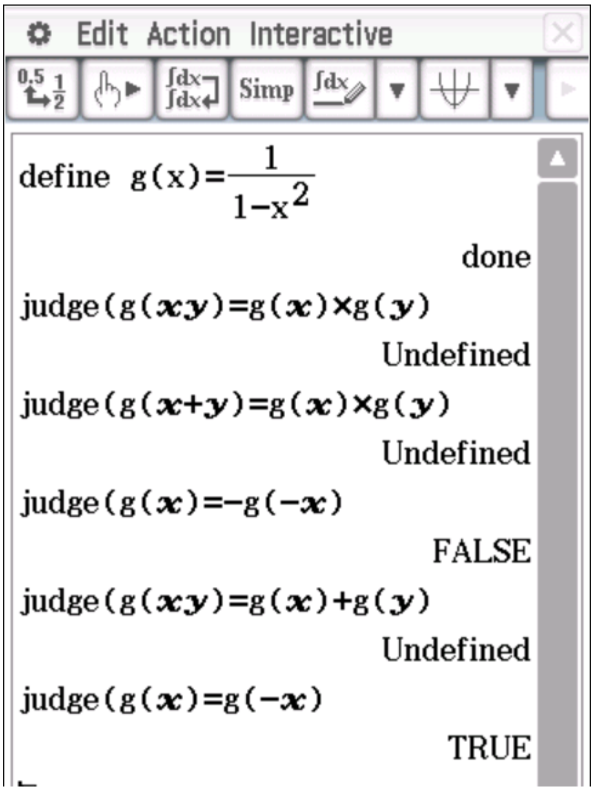
Question 3 **Answer E**

$$g(x)=\frac{1}{1-x^2}$$

g is an even function, hence $g(x)=g(-x)$.

OR

$$g(-x)=\frac{1}{1-(-x)^2}=\frac{1}{1-x^2}=g(x)$$



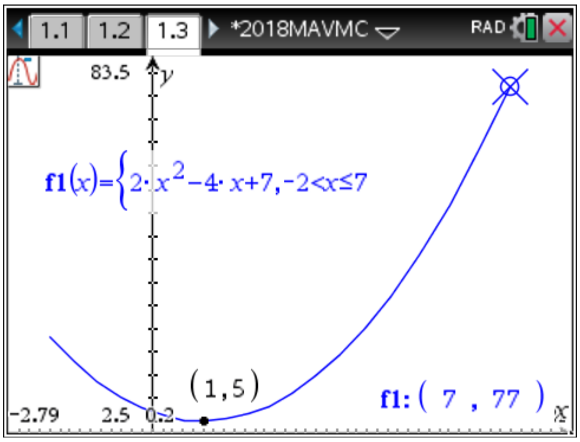
Question 4 **Answer C**

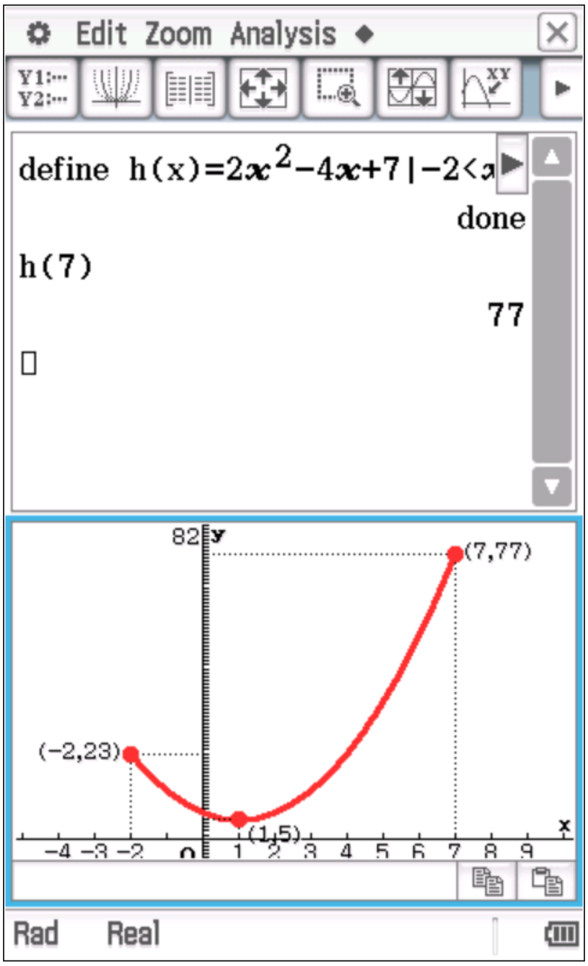
$h:(-2,7]\to R, h(x)=2x^2-4x+7$

$h(x)=2(x-1)^2+5$

The minimum value of h is at the turning point and the maximum occurs when x = 7.

The range is $[5,77]$.


1.1 1.2 1.3 *2018MAVMC
RAD
83.5
y
f1(x)={2·x^2−4·x+7,−2<x≤7
(1,5)
f1: (7 , 77)
-2.79
2.5
0.2
x


Edit Zoom Analysis
define h(x)=2x^2−4x+7|−2<x
done
h(7)
77
82
y
(7,77)
(−2,23)
(1,5)
x
−4 −3 −2 0 1 2 3 4 5 6 7 8 9
Rad Real

Question 5 **Answer C**

$g(x)=\frac{1}{1+3x}$ and $f(x)=(x)^{\frac{2}{3}}$

For $g\left(f(x)\right)$, test that the range $f \subseteq$ domain g

Giving $[0,\infty)\subseteq R\setminus\left\{-\frac{1}{3}\right\}$

The domain of $g\left(f(x)\right)$ is the same as the domain of *f* which is *R*.

The rule is $g\left(f(x)\right)=\frac{1}{1+3x^{\frac{2}{3}}}$.

$g\left(f(x)\right):R\to R,$ where $g\left(f(x)\right)=\frac{1}{1+3x^{\frac{2}{3}}}$

Question 6 **Answer E**

$2\sin(2x)+1=0, x\in[0,\pi]$

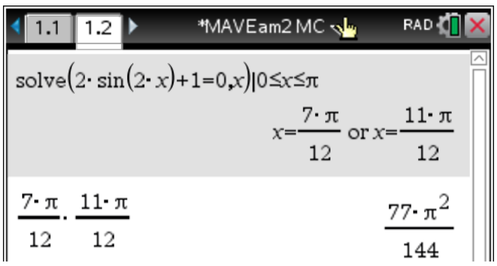
$\sin(2x)=-\frac{1}{2}$

$2x=\pi+\frac{\pi}{6}, 2\pi-\frac{\pi}{6}$

$2x=\frac{7\pi}{6},\frac{11\pi}{6}$

Giving $x=\frac{7\pi}{12},\frac{11\pi}{12}$

Product = $77\left(\frac{\pi}{12}\right)^2$



Question 7 **Answer C**

$(k+1)x^2-2kx-(k-1)=0$

$\Delta=(-2k)^2-4(k+1)\times(-(k-1))$

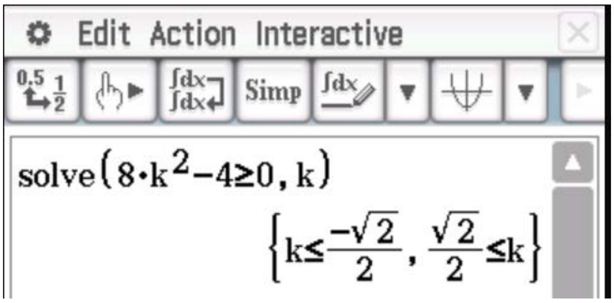
$\Delta=4k^2+4(k^2-1)$

$\Delta=8k^2-4$

Real roots when $\Delta\geq 0$

Giving $8k^2-4\geq 0$

$k\in\left(-\infty,-\frac{\sqrt{2}}{2}\right]\cup\left[\frac{\sqrt{2}}{2},\infty\right)$



Question 8 **Answer E**

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\-2\end{bmatrix}$

Gives the equations

$-x+1=x',\ 4y-2=y'$

Rearrange to get

$-x+1=x',\ x=1-x'$

$4y-2=y',\ y=\frac{y'+2}{4}$

In the equation $y=2\sqrt{x}$

$\frac{y'+2}{4}=2\sqrt{1-x'}$

Giving $y'=8\sqrt{1-x'}-2$

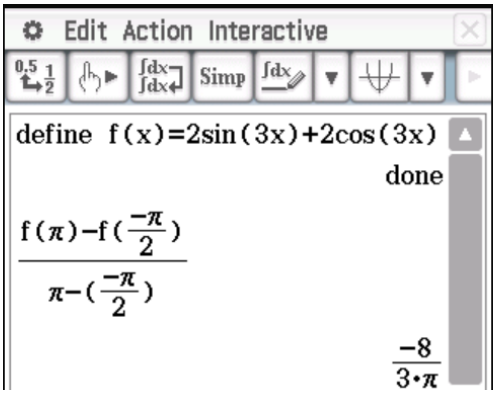
Question 9 **Answer D**

$f(x) = 2\sin(3x) + 2\cos(3x)$

Average rate of change for $x \in \left[-\frac{\pi}{2}, \pi\right]$

$$= \frac{f(\pi) - f\left(-\frac{\pi}{2}\right)}{\pi - \left(-\frac{\pi}{2}\right)}$$

$$= -\frac{8}{3\pi}$$



Question 10 **Answer A**

$f(x) = \sqrt{m - x^2}$

$f'(x) = \frac{1}{2\sqrt{m - x^2}} \times -2x$

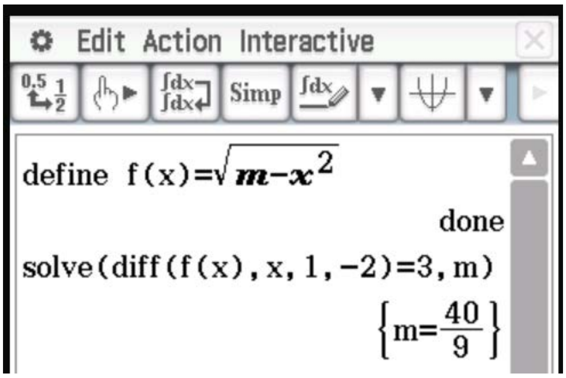
$= \frac{-x}{\sqrt{m - x^2}}$

Tangent has gradient of 3 when $x = -2$

Solve

$\frac{2}{\sqrt{m-4}} = 3$

$m = \frac{40}{9}$

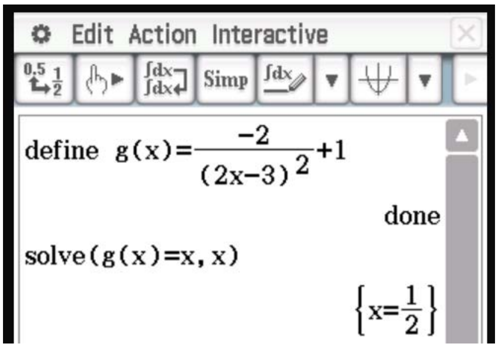


Question 11 **Answer B**

$g(x)=\dfrac{-2}{(2x-3)^2}+1$

Solve $\dfrac{-2}{(2x-3)^2}+1=x$

Intersection g^{-1} and g is $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$



Question 12 **Answer E**

$h(x)=\log_e\left(kx^2+2\right)$

The derivative of $g\left(h(x)\right)$

= derivative of $g\left(h(x)\right)$ × derivative of $h(x)$

$=\dfrac{2kx}{kx^2+2}g'\left(h(x)\right)$

Question 13 **Answer B**

The area enclosed by the graphs of f and g

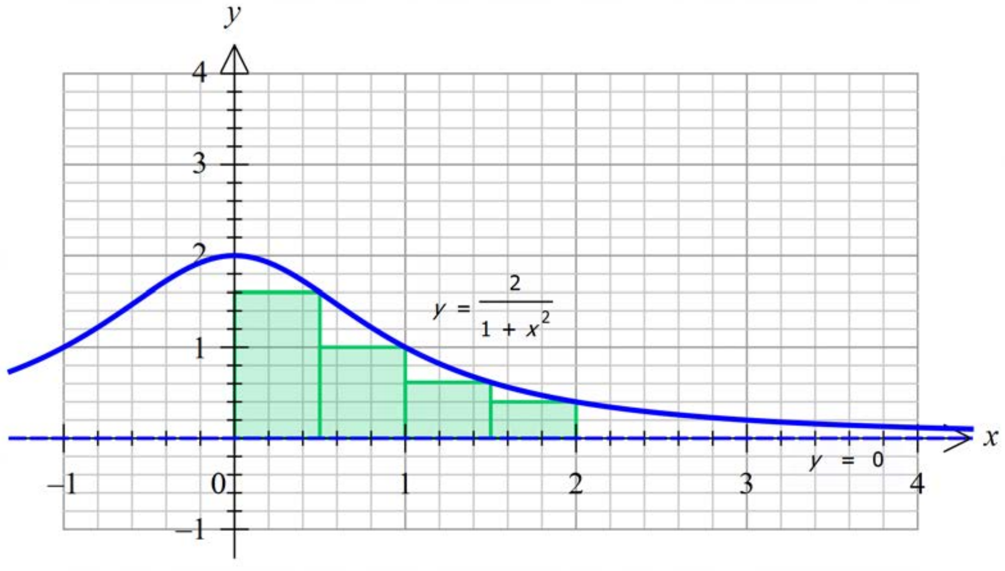
$=\int_2^{\frac{11}{3}}\left(g(x)-f(x)\right)dx$

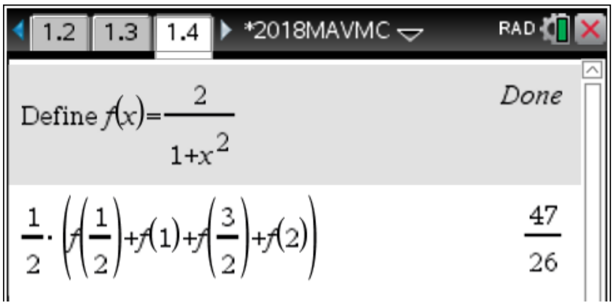
$=\int_{\frac{11}{3}}^{2}\left(f(x)-g(x)\right)dx$

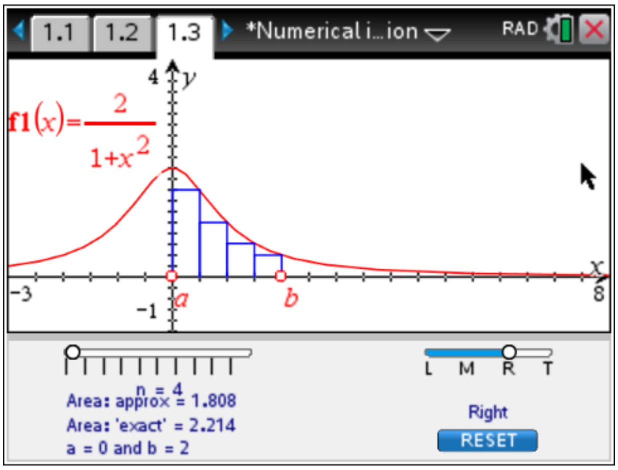
Question 14 **Answer D**

Let $y = f(x) = \dfrac{2}{1+x^2}$

$$A = \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right) = \frac{47}{26}$$







Question 15 **Answer B**

Area of a circle is $A = \pi r^2$

In the first quadrant there is $\frac{1}{4}$ of a circle and $r = 2$.

So the total area is $\frac{1}{4} \times \pi \times 4$.

Rearrange $x^2 + (y-2)^2 = 4$ to make x the subject

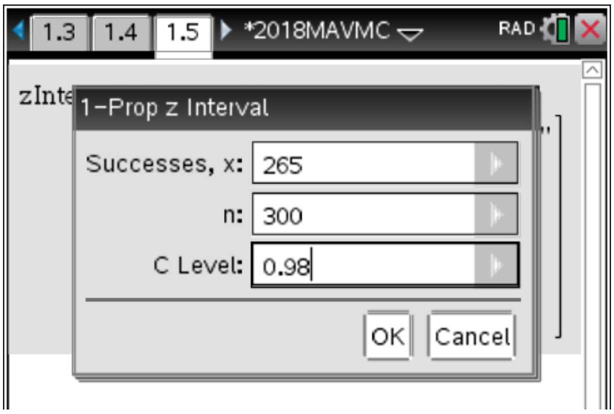
$x = \sqrt{4 - (y-2)^2}$, $x > 0$ in the first quadrant.

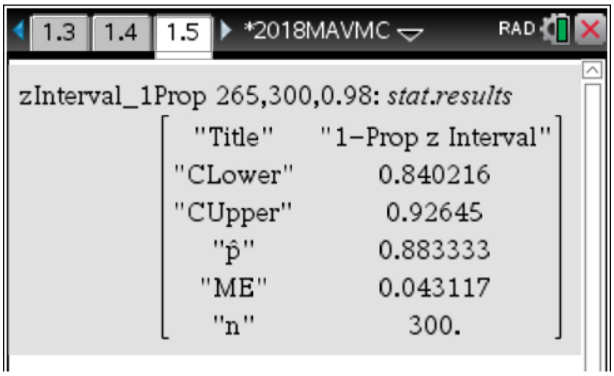
$$\frac{1}{3} \times \frac{1}{4} \times \pi \times 4 = \int_0^d \sqrt{4 - (y-2)^2}\,dy$$

Question 16 **Answer D**

$n = 300$, the number of successes is 265

(0.840, 0.926)





Question 17 **Answer C**

$\Pr(A)=0.6$ and $\Pr(A\cap B)=0.1$

For independent events $\Pr(A)\times\Pr(B)=\Pr(A\cap B)$

$0.6\times\Pr(B)=0.1$

$\Pr(B)=\dfrac{1}{6}$

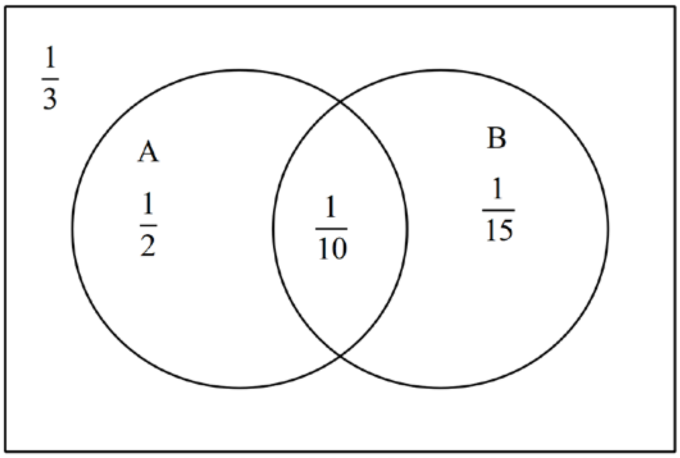
$\Pr(A\cap B')=\dfrac{3}{5}-\dfrac{1}{10}=\dfrac{1}{2}$

then $\Pr(A'\cap B')=\dfrac{5}{6}-\dfrac{1}{2}=\dfrac{1}{3}$

	A	***A'***	
B	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{6}$
B'	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{6}$
	$\frac{3}{5}$	$\frac{2}{5}$	1

Or using a Venn diagram

$\Pr(A\cap B)=\dfrac{1}{10}$, $\Pr(B\cap A')=\dfrac{1}{15}$, $\Pr(A\cap B')=\dfrac{1}{2}$, $\Pr(A'\cap B')=\dfrac{1}{3}$



Question 18 **Answer A**

$X \sim \text{Bi}(n, 0.2)$

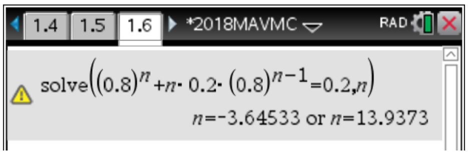
$\Pr(X > 1) > 0.8$

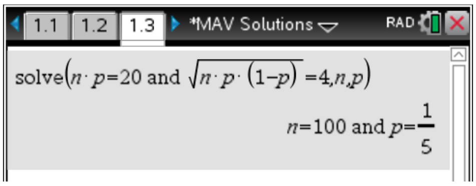
$1 - (\Pr(X = 0) + \Pr(X = 1)) > 0.8$

$\Pr(X = 0) + \Pr(X = 1) < 0.2$

$$0.8^n + \binom{n}{1} 0.2 \times 0.8^{n-1} < 0.2$$

$n = 14$





Question 19 **Answer A**

$X \sim N\left(\mu, \sigma^2\right)$

0.6% of giraffes have a gestation period greater than 155 days

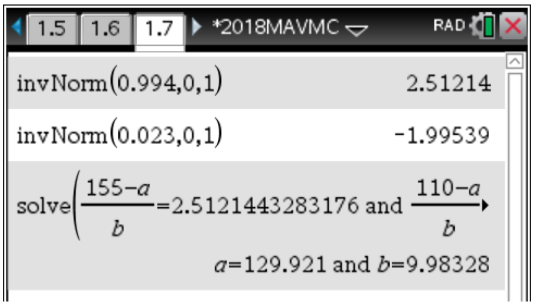
$\Pr\left(X > 155\right) = 0.006$

and 2.3% of giraffes have a gestation period less than 110 days

$\Pr\left(X < 110\right) = 0.023$

Solve $\dfrac{155-\mu}{\sigma} = 2.512...$ and $\dfrac{110-\mu}{\sigma} = -1.995...$ for μ and σ

$\mu = 130$ and $\sigma = 10$

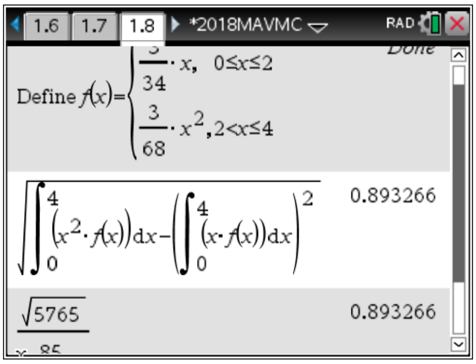


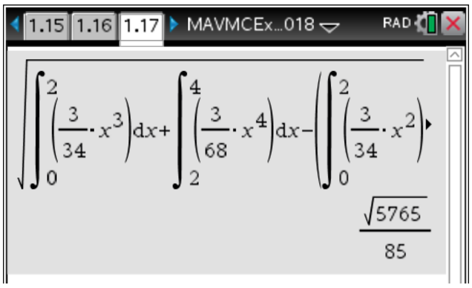
Question 20 **Answer B**

$$f(x) = \begin{cases} \dfrac{3}{34}x & 0 \le x \le 2 \\ \dfrac{3}{68}x^2 & 2 < x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$\operatorname{sd}\left(X\right) = \sqrt{E\left(X^2\right) - \left(E\left(X\right)\right)^2}$$

$$\operatorname{sd}\left(X\right) = \sqrt{\left(\int_0^4 x^2 \times f(x) - \left(\int_0^4 x \times f(x)\right)^2\right)} = \frac{\sqrt{5765}}{85}$$


1.6 1.7 1.8 *2018MAVMC
RAD
Define f(x)=
34
3/68 · x², 2<x≤4
0.893266
√5765
0.893266
85


1.15 1.16 1.17 MAVMCEx...018
RAD
√5765/85

SECTION B

Question 1

$f: R \to R,\ f(x) = 18x^2 - 36x^4$

a. (0, 0) **1A**

b. Solve $f(x) = 0$

$(0,0), \left(\frac{\sqrt{2}}{2}, 0\right), \left(-\frac{\sqrt{2}}{2}, 0\right)$ **1A** any two correct **2A** all correct

c. $f'(x) = 36x - 144x^3$ **1A**

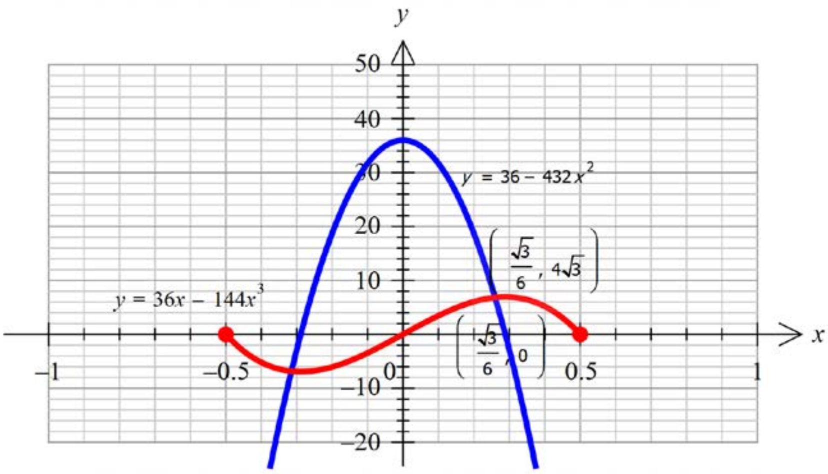
Solve $f'(x) = 0$

Turning points $(0,0)$, $\left(\frac{1}{2}, \frac{9}{4}\right)$, $\left(-\frac{1}{2}, \frac{9}{4}\right)$ **1A**

d. Sketch a graph of the derivative function with equation $f'(x) = 36x - 144x^3$. The maximum occurs at the turning point.

Solving $f''(x) = 36 - 432x^2 = 0$ for points of inflexion gives $x = \pm\frac{\sqrt{3}}{6}$.

From the graph given, gradient at its maximum at $x = \frac{\sqrt{3}}{6}$. **1A**



e. Line joining $A(-k, g(-k))$ and $B(k, g(k))$ where

$g(k) = 18k^2 - 36k^4$, $g(-k) = 18k^2 - 36k^4$

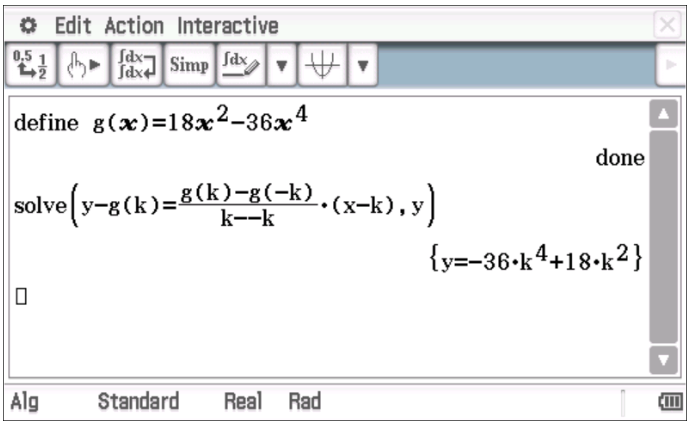
Gradient $= \frac{g(k) - g(-k)}{k - (-k)} = 0$

Equation of the straight line

$y - g(k) = 0(x - k)$

$y = g(k)$

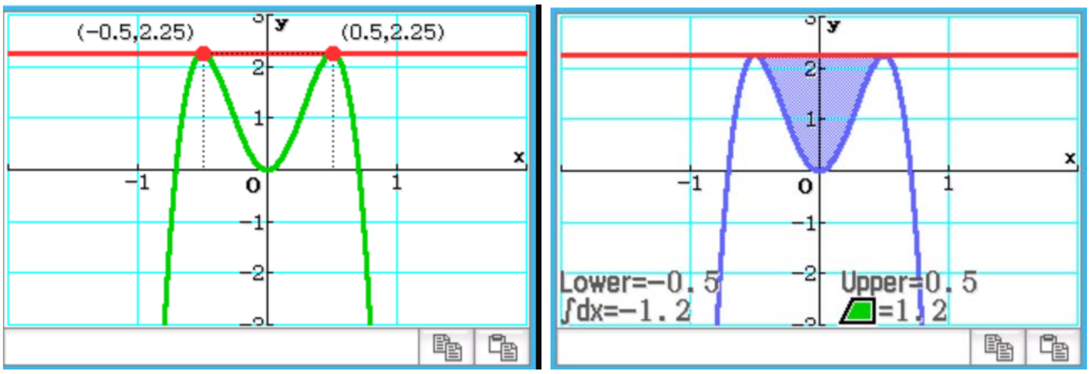
$y = 18k^2 - 36k^4$ **1A**



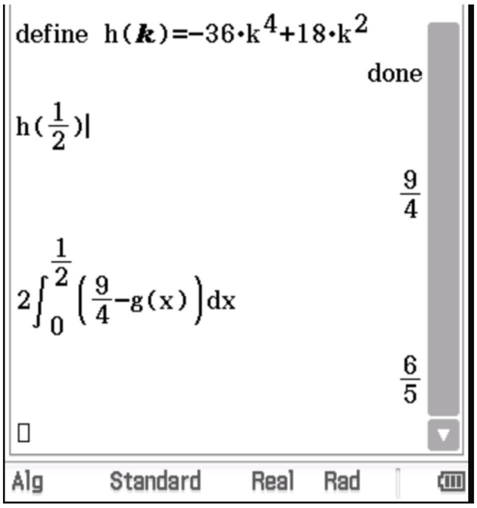
f.i. Area when $k=\frac{1}{2}$

Area $=2\int_0^{\frac{1}{2}}\left(\frac{9}{4}-g(x)\right)dx$

$=\frac{6}{5}$ sq units **1A**



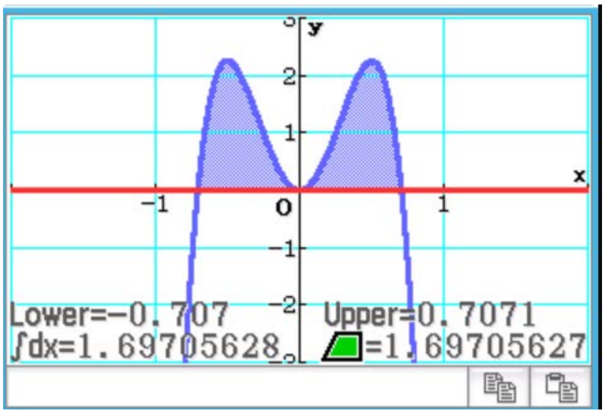
Graph showing intersection of $y=\frac{9}{4}$ and $g(x)$, then showing enclosed Area $=\frac{6}{5}=1.2$



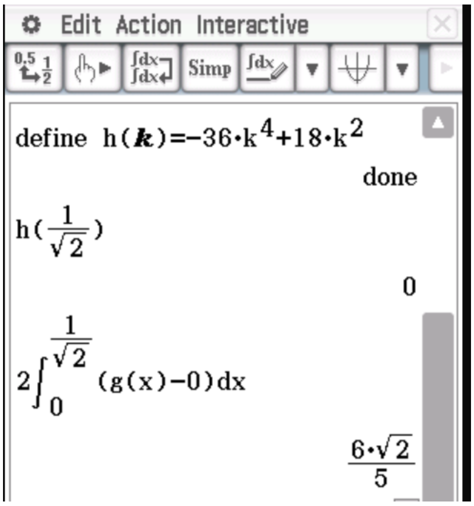
f.ii. Area when $k = \frac{1}{\sqrt{2}}$

Area $= 2\int_0^{\frac{1}{\sqrt{2}}} \left(g(x) - 0\right) dx$

$= \frac{6\sqrt{2}}{5}$ sq units **1A**



Graph showing $y = 0$ and $g(x)$, then showing enclosed Area $\frac{6\sqrt{2}}{5} \approx 1.697$

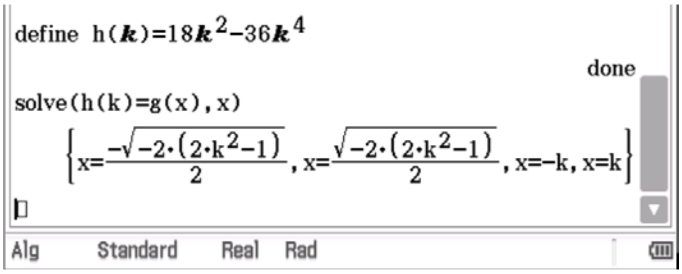


f.iii. Enclosed area when $\frac{1}{2} \le k \le \frac{1}{\sqrt{2}}$

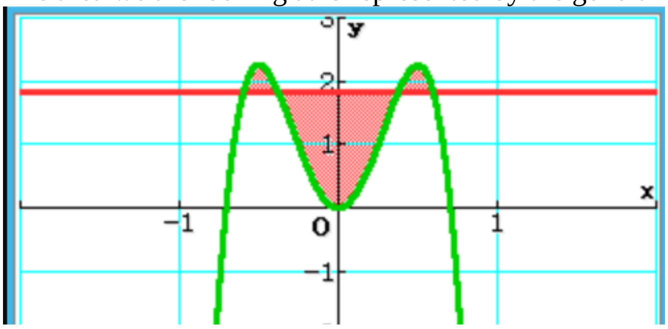
Let equation of line $h(k) = 18k^2 - 36k^4$

Solve $h(k) = g(x)$ to find points of intersection of graph of $g(x)$ and straight line.

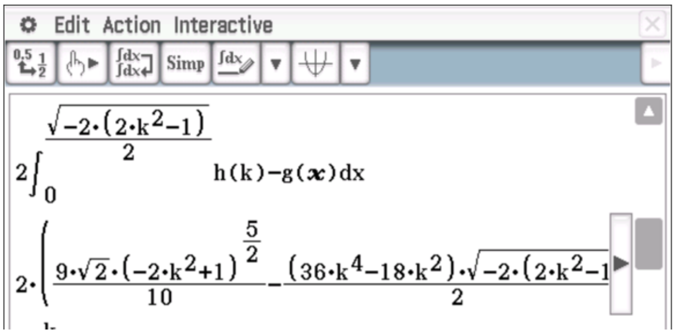
Points of intersection at $x = \pm k,\ x = \pm\frac{\sqrt{-2\left(2k^2 - 1\right)}}{2}$ **1M**

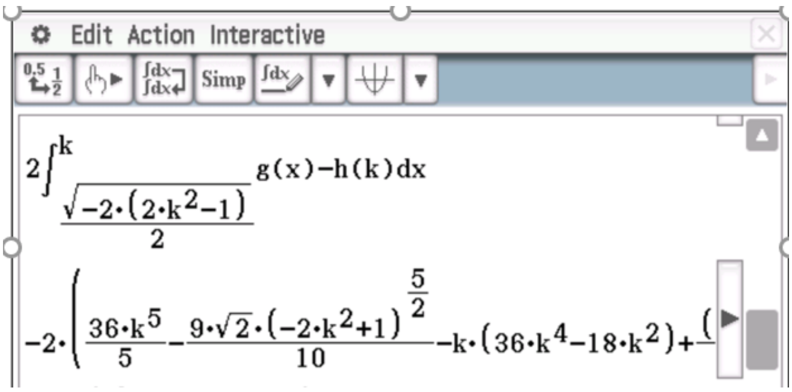


The area we are looking at is represented by the general case shown below.

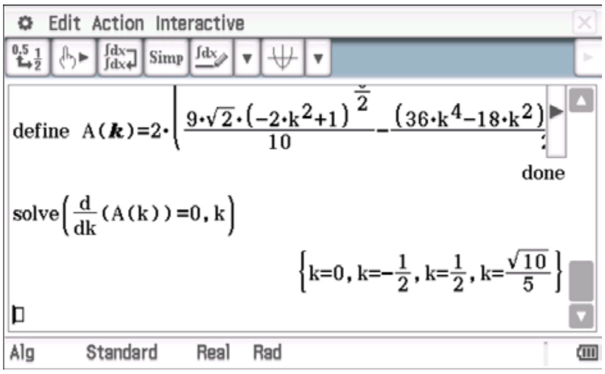


Area $= A(k) = 2\int_{0}^{\frac{\sqrt{-2(2k^2-1)}}{2}} \left(h(k)-g(x)\right)dx + 2\int_{\frac{\sqrt{-2(2k^2-1)}}{2}}^{k} \left(g(x)-h(k)\right)dx$ **1M**





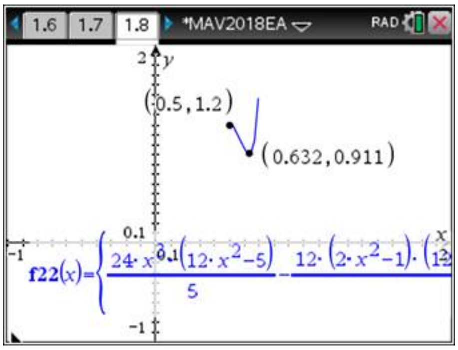
$A'(k) = 0$ gives $k = 0,\ \pm\frac{1}{2},\ \frac{\sqrt{10}}{5}$

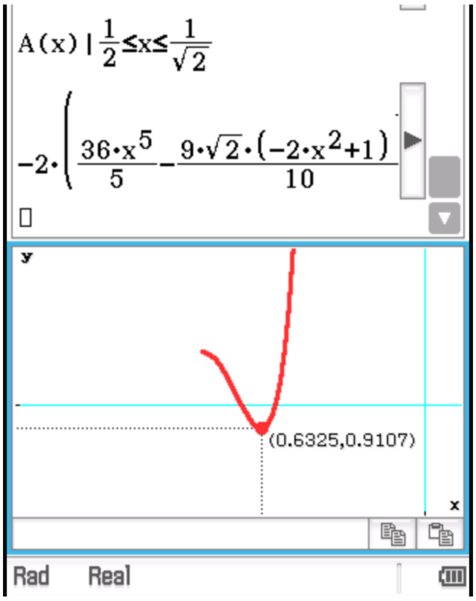


Within domain $\frac{1}{2} \le k \le \frac{1}{\sqrt{2}}$, minimum area occurs at

$k = \frac{\sqrt{10}}{5}$ **1A**

Or to visualise the answer trace value as below.





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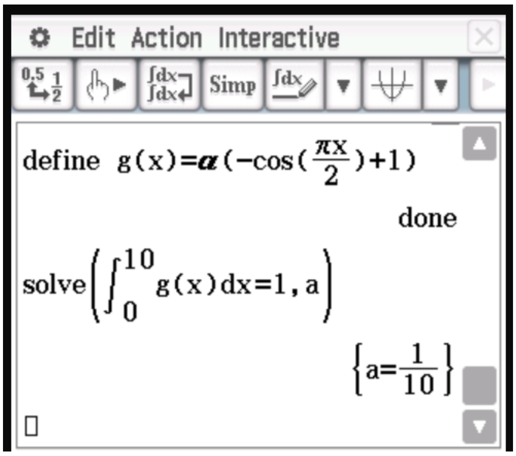
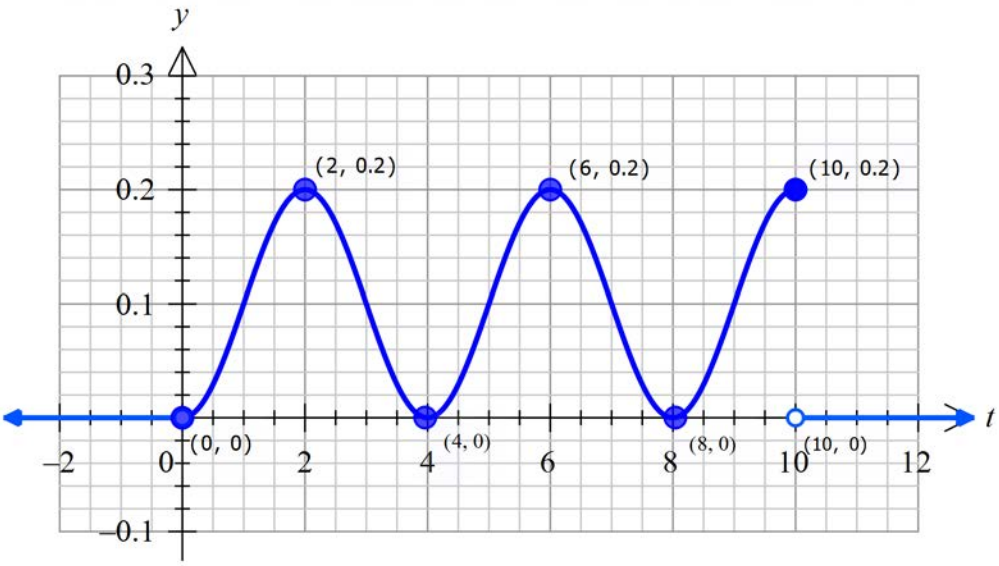
Question 2

a. $\int_{0}^{10} a\left(-\cos\left(\frac{\pi t}{2}\right)+1\right)dt=1$ **1A**

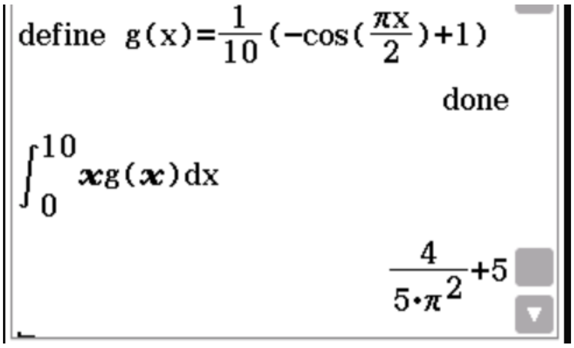


b. shape **1A**, drawing along the axis **1A** , correct coordinates **1A**



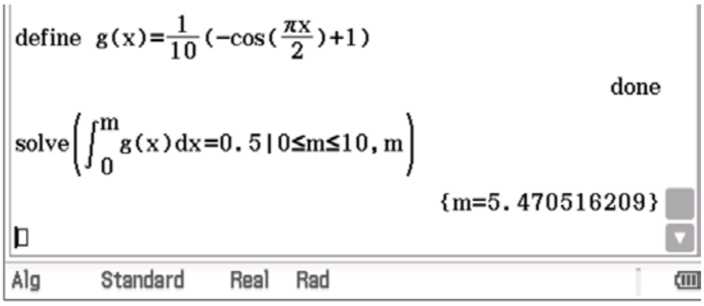
c.i. mean $=\frac{1}{10}\int_{0}^{10} t\left(-\cos\left(\frac{\pi t}{2}\right)+1\right)dt$ **1M**

$=\frac{4}{5\pi^2}+5$ hours **1A**

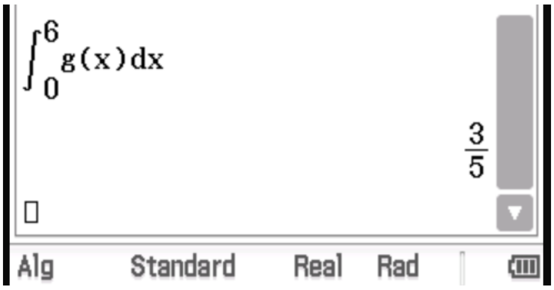


c.ii. Median, *m*, solve $\frac{1}{10}\int_0^m\left(-\cos\left(\frac{\pi t}{2}\right)+1\right)dt=0.5$ **1M**

$m=5.5$ hours **1A**

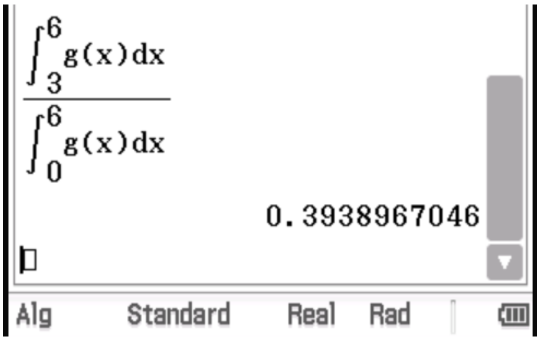


d. $\Pr(T\le 6)=\frac{3}{5}$ **1A**



e. $\Pr(T>3\,|\,T<6)=\frac{\Pr(3<T<6)}{\Pr(T<6)}$ **1M**

$=0.3939$ **1A**



f. $(0.4)^3 = 0.064$ OR $\frac{8}{125}$ **1A**

g. $X \sim \text{Bi}(5, 0.4)$

$\Pr(X = 5 \mid X \geq 3) = \dfrac{\Pr(X = 5)}{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)}$ **1M**

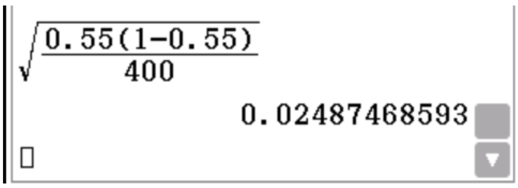
$= \dfrac{0.4^5}{{}^5C_3 0.4^3 0.6^2 + {}^5C_4 0.4^4 0.6^1 + 0.4^5}$ **1M**

$= \dfrac{1}{31}$ **1A**

h. Standard deviation of $\hat{P} = \sqrt{\dfrac{p(1-p)}{n}}$ where $p = 0.55, n = 400$

$= \sqrt{\dfrac{0.55(1-0.55)}{400}}$

$= 0.0249$ **1A**



i. $\dfrac{60}{100} \times 400 = 240$

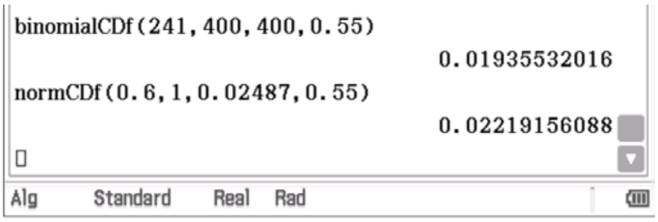
$X \sim \text{Bi}(400, 0.55)$

$\Pr(X > 240) = \Pr(X \geq 241)$ **1M**

$= 0.0194$ **1A**

j. $Y \sim \text{N}\left(0.55, (0.02487...)^2\right)$

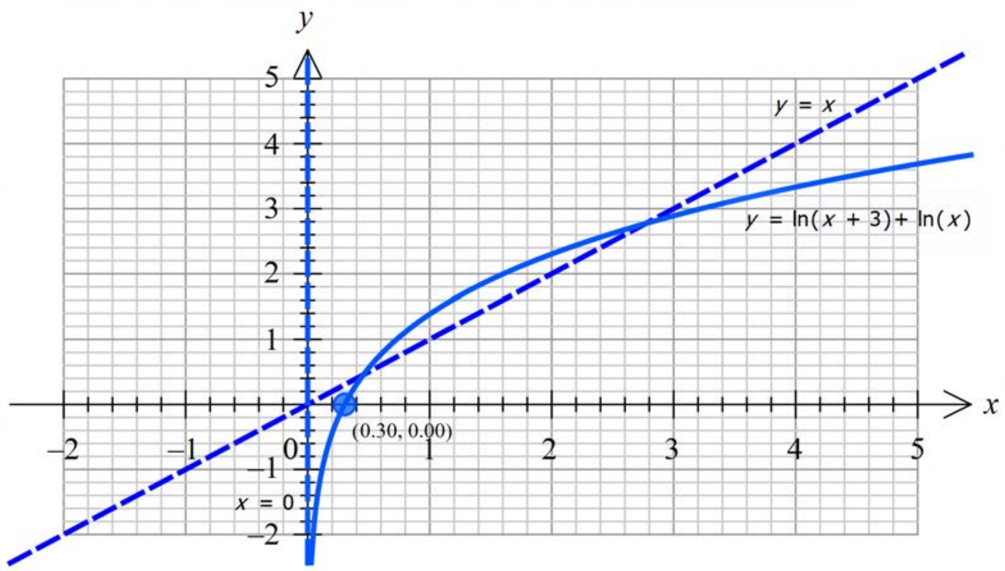
$\Pr(Y > 0.6) = 0.022$ **1A**



Question 3

a. Shape **1A**

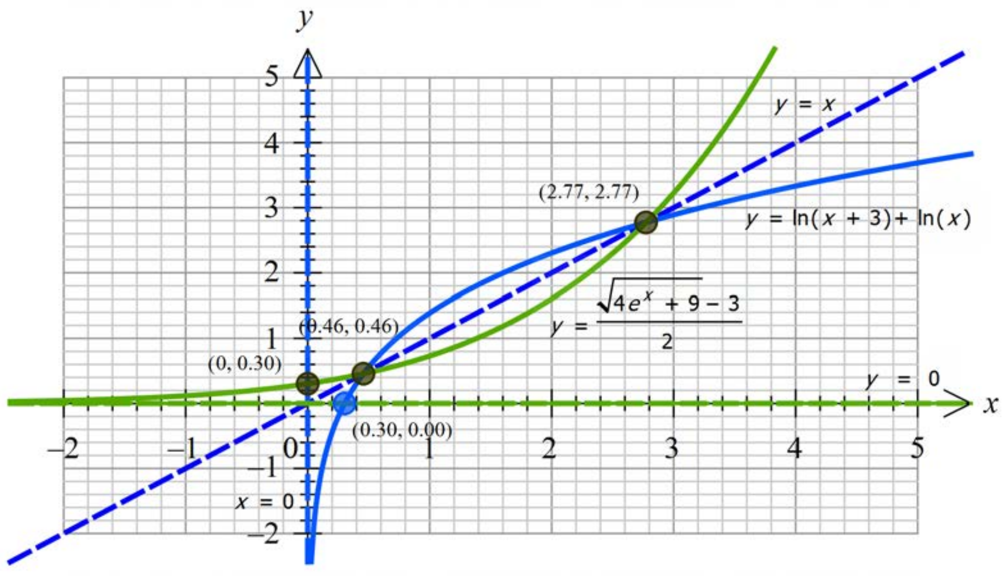
Asymptote $x = 0$ and axial intercept (0.30, 0.00) **1A**

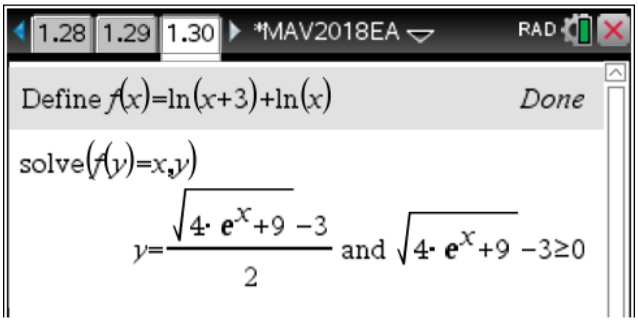


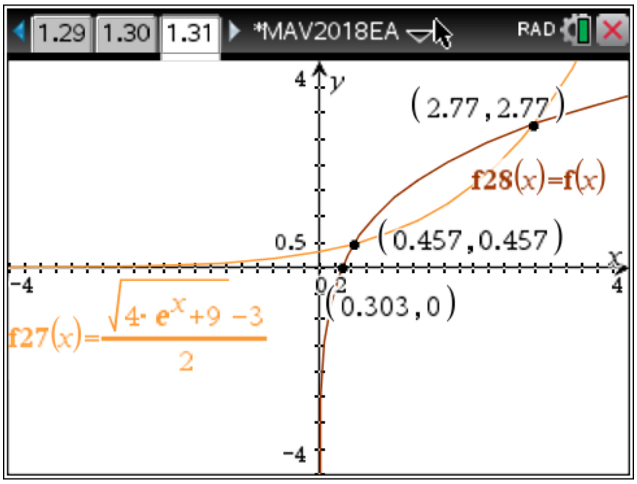
b. $f^{-1}(x) = \dfrac{\sqrt{\left(4e^{x} + 9\right)} - 3}{2}$ **1A**

(0.46, 0.46) and (2.77, 2.77) **1A**

Everything else correct **1A**







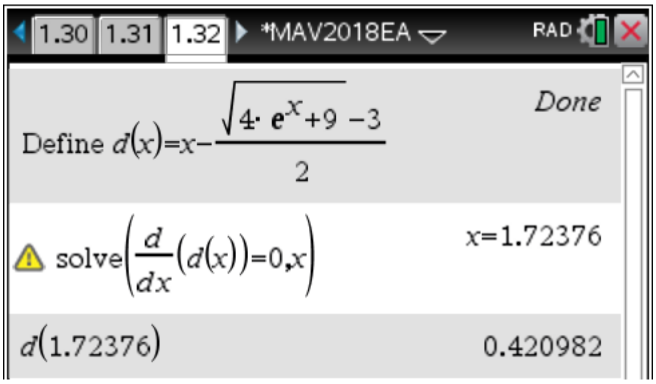
c. Translation of $-k$ units to the right. **1A**

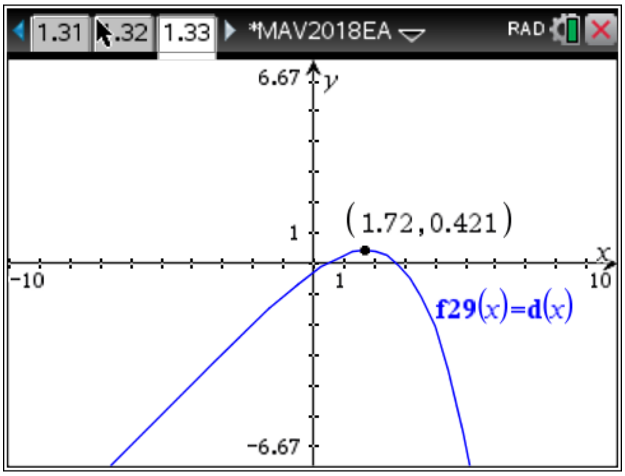
d. $d(x) = x - \frac{\sqrt{(4e^x + 9)} - 3}{2}$ **1A**

e. Solve $d'(x) = 0$ or graph d **1M**

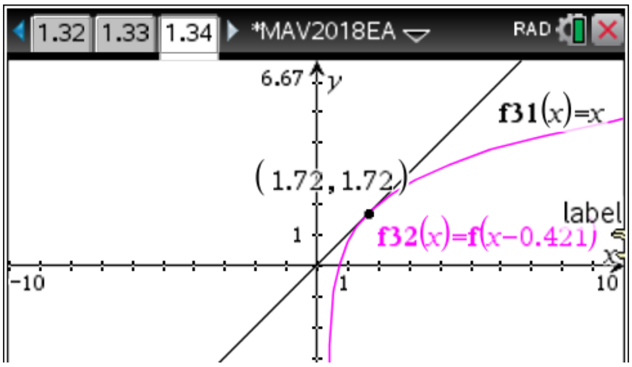
Maximum (1.72..., 0.421…)

$k = -0.421$ **1A**





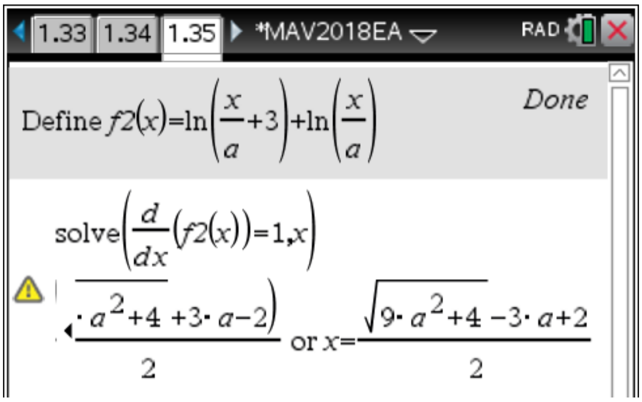
Check with the graph



f. Dilation by a factor of a from the y-axis. **1A**

g. Solve $f_2'(x)=1$ for x.

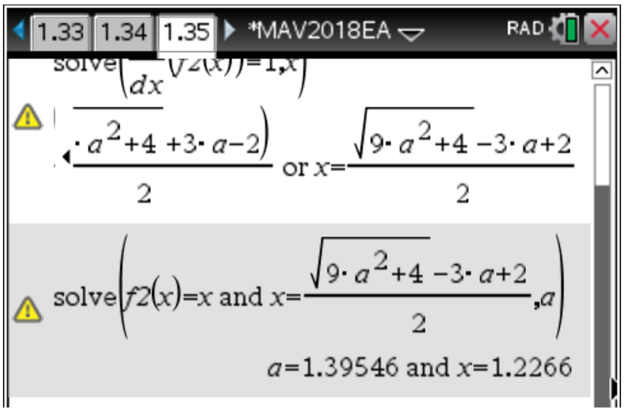
$$x=\frac{\sqrt{9a^2+4}-3a+2}{2}$$ **1A**



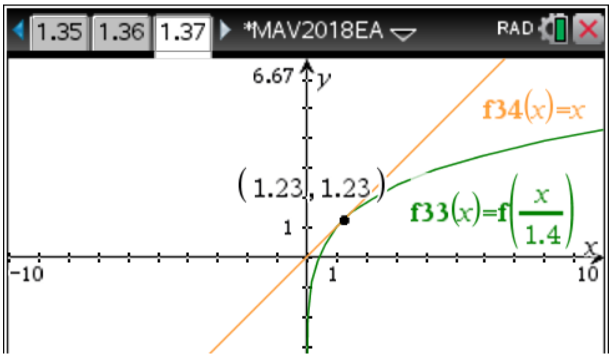
h. Solve $f_2(x) = x$ and $x = \dfrac{\sqrt{9a^2+4}-3a+2}{2}$ for a. **1M**

$a = 1.395\ldots$

$= 1.40$ correct to two decimal places **1A**



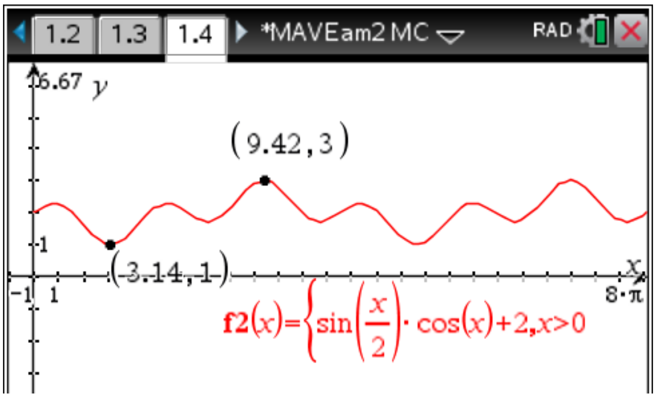
Check with the graph.



Question 4

a. period is 4π **1A**

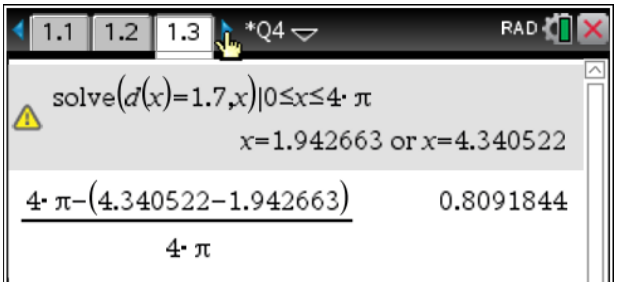
range is $[1,\ 3]$ **1A**

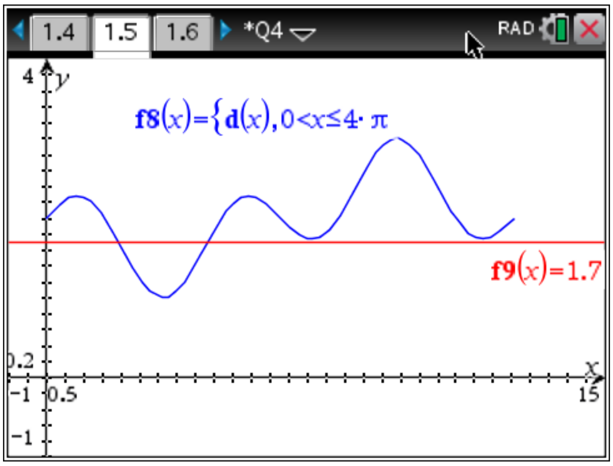


b. Solve $d(t)=1.7$ for t.

$$\frac{4\pi-(4.340...-1.942...)}{4\pi}$$ **1M**

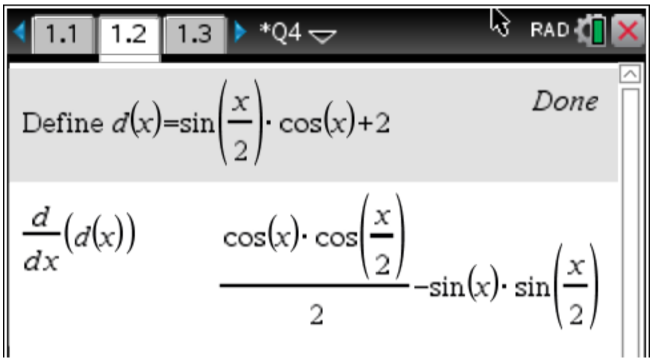
$=0.80918...=0.809$ correct to three decimal places **1A**

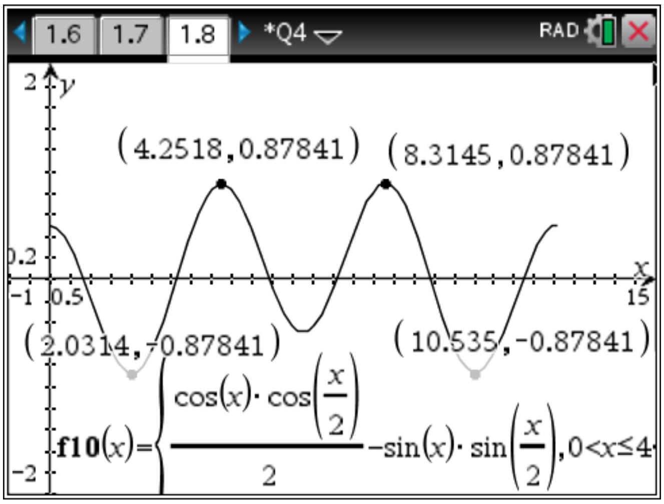


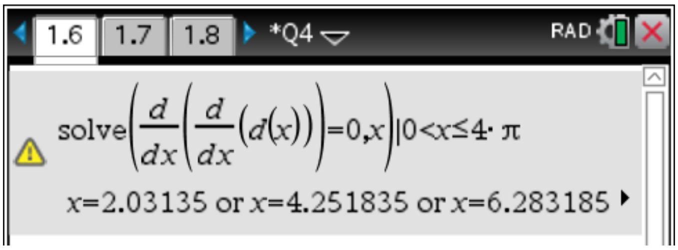


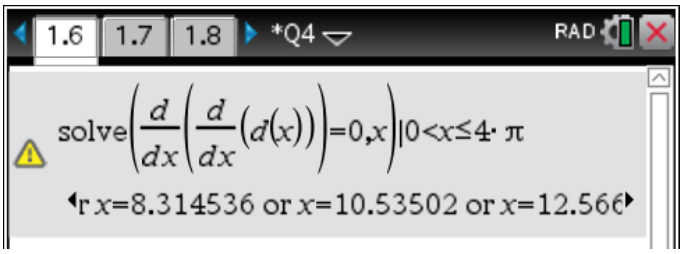
c. $d'(t)=\frac{1}{2}\cos(t)\cos\left(\frac{t}{2}\right)-\sin(t)\sin\left(\frac{t}{2}\right)$ **1A**

t = 2.03, 4.25, 8.31, 10.54 **1A**



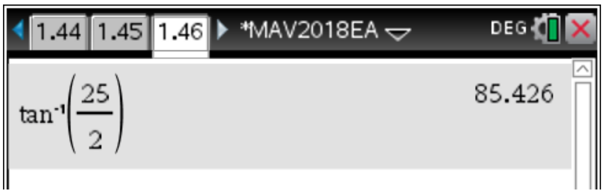






d. $\theta = \tan^{-1}\left(\frac{25}{2}\right)$

$\approx 85.4°$ **1A**

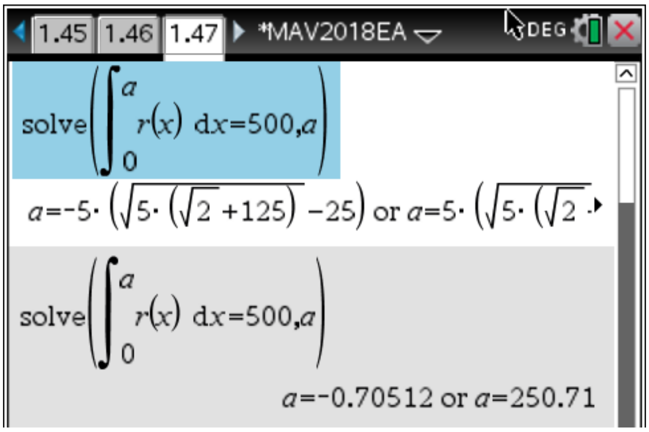


e. Total Volume $= \dfrac{25\times 10}{2}(1+3)\ \text{m}^3$ **1M**

$= 500\ \text{m}^3$ **1A**

f. Solve $\displaystyle\int_0^a \left(\frac{dV}{dt}\right)dt = 500$ for a. **1M**

$a = 250.71$ min **1A**



g. $1\times 25\times 10 = 250\ \text{m}^3$ out

Hence depth 2 m **1A**

h. $V_{pumped} = \displaystyle\int\left(\frac{dV}{dt}\right)dt$

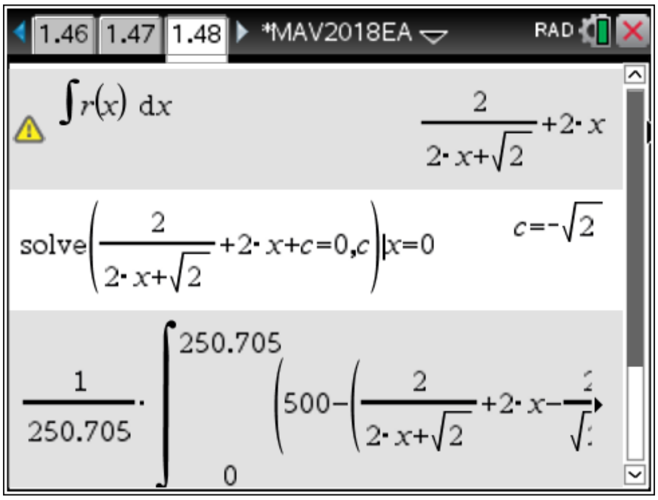
$= 2t + \dfrac{2}{2t+\sqrt{2}} + c$, substitute (0, 0)

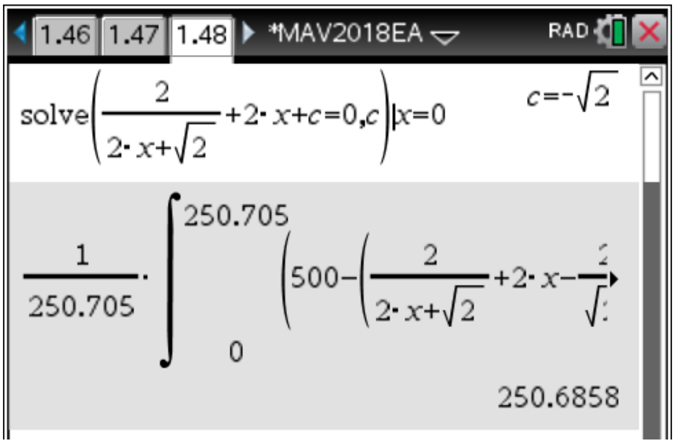
Hence $c = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$ **1M**

$V_{remaining} = 500 - \left(2t + \dfrac{2}{2t+\sqrt{2}} - \dfrac{2}{\sqrt{2}}\right)$ **1H**

Average value $= \dfrac{1}{250.705...}\displaystyle\int_0^{250.705...}\left(500 - \left(2t + \frac{2}{2t+\sqrt{2}} - \frac{2}{\sqrt{2}}\right)\right)dt$ **1H**

$= 250.69\ \text{m}^3$ correct to two decimal places **1A**





END OF SOLUTIONS